Symbolic powers 22/01/2022

Combinatorial Algebra meets algebraic Combinatorics

throughout $R = k[x_1, \dots, x_d]$, k field

I (homogeneous) radical ideal

P prime nth symbolic power of P:

$$P^{(n)} := \{ f \in R \mid sf \in P^n \text{ for some } s \notin P \}$$

$$I = P_1 \cap \cdots \cap P_t \qquad (P_i \text{ primes})$$

$$I^{(n)} := P_1^{(n)} \cap \cdots \cap P_t^{(n)}$$

$$= \{ f \in R \mid sf \in I^n \text{ for some } s \notin \bigcup_{i=1}^{t} P_i \}$$

theorem (Zariski-Nagata) $k = \mathbb{C}$

$$I^{(n)} = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n = \left\{ f \in R \;\middle|\; \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}(f) \in I \text{ for all } a_1 + \cdots + a_d < n \right\}$$

(De Stefani - G - Jeffries, 2020) a version of this for $k = \mathbb{Z}$

Properties for all $a, b, n \geq 1$

1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$

theorem I ci $\Longrightarrow$ $I^{(n)} = I^n$ for all $n \geq 1$

(codim $I = \mu(I)$)

<u>Example</u> $I = (xy, xz, yz) \subseteq k[x,y,z]$
$= (x,y) \cap (x,z) \cap (y,z)$

$$I^2 \subsetneq I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2$$

$$\underset{xyz \,\in\, I^{(2)}}{\cup \! \! \! \! \! \! \quad}$$

$xyz \in I^{(2)}$
but $\notin I^2$

<u>Example</u> X 3×3 generic matrix $R = k[X]$

$I = I_2(X)$ 2×2 minors of X

$\det X \in I^{(2)}$

$= x_{11} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix}$ + element in $(x_{12}, \dots, x_{33})$ other vars

$\partial/\partial x_{11} (\det X) = \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \in I$

But $\deg(\det X) = 3 < 4 \Rightarrow \det X \notin I^2$

So $I_2 \subsetneq I^{(2)}$

Some difficult/open questions

- <u>Equality</u> When is $I^{(n)} = I^n$? (for all/some n)

$\to$ Characterize I with $I^{(n)} = I^n$ for all $n \geq 1$

$\to$ Is there d such that $I^{(n)} = I^n$ for all $n \leq d$
$\Rightarrow I^{(n)} = I^n$ for all n?

<u>theorem</u> (Montaño — Núñez Betancourt, 2021)
I squarefree monomial ideal
$\mu = \mu(I)$ = minimal number of generators of I

$I^{(n)} = I^n$ for all $n \leq \left\lceil \frac{\mu(I)}{2} \right\rceil \Rightarrow I^{(n)} = I^n$ for all $n \geq 1$

(<u>Packing Problem</u>) I squarefree monomial ideal

Regular sequence of monomials inside I
= monomials in I with no common variables
length of such a sequence $\leq$ codim I
<u>Ex</u> $I = (xy, xz, yz)$ length $1 <$ codim $I = 2$

I is <u>packed</u> if whenever we set any number of variables equal to 0 or 1 (or do nothing), the resulting ideal J has c = codim J = length of a regular sequence of monomials

<u>Conjecture</u> $I^{(n)} = I^n$ for all $n \geq 1$ $\Leftrightarrow$ I is packed

($\Rightarrow$ is easy)

<u>theorem</u> (Gitler–Valencia – Villarreal, 2005)
true for I = edge ideal of a finite graph G
$\Leftrightarrow$ G bipartite

- Finite generation of symbolic Rees algebras

$\bigoplus I^{(n)} t^n \subseteq R[t]$ is not always a fg R-algebra (but it is for monomial ideals)

- <u>Degrees</u> I homogeneous ideal

$$\alpha(I) := \min \{ \deg f \mid 0 \neq f \in I \text{ homogeneous} \}$$

<u>Question</u> What is $\alpha(I^{(n)})$?

<u>Note</u> $I^n \subseteq I^{(n)} \Rightarrow \alpha(I^{(n)}) \leq \alpha(I^n) = n\alpha(I)$

<u>Special case</u> $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^N$

$$I = I(X) = \bigcap_{i=1}^{s} I(P_i)$$

$$I^{(n)} = \bigcap_{i=1}^{s} I(P_i)^n$$

<u>Conjecture</u> (Chudnovsky, 1981)

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I) + N - 1}{N}$$

<u>theorem</u> (Bisui – G – Hà – Nguyễn, 2022) $k = \bar{k}$ any char

Chudnovsky's Conjecture holds for $s \geq 4^N$ general points (it holds in an open dense set of the Hilbert scheme of s points)

• Containment Problem: When is $I^{(a)} \subseteq I^b$?

theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2018)

$h := \max\{\operatorname{codim} P_i\}$ $\qquad I = \underbrace{P_1 \cap \cdots \cap P_t}_{\text{primes}}$

$I^{(hn)} \subseteq I^n$ for all $n \geq 1$

Ex: $I = (xy, xz, yz)$ $\qquad h = \operatorname{codim} I = 2$

$I^{(2n)} \subseteq I^n \Rightarrow I^{(4)} \subseteq I^2$ but $I^{(3)} \subseteq I^2$

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$

Fact: In char p, $q = p^e \Rightarrow I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$

(Dumnicki–Szemberg–Tutaj-Gasińska, 2013) counterexample

theorem (G-Huneke, 2019)

If R/I is F-pure, then Harbourne's Conjecture holds.

eg: • I squarefree monomial

• $I = I_t(X)$, X generic matrix

• $R/I = k[\text{all monomials of degree } d \text{ in } v \text{ vars}]$ Veronese

(Also: the local rings of a Schubert variety are F-pure)

Conjecture (Stable Harbourne) $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.