SOME APPLICATIONS OF AND OPEN QUESTIONS IN CATEGORICAL MODEL THEORY

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CATEGORICAL MODEL THEORY

"IS"

THE STUDY OF THE <u>2-CATEGORY</u>

<u>ACC</u> OF ACCESSIBLE CATEGORIES

AND ACCESSIBLE FUNCTORS

<u>RECALL</u>

ACCESSIBLE CATEGORIES CAN BE VIEWED IN 3 WAYS:

1. AS CATEGORIES OF MODELS OF THEORIES IN $L_{\infty\infty}$
2. AS CATEGORIES OF MODELS OF (MIXED) SKETCHES
3. AS CATEGORIES FOR WHICH THERE IS A REGULAR CARDINAL κ

SUCH THAT

(a) $\exists$ κ-FILT COLIMITS

(b) THE FULL SUBCATEGORY OF κ-PRESENTABLES IS SMALL AND κ-DENSE.

ACCESSIBILITY IS A STRONG SMALLNESS CONDITION WHICH IS REMARKABLY STABLE UNDER CATEGORICAL CONSTRUCTIONS

<u>LIMIT THEOREM</u>:

IF $\Gamma : \underline{\underline{I}} \to \underline{\underline{Acc}}$ IS A SMALL WEIGHTED (= INDEXED) 2-DIAGRAM, THEN $\varprojlim \Gamma$ (TAKEN IN $\underline{\underline{CAT}}$) IS ACCESSIBLE.

<u>E.G.</u>

$$
\begin{tikzcd}
\underline{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & \underline{A} \arrow[d, "F"] \\
\underline{B} \arrow[r, "G"'] & \underline{C}
\end{tikzcd}
,\qquad
\begin{tikzcd}
\underline{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \underline{A} \arrow[d] \\
\underline{B} \arrow[r] & \underline{C}
\end{tikzcd}
,\qquad
\underline{A}^{\mathbb{C}}
$$

"COMMA"

SMALL

APPLICATIONS

① IF $\underline{A}$ ACC. & HAS COKER PAIRS THEN $\underline{A}$ IS COWELL-POWERED (see FREYD – AB. CATS)

PROOF: FOR $A \in \underline{A}$, EPI(A) IS EQUIVALENT TO THE $\varprojlim$ OF

$$\begin{tikzcd}
 & & \underline{A}^{(\rightrightarrows)} \\
\underline{A}^{(\to)} \arrow[urr, shift left, "\text{COKER}"] \arrow[urr, shift right, "\Delta\circ\text{COD}"'] \arrow[drr, "\text{DOM}"] & & \\
 & & \underline{A} \\
\mathbb{1} \arrow[urr, "\ulcorner A \urcorner"'] & &
\end{tikzcd}$$

$\Rightarrow$ ACC POSET $\Rightarrow$ SMALL. ∎

② DEF: (GABRIEL-ULMER)

$\underline{A}$ LOCALLY PRESENTABLE $\Leftrightarrow$ $\underline{A}$ ACC. + ALL $\varinjlim$

PROP: $\Leftrightarrow$ $\underline{A}$ ACC + ALL $\varprojlim$

LOC. PRES MEANS ALGEBRAIC

<u>PROP</u>: $\underline{A}$, $\underline{B}$ LOC. PRES. $F : \underline{A} \to \underline{B}$ ACC.

(i) $\exists$ LADJ $\Leftrightarrow$ PRES $\varprojlim$

(ii) $\exists$ RADJ $\Leftrightarrow$ PRES $\varinjlim$.

R = COMM RING

R-<u>MOD</u> = CAT OF R-MODULES

$\otimes_R$ = TENSOR PROD

<u>COALG</u> = CAT OF COMM COALGEBRAS

(C, ε, δ) $\quad \varepsilon : C \to R \quad \delta : C \to C \otimes_R C$

$\ni$ COASSOC, COUNIT, COCOMM.

(BARR ≈ 1970) (i) COALG HAS $\varprojlim$

(ii) COALG CART. CL.

(iii) $\exists$ COFREE !

<u>NOW</u>:

R-<u>MOD</u> LOC. PRES., $\otimes_R$ ACC

<u>COALG</u> CAN BE CONSTRUCTED WITH $\varprojlim$

$\Rightarrow$ ACC $\Rightarrow$ LOC. PRES. $\Rightarrow$ (i) – (iii). $\square$

LAX COLIMIT THEOREM

$\Phi : \underline{B}^{op} \longrightarrow \underline{\underline{ACC}}$ PSEUDO-FUNCTOR $\ni$.

$\underline{B}$ κ-ACC & $\forall$ κ-FILT $\Gamma : \underline{I} \longrightarrow \underline{B}$

$$\Phi(\varinjlim \Gamma) \xrightarrow{\simeq} \underleftarrow{\mathrm{LIM}}\, \Phi\Gamma$$

THEN THE FIBRATION ASSOC TO Φ BY GROTH. CONSTR. IS ACCESSIBLE.

<u>EX</u>: $\underline{A}$ ACC. $\underline{CAT}^{op} \xrightarrow{\underline{A}^{(\ )}} \underline{\underline{ACC}}$

$$\mathbb{C} \longmapsto \underline{A}^{\mathbb{C}}$$

SATISFIES ABOVE HYPOTHESES.

GROTH. CONSTR. GIVES DIAG($\underline{A}$)

<u>MORPH</u> $\rightsquigarrow$
$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "\Phi"] \arrow[dr, "\Gamma"'] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha}{\Rightarrow}"] & & \mathbb{D} \arrow[dl, "\Phi"] \\
& \underline{A} &
\end{tikzcd}$$

DIAG($\underline{A}$) IS ACC.

THERE IS A FUNCTOR $C : \underline{A} \longrightarrow$ DIAG($\underline{A}$)

($C(A) = \ulcorner A \urcorner : \mathbb{1} \longrightarrow \underline{A}$) WHOSE LADJ IS $\underrightarrow{\mathrm{COLIM}}$

DEF: $\underline{A}$ HAS DETECTABLE COLIMITS IF THE FULL SUBCAT OF DIAG($\underline{A}$) DETERMINED BY THE DIAGRAMS THAT HAVE COLIMS IS ACCESSIBLE.

NOTE: LOC. PRES. $\Rightarrow$ DET$^{\text{ABLE}}$ COLIMS.

THM: $\underline{A}$ HAS DETECTABLE COLIMS IFF IT HAS A COMPLETION (cf. LAMBEK) TO A LOC PRES CAT.

DEF: $\underline{A}$ IS AN ELEMENTARY CAT IF IT IS MODELS OF A THEORY IN $L_{\omega\omega}$ (ORDINARY 1st ORD. LOGIC).

THM: $\underline{A}$ ELEM $\Rightarrow$ $\underline{A}$ HAS DET$^{\text{ABLE}}$ COLIM.

THERE ARE SMALL CATS WHICH DON'T HAVE DET$^{\text{ABLE}}$ COLIM (MAYBE EVEN (COUNTABLE SET)^{OP}).* [IF $\nexists$ MEAS CARDS.]

<u>COR</u>: $\underline{A}$ ELEM $\Rightarrow \exists$ LOC PRES COMPLETION

THE COMPLETION IS ALSO COCOMP.

<u>QUESTIONS</u> :

(1) IS THERE A <u>CATEGORICAL</u> CHAR. OF ELEM. CAT.?

- ELEM $\not\Rightarrow$ $\aleph_0$-ACC
- $\not\Leftarrow$
- ELEM $\Leftarrow$ $\aleph_0$-LOC PRES
- <u>EX</u>: CONN. GRAPHS: $\mathrm{MOD}\boxed{\bullet \rightrightarrows \bullet \xrightarrow[\text{COEQ}]{} 1}$?

(2) IS THERE A <u>CATEGORICAL</u> CHAR. OF CATEGORIES OF POINTS OF A GROTH. TOPOS ?

- $\exists$ filt COLIMS (NOT SUFF.)
- $\underline{A}$ ACC + FILT COLIM
  FILT ($\underline{A}$, <u>SET</u>) IS GROTH TOPOS.
- $\underline{A} \longrightarrow$ PT (FILT ($\underline{A}$, <u>SET</u>))