Jordan type and Artinian Gorenstein algebras

Anthony Iarrobino

Northeastern University

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Abstract

Let ℓ be an element in the maximal ideal of a local Artinian algebra A. The Jordan type P_ℓ of the multiplication map $m_\ell : A \to A$ is the partition giving the sizes in the Jordan block decomposition for m_ℓ, which is nilpotent. The Jordan degree type is a finer invariant, which specifies the initial degrees of "strings" comprising P_ℓ.

I. When A is graded Gorenstein, results of T. Harima and J. Watanabe concerning central simple modules show that there is a reflexive symmetry in the Jordan degree type. We outline this symmetry and give examples.

II. When A is non-graded Gorenstein, the associated graded algebra A^* of A has a symmetric decomposition; this, combined with the deformation properties of Jordan type leads to families of Gorenstein algebras of Hilbert function H) in codimension three having several irreducible components. Joint with Pedro Macias Marques.

Plan

1. Multiplication maps on an Artinian algebra A, Jordan type, Jordan degree type (JDT).
2. Symmetry of JDT for graded AG algebras.
3. Cells $\mathbb{V}(E_P), P \in \mathcal{P}(T)$.
4. Kinds of Artinian algebras and Jordan type.
5. Families Gor(H) with several irreducible components (non-graded Gorenstein, codim. 3).
6. Compatible sets of partitions $P_{\ell_i}, \ell_i \in A_1$. if time

1. Multiplication maps on an Artinian algebra A

Given an element $\ell \in \mathfrak{m}_A$, the maximum ideal of an Artinian algebra A over a field k, with $\dim_{\mathsf{k}} A = n$, we denote by m_ℓ the multiplication map

$m_\ell : A \to A, m_\ell(a) = \ell \cdot a$, and by

P_ℓ: the *Jordan type* of ℓ, the partition of n giving the sizes of the Jordan blocks of m_ℓ.

Since m_ℓ is nilpotent, all eigenvalues are zero, and the conjugacy class of m_ℓ is given by the partition P_ℓ.

See [IMM] for a detailed introduction to Jordan type.

Example (P_ℓ depends on the choice of $\mathrm{char}\ \mathsf{k}$)

$B = \mathsf{k}[x,y]/(x^3,y^3)$. $B = \begin{array}{ccccc} & & x^2 & x^2y & x^2y^2 \\ & x & xy & xy^2 & \\ 1 & y & y^2 & & \end{array}$

$H(B) = (1,2,3,2,1)$.

Letting $\ell = x+y$ we have that m_ℓ has strings

• $m_\ell : 1 \to \ell \to \ell^2 \to \ell^3 \to \ell^4 \to 0$, $(\mathrm{char}\ \mathsf{k} \neq 2,3)$

$(x-y) \to (x^2-y^2) \to (x^2y - xy^2) \to 0$

$(x^2+xy+y^2) \to 0$:

$\mathrm{P}_\ell = (5_0, 3_1, 1_2) = H(B)^\vee$ when $\mathrm{char}\ k \neq 2,3$.

• m_x has strings $\underline{1, x, x^2}; \underline{y, yx, yx^2}; \underline{y^2, y^2x, y^2x^2}$.

$P_x = (3,3,3)$.

Jordan degree type (JDT)

JDT: Lumini work group: N. Abdallah, N. Altafi, S. Faridi, I., L.Khatami,

P. Macias Marques, J. Yaméogo. Also, B. Costa & R. Gondim.

$B = \mathsf{k}[x, y]/(x^3, y^3)$.

$B = \langle 1; x, y; x^2, xy, y^2; xy^2, yx^2; x^2y^2\rangle$

$H(B) = (1, 2, 3, 2, 1)$. Socle degree $j_B = 4$.

JDT: 5_0: string of length 5 beginning degree zero.

JDT for $\ell = x + y$ on B: $\mathrm{P}_\ell = (5_0, 3_1, 1_2)$,

End degrees $\mathrm{Q}_\ell = (5_{0+4}, 3_{1+2}, 1_{2+0}) = (5_4, 3_3, 1_2)$.

Symmetry: $a_k \in \mathrm{P}_\ell \Leftrightarrow a_{j-k} \in \mathrm{Q}_\ell$.

JDT for m_x on B: $\mathrm{P}_x = (3_0, 3_1, 3_2)$.

End degrees $\mathrm{Q}_x = (3_2, 3_3, 3_4)$.

2. Symmetry of JDT for graded Artinian Gorenstein

Let A be Artinian Gorenstein (AG) of socle degree $j : A_j = 1, A_{j+1} = 0$.
JDT: $\mathrm{P}_\ell = (\{\mathrm{p}_1\}_{i_1}, \dots, \{\mathrm{p}_t\}_{i_t})$; here $i_k =$ degree.
EDT: $\mathrm{Q}_\ell = (\{\mathrm{p}_1\}_{i_1+\mathrm{p}_1-1}, \dots, \{\mathrm{p}_t\}_{i_t+\mathrm{p}_t-1})$.

Theorem (JDT Symmetry, by Harima-Watanabe))

Let A be graded AG of socle degree j.
A. The end degree type Q_ℓ satisfies

$$\mathrm{Q}_\ell = (\{\mathrm{p}_1\}_{j-i_1}, \dots, \{\mathrm{p}_t\}_{j-i_t}).$$

B. The Jordan degree type is invariant under the substitution of indices, $i_k \to j + 1 - i_k - p_k$.

Example

Let $H(A) = (1, 2, 3, 2, 1)$. Then there are 9 partitions of diagonal lengths T. Of these, four: $(5_0, 3_1, 1_2), (5_0, 2_1, 2_2), (4_0, 4_1, 1_2)$ and $(3_0, 3_1, 3_2)$ can occur for CI algebras.

One more, $(3, 3, 1^3)$ construed as $(3_0, 3_3, 1_1, 1_2, 1_3)$ satisfies the condition of symmetry, but is not possible for a CI.

Four more, $(5, 2, 1^3), (4, 2, 1^3), (3, 2^3), (3, 2, 2, 1^2)$ fail the JDT symmetry condition for a CI partition.

From N. Altafi, I., L. Khatami CIJT - arXiv 1810.00716

Claim: $(4,4,1)$ is a CIJT.

1	x	x^2	x^3
y	yx	yx^2	yx^3
y^2			

Figure: Difference-one hook in the Ferrers diagram of partition $(4,4,1)$. (from [CIJT])

Example The ideal (y^3+x^3, xy^2) is a CI having initial ideal in x-direction $I^* = (y^3, xy^2, x^4)$, whose cobasis has shape $P=(4,4,1)$, and JDT $(4_0, 4_1, 1_2)$.

Ostensibly, we need 3 generators, with initial terms y^3, xy^2, x^4, but

$x^4 = x(y^3+x^3) - y(xy^2)$ is kicked out.

CI partitions of diagonal lengths $H = (1,2,3,2,1)$.

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & & \\ * & & & & \end{array}$$

$(5,3,1)$

$(y^3, xy^2 + x^3)$

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & & & \\ * & * & & & \end{array}$$

$(5,2,2)$

$(y^3 + x^3, x^2y)$.

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & & & \end{array}$$

$(4,4,1)$

$(y^3 + x^3, xy^2)$

$$\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}$$

$(3,3,3)$.

(y^3, x^3).

CIJT partitions in height two (Altafi, I.,Khatami)

Hilbert function $H(A)$ for CI $A = \mathsf{k}[x, y]/I$ satisfies $H = (1, 2, \ldots, d-1, d^k, d-1, \ldots, 2, 1), k \geq 1$ (H is symmetric – F.H.S. Macaulay).

Theorem (CIJT)

When $k = 1$ there are 3^{d-1} partitions of diagonal lengths H, of which 2^{d-1} are possible for CI. When $k \geq 2$ there are $2 \cdot 3^{d-1}$ partitions in $\mathcal{P}(H)$ of which 2^d are CI. $\mathcal{P}(H)$: *all partitions P of diagonal lengths H.*

[Count of $\# \mathcal{P}(H)$ by the *hook code* introduced by J. Yaméogo, I. and extends to all H of height two.]

3. Cells $\mathbb{V}(E_P)$ for $A = R/I, R = \mathsf{k}[x, y], H(A) = H$.

Point: fix a C.I. Hilbert function graded height two, $H = (1, 2, \ldots, d-1, d, \ldots, d, d-1, \ldots, 2, 1)$. Choose a partition P of diagonal lengths T (so $P \in \mathcal{P}(H)$). We define an affine space cell $\mathbb{V}(E_P)$.

Theorem

i. $GGor(H) = \bigcup_{P\in\mathcal{P}(H)} \mathbb{V}(E_P)$.

ii. *For* $A \in \mathbb{V}(E_P)$, $\ell \in A_1$ *generic* $\Rightarrow P_{\ell,A} = P$.

This cellular decomposition is an element in our study of which Jordan types are possible for a C.I. quotient of R (N.Altafi, I.,L. Khatami, arXiv 1810.00716)

Definition (Initial ideal of I, and the Cell $\mathbb{V}(E_P)$)

The *initial monomial* $\mu(f) = \mathrm{in}(f)$ of a form $f = \sum_k a_k y^k x^{i-k}$, $a_k \in \mathsf{k}$ in the y-direction is the monomial $\mu(f) = y^s x^{i-s}$ of highest y-degree s among those with non-zero coefficients a_k. Given an ideal $I \subset R = \mathsf{k}[x, y]$, we denote by $\mathrm{in}(I)$ the ideal

$$\mathrm{in}(I) = (\{\mathrm{in}(f), f \in I\})$$

generated by the initial monomials of all elements of I. We may identify $\mathrm{in}(I)$ with an initial ideal E_P for a partition $P \in \mathcal{P}(T)$. See next slide
We denote by $\mathbb{V}(E_P)$ the affine variety parametrizing all ideals $I \subset R$ having initial ideal E_P.

$P \in \mathcal{P}(T)$, the partitions of diagonal lengths T

Given a partition $P = (p_1 \geq p_2 \geq \cdots \geq p_s)$ we denote by C_P the set of monomials:

$$\begin{array}{l} 1, \quad x, \quad x^2, \qquad \ldots \qquad , x^{p_1-1}; \\ y, \quad yx, \quad yx^2, \quad \ldots \quad , yx^{p_2-1} \\ \ldots \\ y^{s-1}, y^{s-1}x, \ldots, y^{s-1}x^{p_s-1}. \end{array}$$

Let $E_P = (x^{p_1}, yx^{p_2}, \cdots, y^{s-1}x^{p_s}, y^s)$.
The Hilbert function $H = H(R/E_P)$ is the *diagonal lengths* of P: we write $P \in \mathcal{P}(T)$.

Example: Cell $\mathbb{V}(E_P)$, $P = (5,2,2)$, $H = (1,2,3,2,1)$.

$$C_P: \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \\ y & yx & & & \\ y^2 & y^2x & & & \end{array} \qquad E_P = (x^5, yx^2, y^3).$$

$A = R/I, I = I_{a,b,c}$: (Actually, $I = (f_3, f_1)$ so is C.I.)

$$\begin{aligned} f_0 &= x^5; \\ f_1 &= yx^2 + cx^3 = x^2 g_1, \text{ where } g_1 = y + cx; \\ f_2 &= yf_1 = yx^2 g_1; \\ f_3 &= y^3 + ay^2x + bx^3 = g_3. \end{aligned}$$

Key: The multiplication m_x has strings on A

$1, x, x^2, x^3, x^4; g_1, xg_1; yg_1, yxg_1$. So $P_{A,x} = (5,2,2)$.

Use of Hessians and mixed Hessians

Theorem (B. Costa and R. Gondim[1])

The Jordan type of any standard graded Artinian Gorenstein algebra $A = Q/\operatorname{Ann} f$ depends only on the ranks of certain mixed Hessians of f.

They use string diagrams to visualize the Jordan degree type of A and determine the possible Jordan types for some low socle degree examples of Hilbert functions, such as $H = (1, r, r, 1)$.

[1] Advances in Applied Math. 111 (Oct 2019) 101941.

JDT for $H = (1, n, n, 1)$ (B. Costa, R. Gondim).

We denote by P_A the JDT for $\ell \in A_1$ general enough.

Proposition

Let $H = (1, n, n, 1)$, that $A = R/\operatorname{Ann} f$ is graded AG, and suppose that the rank of the Hessian $H(f)$ is $r \le n$. Then the JDT of A is
$\mathrm{P}_A = (4_0, 2_1^{r-1}, 1_1^{n-r}, 1_2^{n-r})$.
In particular, when $n \ge 5$, we may have $r = n - 1$.

Example (Perrazo cubic)

$F = XU^3 + YUV^2 + ZU^2V$, $H_f = (1, 5, 6, 5, 1)$ and $\mathrm{P}_A = (5_0, 3_1^3, 2_1, 2_2)$.
But for a generic f, $P_A = (5_0, 3_1^4, 1_2)$ (strong Lefschetz case).

B. Costa and R. Gondim determine the possible JDT for graded AG of Hilbert functions $H = (1, n, a, n, 1)$ and $H = (1, n, a, a, n, 1)$, $a \geq n$ in terms of the ranks of Hessian and mixed Hessians.

4. Kinds of Artin algebras to consider for JT:

A. Graded Artinian Gorenstein: H symmetric:
2 variables: (Macaulay) $H = (1, 2, \ldots d, d \ldots, 2, 1)$,
Variety GGor(H), fibration to $\mathbb{P}^d$ understood.
(N. Altafi, L. Khatami,I.: JT possible for CI, any ℓ.)
3 variables (Buchsbaum-Eisenbud) H understood,
Variety GGor(H) irreducible. Catalecticants.
4 variables (Srinivasan-I, Boij): Some H understood,
GGor(H) has several irred. components, related
to space curves when $H = (1, 4, 7, \ldots, 7, 4, 1)$.
5 variables: non-unimodal Hilbert functions.

Kinds of Artin algebras . . .

B. Graded Artinian, G_T
2 variables: G_T smooth, projective:
(J. Briançon;J. Yameogo-I. Fibration $Z_T \to G_T$.
$r \geq 3$ variables: poorly understood.
B′. Non-standard graded Artinian: T. Harima, J. Watanabe- study Lefschetz Properties.
C. Gorenstein local: use symmetric decomposition (P. Macias Marques-I.).
C′. Local Artinian: understood in 2 variables.

Questions on Jordan type of $P_\ell, \ell \in \mathfrak{m}_A$.

a. Compare P_ℓ with $H(A)^\vee$. Is $P_\ell \leq H(A)^\vee$ in dominance partial order? Open in general!

b. For which A is $P_\ell = H(A)^\vee$ (strong Lefschetz), for generic ℓ? (See [H-W, SLN 2080],[MW], and [IMM, arXiv 1802.07383: Artinian algebras and Jordan type.]

c. Fix H, consider $GGor_H$, graded Gorenstein: which P_ℓ occur? (2 vars. N. Altafi, L. Khatami, I.). Open in $r \geq 3$ variables! despite Pfaffian structure theorem of D. Buchsbaum-D. Eisenbud for $r = 3$!

Questions on Jordan type of P_ℓ ...

d. Behavior under tensor product $A \otimes_{\mathsf{k}} B$, or free extensions (a generalization of TP), or for connected sums $A\#_T B$ (C. McDaniel, A. Secealanu, I -).

e. Which pairs of Jordan types can occur in the same algebra A? Thorny area of commuting nilpotent Jordan types, very basic questions open:
e$'$. P. Oblak conjecture on the maximum commuting Jordan type $Q(P)$ for P. [known when $Q(P)$ has ≤ 3 parts – P. Oblak, L. Khatami.]
Fact: Parts of $Q(P)$ differ pairwise by at least two (P. Oblak, T. Košir): CI property of $\mathsf{k}\{A, B\}, B$ generic $\in \mathcal{N}_A$.

Inequality between P_ℓ and $H(A)^\vee$.

Dominance partial order: Given $P = (p_1, \dots, p_s), p_1 \geq p_2 \geq \cdots \geq p_s$: then $P \geq P'$ if $\sum_1^k p_i \geq \sum_1^k p'_i$ for all k.

Thm. If A is graded and $\ell \in A_i, i > 0$ is homogeneous, then $P_\ell \leq H(A)^\vee$.(Dominance partial order)

Open whether this is true if we take $\ell \in \mathfrak{m}_A$;

Thm.[P. Marques, C. McDaniel, I.] Let A be (non-standard) graded, and denote by $\kappa(A) = \mathcal{A}$ the "isomorphic" local algebra, regrade generators to degree one. Then for $\ell \in \mathfrak{m}_A$ we have $P_\ell \leq H(\mathcal{A})^\vee$.

5. Families $\mathrm{Gor}(H)$ with several irreducible components (with P. Marques)

Let $H(A) = (1,3,3,2,2,1)$, which has two possible symmetric decompositions. The first is $\mathcal{D}$:

$$\big(H(0) = (1,2,2,2,2,1),\ H(1) = 0,\ H(2) = (0,1,1,0)\big).$$

Ex. $F_A = X^5 + Y^5 + Z^3$. $P_{\ell,A} = (6,4,2)$.

The second is $\mathcal{S}$: $\big((H(0) = (1,2,2,2,2,1)$
$H(1) = (0,0,1,0,0),\ H(2) = 0, H(3) = (0,1,0)\big)$.
Ex. $G_B = X^5 + Y^5 + (X+Y)^4 + Z^2$.
$P_{\ell,B} = (6,4,1,1)$. Compare the algebras A, B.

$H(A) = (1, 3, 3, 2, 2, 1)$: non-graded AG.
$\mathcal{D}$: $\big(H(0) = (1, 2, 2, 2, 2, 1),$
$H(1) = 0,\ H(2) = (0, 1, 1, 0)\big)$. $F_A = X^5 + Y^5 + Z^3$.
$P_{\ell,A} = (6, 4, 2)$.
$\mathcal{S}$: $\big((H(0) = (1, 2, 2, 2, 2, 1),$
$H(1) = (0, 0, 1, 0, 0),\ H(2) = 0, H(3) =$
$(0, 1, 0)\big)$.ot $G_B = X^5 + Y^5 + (X + Y)^4 + Z^2$.
$P_{\ell,B} = (6, 4, 1, 1)$. Compare the algebras A, B.
$P_{\ell,A} = (6, 4, 2) > P_{\ell,B} = (6, 4, 1, 1)$:
$\overline{\mathrm{Gor}(\mathcal{S})} \cap \mathrm{Gor}(\mathcal{D}) = \emptyset$.
$H_{\mathcal{S}}(1)_2 = 1 > H_{\mathcal{D}}(1)_2 = 0$: $\overline{\mathrm{Gor}(\mathcal{D})} \not\supset \mathrm{Gor}(\mathcal{S})$. So
$\overline{\mathrm{Gor}(\mathcal{D})}, \overline{\mathrm{Gor}(\mathcal{S})}$ are two irred. comps. of $\mathrm{Gor}(H)$.
$\dim \mathrm{Gor}(\mathcal{D}) = 20, \quad \dim \mathrm{Gor}(\mathcal{S}) = 21.$

6. What sets of partitions $\{P_a, a \in \mathcal{A}\}$ are possible?

Question 1 (Compatibility) Given partitions P, P' of n, can we find $\mathcal{A}; a, a' \in \mathcal{A}$ with $P_a = P, P_{a'} = P'$?

Answer 1. Not all pairs can occur: for example $P_a = 4, P_{a'} = (3, 1)$ is impossible (see Maximum commuting orbit, below). See P. Oblak [Ob]).

Question 2 (Intermediate P) For which $\mathcal{A}$ does a generic $\ell \in \mathfrak{m}_{\mathcal{A}}$ give P_ℓ that are WL but not SL?

Answer 2. Examples have been provided by A.I., R. Gondim, and by E. Mezzetti, R.M Miro, J. Vallés. R. Gondim's "On higher Hessians and the Lefschetz properties" gives many examples and results.

Moral: we all see Artinian algebras (quotient out your favorite algebra C by a general enough s.o.p. to make $A = C/(f_1, \ldots, f_s)$): then the Jordan type of multiplication maps on A is an invariant of C.

Thank you!

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