Multi-Rees Algebras of Strongly Stable Ideals

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~ Combinatorial Algebra meets Algebraic Combinatorics ~
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Joint work with Kuei-Nuan Lin and Gabriel Sosa

Blow-up/Rees Algebras

- $W \subseteq V$ varieties
- S: coordinate ring of V
- $S/_I$: coordinate ring of W

$$\boxed{\text{Blow-up of } V \text{ along } W = \mathrm{Proj}(R(I))}$$

where

$$\boxed{R(I) = \bigoplus_{n=0}^{\infty} I^n t^n \subseteq S[t]}$$ (graded ring)

Rees Algebra of I

Blow-up/Rees Algebras

example!

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- S: coordinate ring of V
- S/I: coordinate ring of W

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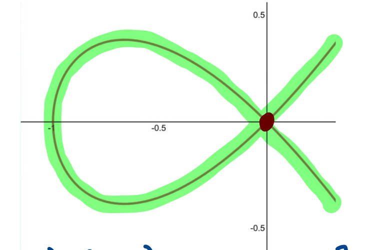
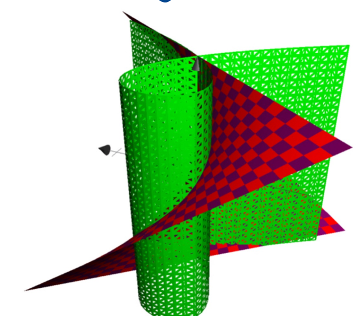
Rees Algebra of I



$V(f) = \{(x,y) : x^3 + x^2 - y^2 = 0\}$

Blow-up $\mathbb{R}^2$ at $(0,0)$



$\mathrm{Proj}(R(I))$

- $I = (x, y)$
- $S = \Bbbk[x,y]/x^3+x^2-y^2$
- $R(I) = S[u,v]/(xv - yu)$

Multi-Rees Algebras

- S: commutative ring
- $I_1, \dots, I_r$ collection of ideals
- The multi-Rees algebra of $I_1, \dots, I_r$ is defined as

$$\boxed{R(I_1 \oplus \cdots \oplus I_r) = \bigoplus_{a_1, \dots, a_r \geq 0} I_1^{a_1} \cdots I_r^{a_r} t_1^{a_1} \cdots t_r^{a_r} \subseteq S[t_1, \dots, t_r]}$$

* it is also the Rees algebra of the module $I_1 \oplus \cdots \oplus I_r$

Defining Ideal of the Multi-Rees Algebra

- $I_1,\dots,I_r$ collection of monomial ideals in $S=\Bbbk[x_1,\dots,x_n]$
- $I_i = \langle \underbrace{u_{i,1},\dots,u_{i,s_i}}_{\text{same degree}} \rangle$ and $\mathcal{G} = \bigcup_{i=1}^{r} \{u_{i,j} : 1\le j\le s_i\}$
- Consider the S-algebra homomorphism

$$\begin{aligned}\varphi : S[T_{i,j} : u_{i,j}\in \mathcal{G}] &\longrightarrow S[t_1,\dots,t_r]\\ T_{i,j} &\longmapsto u_{i,j}t_i\end{aligned}$$

- $R(I_1\oplus\cdots\oplus I_r) \simeq S[T_{i,j}]/\ker\varphi$ ⇝ defining ideal of the Rees algebra

- Special (multi)-fiber ring: $F(I_1\oplus\cdots\oplus I_r) = R(I_1\oplus\cdots\oplus I_r)\otimes_S \Bbbk$

⇝ defining ideal $J = \ker\varphi'$ where $\varphi' = \varphi|_{\Bbbk}$.

Main Question

- Find implicit equations for the defining ideals of the multi-Rees algebra and its special fiber ring

$$(\mathcal{I} = \ker \varphi \text{ and } \mathcal{J} = \ker \varphi')$$

- Side question: investigate Koszulness

$\rightsquigarrow$ Need to focus on special classes of ideals

Strongly Stable Ideals

- A monomial ideal $I \subseteq S = \Bbbk[x_1, \dots, x_n]$ is called **strongly stable** if for each monomial $m \in I$, we have

$$x_j \frac{m}{x_i} \in I \quad \text{whenever } x_i \text{ divides } m \text{ and } j < i.$$

Borel move

$$m \longrightarrow x_j \frac{m}{x_i}$$

for $x_i \mid m$ and $j < i$

- *example*: $S = \Bbbk[x_1, x_2]$

$I_1 = \langle x_1^2, x_1x_2, x_2^2 \rangle$ strongly stable

$I_2 = \langle x_1^2, x_2^2 \rangle$ not strongly stable

$$\begin{array}{c} x_2^2 \\ | \\ x_1x_2 \\ | \\ x_1^2 \end{array}$$

Hasse diagram

Borel generators

monomials on top of the Hasse diagram

$$I_1 = \underset{\text{Borel generated}}{\underset{\downarrow}{B}}(x_2^2)$$

Our Strategy

- Given a collection of strongly stable ideals $I_1, \dots, I_r$ focus on three parameters:

  - r : number of ideals
  - g_i : number of Borel generators
  - d_i : degrees of Borel generators of I_i where $d_i = \deg(u_{i,j})$ for $I_i = \langle u_{i,1}, \dots, u_{i,s_i} \rangle$

What is known? (Rees world $\rightsquigarrow r=1$)

- [DeNegri, 99] $F(I)$ is Koszul when $g=1$.

- [Herzog-Hibi-Vladoiu, 2005] $R(I)$ is of fiber type for $r=1$.

  $\left(\mathcal{I} = \mathcal{J} + (\text{relations of the symmetric algebra})\right)$

- [Conca-DeNegri, early 90's] Consider $I = B(x_1^3x_3^3, x_2^6, x_1^2x_2^2x_3^2)$.

  $$\boxed{T^2_{x_1^3x_3^3}T_{x_2^6} - T^3_{x_1^2x_2^2x_3^2}} \text{ is a minimal generator of } \mathcal{J}$$

  $\rightsquigarrow$ $F(I)$ is not always Koszul when $g \geq 3$.

- [DiPasquale-Francisco-Mermin-Schweig-Sosa, 2018] $R(I)$ is Koszul for $g=2$.

(Multi-Rees World $\rightsquigarrow$ $r > 1$)

Multi-Rees algebra and its special fiber ring are Koszul

- [Lin-Pollini, 2013] powers of the maximal ideal
- [Sosa, 2014] principal strongly stable ideals satisfying an "ordering condition"
- [DiPasquale-Jabbar Nezhad, 2020] principal strongly stable

Our Results [K-Lin-Sosa, 2021]

(1) The multi-Rees algebra of strongly stable ideals is of fiber type.

If G is a Gröbner basis of J, then

$$G \cup \{ x_i T_u - x_j T_v : i < j,\ u, v \in I_k \text{ and } x_i u = x_j v \}$$

is a Gröbner basis of $\mathcal{I}$.

(2) We found examples in the spirit of Conca-DeNegri to eliminate classes of strongly stable ideals.

$r \geq 3,\ g_i \geq 2,\ d_i \geq 2$

$r = 2,\ g_1 = g_2 = 2,\ d_1, d_2 \geq 4$

$r = 2,\ g_1 = g_2 = 2,\ d_1 = 2,\ d_2 \geq 4$

Potential Collection of Strongly Stable Ideals whose multi-Rees algebras are always Koszul

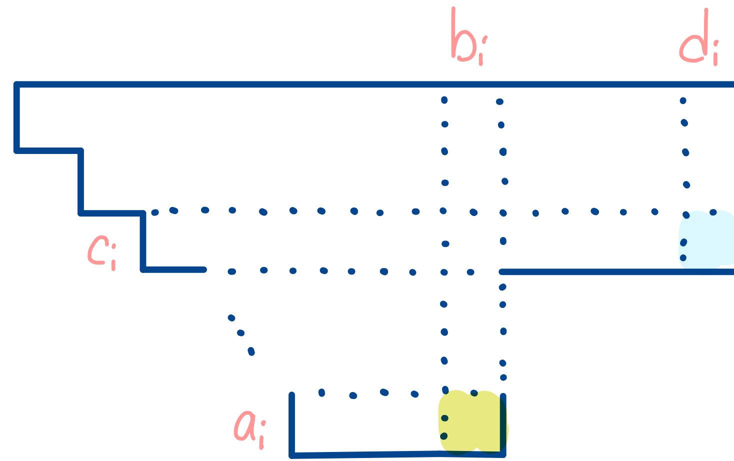
r : # of ideals	g_i : # of Borel generators	d_i : degrees of Borel gens.
$r=2$	$g_1=g_2=2$	$2\leq d_1\leq d_2\leq 3$
$r>2$	$g_1=\cdots=g_{r-2}=1,\ g_{r-1}=g_r=2$	$2\leq d_{r-1}\leq d_r\leq 3$
$r\geq 2$	$g_1=\cdots=g_{r-1}=1,\ g_r\leq 2$	anything

(3) $R(I_1 \oplus I_2)$ is Koszul when $\boxed{g=d=2}$

[I_1 and I_2 are strongly stable ideals with two quadratic Borel generators.]

$$I_1 = B(x_{a_1}x_{b_1},\ x_{c_1}x_{d_1})$$

$$I_2 = B(x_{a_2}x_{b_2},\ x_{c_2}x_{d_2})$$



$c_i < a_i \leq b_i < d_i$ for $i = 1, 2$

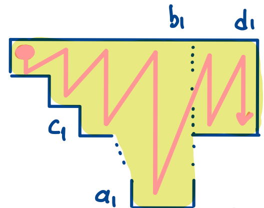
Key Ideas and Tools

- Recall the toric map $\varphi' : \overbrace{\Bbbk[T_u, z_v : u \in G(I_1), v \in G(I_2)]}^{R} \longrightarrow S[t,z]$

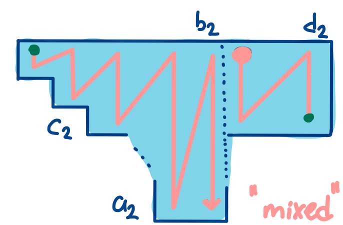
$$T_u \longmapsto ut$$
$$z_v \longmapsto vz$$

- We find a quadratic Gröbner basis of $J = \ker \varphi'$ wrt to "$>_{ht}$" (head and tail order)

  - variable order on R:



rev-lex (rlex)



"mixed" rev-lex (mrlex)

$$T_{11} > T_{12} > T_{22} > \dots > T_{c_1 d_1} > z_{1 b_2+1} > z_{2 b_2+1} > \dots > z_{c_2 d_2} > z_{11} > \dots > z_{a_2 b_2}$$

  - term order: rev-lex on R induced by this variable order

Gröbner basis: $G = G_1 \cup G_2 \cup G_3$ is a Gröbner basis of $\tilde{J}$ wrt "$>_{nt}$" order where

- $G_1 = \{ T_u T_v - T_{u'} T_{v'} \;:\; uv = u'v' \text{ and } u,v >_{rlex} v' \}$ GB of $J(I_1)$ wrt $>_{rlex}$
- $G_2 = \{ z_u z_v - z_{u'} z_{v'} : uv = u'v' \text{ and } u,v >_{mrlex} v' \}$ GB of $J(I_2)$ wrt $>_{mrlex}$
- $G_3 = \{ T_u z_v - T_{u'} z_{v'} \;:\; uv = u'v' \text{ and } v >_{mrlex} v' \}$

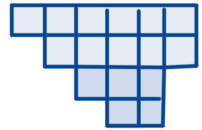
Fiber graphs: $\Gamma_\mu(I_1, I_2)$: fiber graph of I_1 and I_2 at μ wrt G. $(\mu \in S[t,z])$

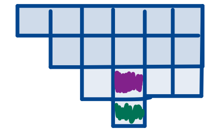
$\varphi' : R \longrightarrow S[t,z]$

- $\Gamma_\mu(I_1, I_2)$ is a directed graph with
  - vertices: monomials $V \in R$ s.t. $\varphi'(V) = \mu$
  - edges: $V \longrightarrow V'$ if V' is a one step reduction of V wrt G.

example: $I_1 = B(x_4x_5, x_2x_6)$ and $I_2 = B(x_4^2, x_3x_6)$

$$\boxed{\mathcal{M} = x_2x_3x_4^2x_5x_6t^2z}$$





$\Gamma_{\mathcal{M}}(I_1, I_2)$:

$$\begin{tikzcd}[column sep=huge, row sep=large]
T_{26}T_{44}Z_{35} \arrow[r, "T_{26}Z_{35} \to_{\mathcal{G}} T_{25}Z_{36}"] \arrow[d, "T_{44}Z_{35} \to_{\mathcal{G}} T_{35}Z_{44}"] & T_{25}T_{44}Z_{36} \arrow[d, "T_{44}T_{25} \to_{\mathcal{G}} T_{24}T_{45}"] & T_{35}T_{44}Z_{26} \arrow[l, "T_{35}Z_{26} \to_{\mathcal{G}} T_{25}Z_{36}"'] \arrow[d, "T_{44}T_{35} \to_{\mathcal{G}} T_{34}T_{45}"] \\
V' = T_{26}\underline{T_{35}Z_{44}}\ \text{sink} & T_{24}T_{45}Z_{36} \arrow[d, "T_{24}Z_{36} \to_{\mathcal{G}} T_{26}Z_{34}"] & T_{34}T_{45}Z_{26} \arrow[dl, "T_{34}Z_{26} \to_{\mathcal{G}} T_{26}Z_{34}"] \\
& V = T_{26}\underline{T_{45}Z_{34}} \arrow[ul, "T_{45}Z_{34} \to_{\mathcal{G}} T_{35}Z_{44}"] &
\end{tikzcd}$$

- $V \to V'$ b/c $T_{45}Z_{34} - T_{35}Z_{44} \in G_3$ b/c $\boxed{x_3x_4} >_{mrlex} \boxed{x_4^2}$

Why fiber graphs?

The collection of (marked) binomials $\mathcal{G}$ is a Gröbner basis of $\tilde{J}$

if and only if

$\Gamma_\mu(I_1, I_2)$ is either empty or has a unique sink for every multidegree μ.

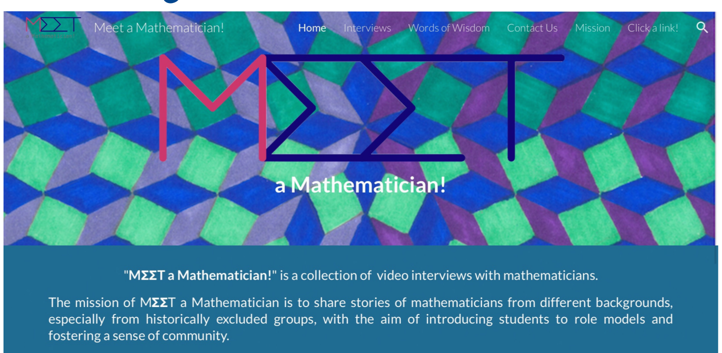
(Proved in different contexts by different groups)

Concluding Remarks

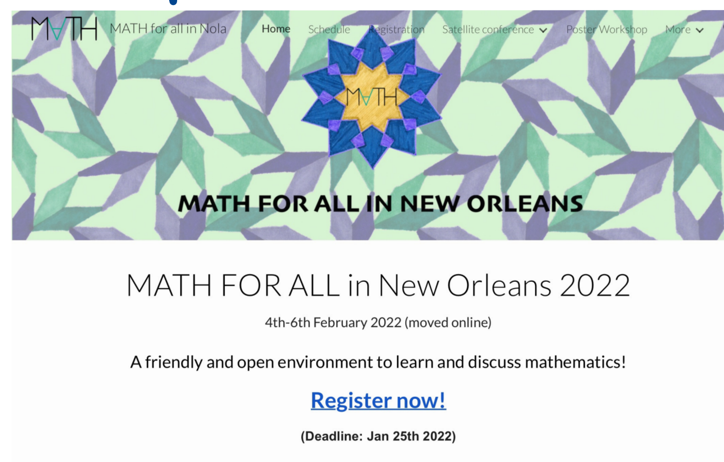
- What about remaining classes of SS ideals from our list?
- Fiber graphs efficient/useful in higher degrees?
- What about normalness, CMness?

Thanks!

Project Advertisement



Conference Announcement



Examples

(1)

$$I_1 = \mathcal{B}(x_3^2 x_6^a, x_1 x_5 x_6^a),\ I_2 = \mathcal{B}(x_3^2 x_6^b, x_2 x_4 x_6^b),\ I_3 = \mathcal{B}(x_2 x_4 x_6^c, x_1 x_5 x_6^c),$$

$r \geq 3$

$g_i \geq 2$

$d_i \geq 2$

gives a cubic syzygy

$$(x_1 x_5 x_6^a)(x_3^2 x_6^b)(x_2 x_4 x_6^c) = (x_3^2 x_6^a)(x_2 x_4 x_6^b)(x_1 x_5 x_6^c)$$

(2)

$$I_1 = \mathcal{B}(x_1^2 x_3^2 x_4^a, x_1 x_2^2 x_3 x_4^a),\ I_2 = \mathcal{B}(x_1^2 x_3^2 x_4^b, x_2^4 x_4^b)$$

$r = 2$

$g_1 = g_2 = 2$

$d_1, d_2 \geq 4$

gives a cubic syzygy

$$(x_1^2 x_3^2 x_4^a)^2 (x_2^4 x_4^b) = (x_1 x_2^2 x_3 x_4^a)^2 (x_1^2 x_3^2 x_4^b)$$

(3)

$$I_1 = \mathcal{B}(x_1 x_3, x_2^2),\ I_2 = \mathcal{B}(x_1^2 x_3^2 x_4^a, x_2^4 x_4^a)$$

$r = 2$

$g_1 = g_2 = 2$

$d_1 = 2,\ d_2 \geq 4$

gives a cubic syzygy

$$(x_1 x_3)^2 (x_2^4 x_4^a) = (x_2^2)^2 (x_1^2 x_3^2 x_4^a)$$