Some Matrix Factorizations of Discriminants

$\rightsquigarrow$ NC resolutions

Discriminant

$$\underset{\substack{\| \\ 1}}{a}t^2+\underset{\substack{\| \\ 0}}{b}t+c \longrightarrow \underset{\substack{\| \\ \Delta}}{b^2-4ac}$$

$\{$ when poly has multiple roots

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roots of polys $\xrightarrow{\ p\ }$ poly

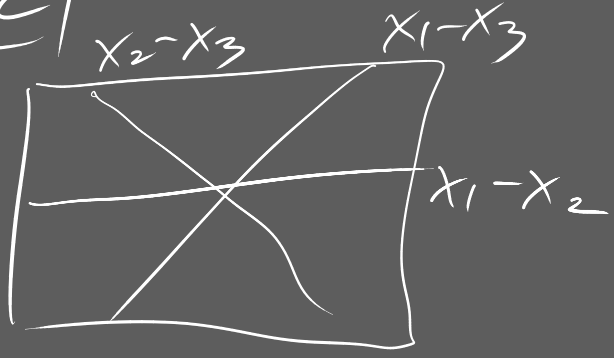
$(x_1,\dots,x_n) \longmapsto (t+x_1)\cdots(t+x_n)$

$= t^n+\sigma_1 t^{n-1}+\cdots+\sigma_n$

$\sum x_i = 0$ $\qquad \sigma_1 = 0$

Cubic

$S_3 \circlearrowleft$

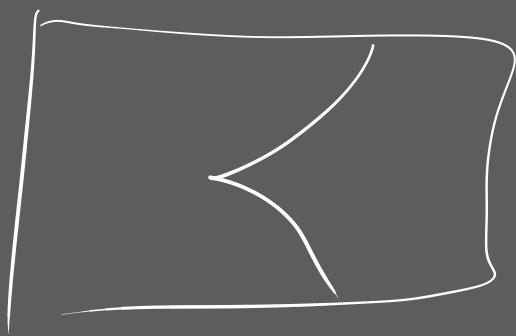


lines
reflection
Hyperplanes

$p \downarrow$

$S = k[x_1,x_2,x_3]/\sigma_1$

$\uparrow$



$$R = S^{S_3} = k[\sigma_1, \sigma_2, \sigma_3]$$

$$\Delta = -4\sigma_2^3 - 27\sigma_3^2$$ discrimint

$$= (x_1-x_2)^2(x_2-x_3)^2(x_1-x_3)^2$$

$$p_i = \tfrac{1}{i}(x_1^i + x_2^i + x_3^i)$$ power sums

(Bring back $\sigma_1 = p_1$)

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}}_{Jac^T} \underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}}_{\left(\frac{\partial p_i}{\partial x_j}\right) \; Jac} = \underbrace{\begin{pmatrix} 3 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{pmatrix}}_{= A} \Big\downarrow \det$$

$$|Jac^T|\,|Jac| = (x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$$
$$= \Delta$$

$$B = \det A\,(A^{-1}) \qquad (B, A)$$

(B, A) form a matrix factorization of Δ

Defn (B, A) is a matrix fact. of Δ if $R^n \xrightarrow{B} R^n \xrightarrow{A} R^n$

$$\underline{BA} = \Delta\, id$$
$$\underline{AB} = \underline{\underline{\Delta\, id}}$$

$MF(\Delta)$ is a category with $\oplus$

Theorem (Eisenbud) Max, Cohen Macaulay

$$\frac{MF(\Delta)}{\{(\underline{\underline{1}}, \Delta)\}} \simeq \underline{MCM\left(R/(\Delta)\right)}$$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} \Delta\, id & 0 \\ 0 & \Delta\, id \end{pmatrix} = \Delta\, id$$

$$(B, A) \longmapsto \operatorname{cok} B$$

$$(B, \det B\, B^{-1}) \longleftarrow\!\shortmid M$$

$$0 \to R^{n\beta} \to R^{n} \to M \to 0$$

Important in Landau Ginzburg Models

String Theory

Problem Given $\Delta \in R$

find $\{M_1, \dots, M_k\} \subseteq MCM(R/\Delta)$

$\cong$

$MF(\Delta)/(1,\Delta)$

So $\mathrm{End}_{R/\Delta}\left(\bigoplus_{i=1}^{k} M_i\right)$ has finite global dim

NC resolution of $R/(\Delta)$

Known $\exists$ NCR

Finite MCM-type - Auslander

$\Leftrightarrow$ simple singularity ADE

McKay correspondence dim 2

Quotient Singularities

Toric — Faber, Smith
Spenko V den Bergh

Generic det. Varieties — Buchweitz
Leuschke VdB

Discriminants of Reflection Groups
— Buchweitz, Faber, I

$S_n \circlearrowright \left(k[x_1 \cdots x_n] \big/ \sigma_1 \right) = S$

$\Delta = \prod_{i<j} (x_i - x_j)^2 = z^2$

$z = \prod_{i<j} (x_i - x_j)$

$R = S^{S_n}$

$$\begin{tikzcd} S \arrow[r, "z"] \arrow[d, "\wr"] & S \arrow[r, "z"] \arrow[d] & S \arrow[d] \\ R^{n!} \arrow[r, "p_* z"] & R^{n!} \arrow[r, "p_* z"] & R^{n!} \end{tikzcd}$$

$z^2 = \Delta \in R$

$\underline{(p_* z, p_* z)}$ is $MF(\Delta)$

$\mathrm{End}_{R/\Delta}\left(\boxed{S/z}\right)$ fin. global dim
NCR of Δ — BFI

? $p_* z$? — need basis of S/R to write a matrix

S_3 $\begin{pmatrix} 0 & & \boxed{1} \\ & \boxed{M} \; \boxed{M} & \\ \boxed{\Delta} & & 0 \end{pmatrix}$ $M = \begin{pmatrix} p_2 & p_3 \\ p_3 & p_4 \end{pmatrix}$

$(p_* z, p_* z))$ <u>splits</u>

S_4 $\begin{pmatrix} & & & & & & & & & 1 \\ & & & & & & & & A & \\ & & & & & & & A & & \\ & & & & & & A & & & \\ & & & & & C & & & & \\ & & & & C & & & & & \\ & & & B & & & & & & \\ & & B & & & & & & & \\ & B & & & & & & & & \\ \Delta & & & & & & & & & \end{pmatrix}$

$(C, C)^{\oplus 2}$
$\oplus (A, B)^{\oplus 3}$
$\oplus (B, A)^{\oplus 3}$
$\oplus (1, \Delta)$
$\oplus (\Delta, 1)$
of $(p_* z, p_* z)$

A, B, C
explicit

<u>Joint work with Faber, May, Talarico</u>

<u>Higher Specht polynomials</u>

$S \cong R \otimes \frac{\Sigma}{(\sigma_2, \dots, \sigma_n)} \cong R \otimes k S_n$ as S_n-reps

$\cong R \otimes \left(\bigoplus_{\lambda \vdash n} \underline{V_\lambda^{\dim V_\lambda}} \right)$

$\cong R \otimes \left(\bigoplus_{\lambda \vdash n} \bigoplus_{V, T \in ST(\lambda)} k F_T^V \right)$

$ST(\lambda) = \{$ stand tableau shape $\lambda\}$

$F^\vee_T \in S$ Higher Specht polys

Ariki, Terasoma, Yamada.

$F^\vee_T$ + variant

$T \in ST(\lambda)$ $\lambda \vdash n$

$C(T) =$ Col. stab. $\leq S_n$

$R(T) =$ Row. stab $\leq S_n$

$$c_T = \sum_{c \in C(T)} \operatorname{sgn}(c)\, c \qquad r_T = \sum_{r \in R(T)} r$$

$$\varepsilon_T = c_T r_T \qquad \sigma_T = r_T c_T$$

Youngs symmetrizer

Charge $i(T)$ vector length n.

$$x^\vee_T = x_{i(T)}^{w(T)} := \prod_j x_{i(T)_j}^{w(T)_j}$$

$\boxed{\text{High. Specht Polys}}$ $F^\vee_T = \varepsilon_T x^\vee_T$

$H^\vee_T = r_T x^\vee_T$

Ordered Least letter ordering.

$\boxed{\text{Thm}}$ $$(p_* \mathbb{Z}, p_* \mathbb{Z}) \simeq \bigoplus_{\lambda \vdash n} \bigoplus_{T \in ST(\lambda)} \left(p_* \mathbb{Z}|_{H_T^{\circ}}, p_* \mathbb{Z}|_{F_T^{\circ}} \right)$$

$T_1, T_2 \in ST(\lambda)$

& $$\left(p_* \mathbb{Z}|_{H_{T_1}}, p_* \mathbb{Z}|_{F_{T_1}} \right) \simeq \left(p_* \mathbb{Z}|_{H_{T_2}}, p_* \mathbb{Z}_{F_{T_2}} \right)$$

$\boxed{\text{Using pairing}}$

$$\langle \, , \, \rangle : S \otimes_R S \to R$$

$$\langle f, g \rangle = \frac{1}{z} \sum_{\pi \in S_n} \text{sgn}(\pi) \pi(fg)$$

$$\langle f_i, f_j \rangle = \begin{pmatrix} 0 & \\ & * \end{pmatrix}$$

$\longrightarrow$ Formula

In progress $\rightarrow$ $G(n, 1, n) = S_n \wr \mu_n$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline 6 & & \\ \hline \end{array}$$

$$w(T) = (6\ 3\ 1\ 5\ 2\ 4)$$

$$i(T)\ (3\ 1\ 0\ 2\ 0\ 1)$$

$$F^{V}_{T} = \varepsilon_T\, x^{i(T)}_{w(T)}$$

$$V, T \in ST(\lambda)$$