

SOME APPLICATIONS OF
AND OPEN QUESTIONS IN
CATEGORICAL MODEL THEORY

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CATEGORICAL MODEL THEORY

"IS"

THE STUDY OF THE 2-CATEGORY

ACC OF ACCESSIBLE CATEGORIES

AND ACCESSIBLE FUNCTORS

RECALL

ACCESSIBLE CATEGORIES CAN BE VIEWED IN 3 WAYS:

- ① AS CATEGORIES OF MODELS OF THEORIES IN $L_{\infty\infty}$
- ② AS CATEGORIES OF MODELS OF (MIXED) SKETCHES
- ③ AS CATEGORIES FOR WHICH THERE IS A REGULAR CARDINAL κ

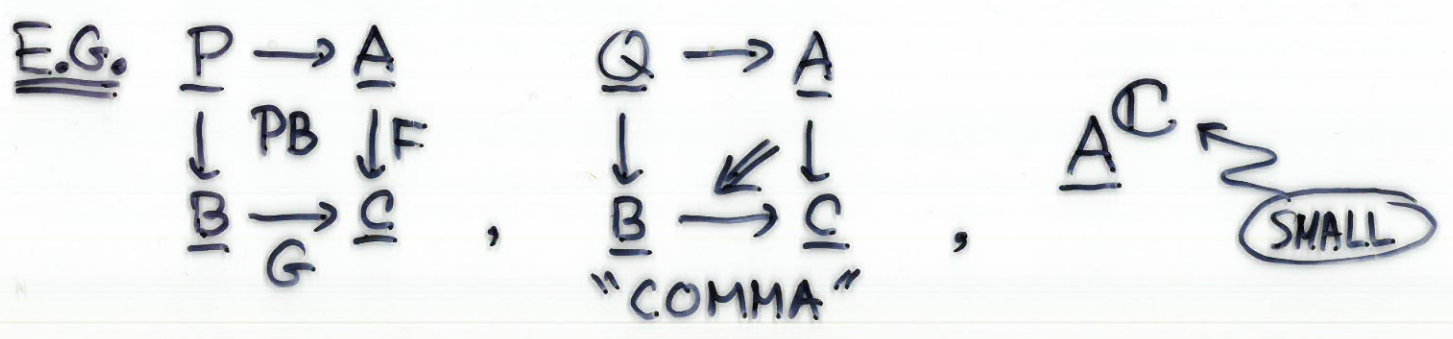
SUCH THAT

- (a) \exists k -FILT COLIMITS
- (b) THE FULL SUBCATEGORY OF k -PRESENTABLES IS SMALL AND k -DENSE.

ACCESSIBILITY IS A STRONG SMALLNESS CONDITION WHICH IS REMARKABLY STABLE UNDER CATEGORICAL CONSTRUCTIONS

LIMIT THEOREM:

IF $\Gamma: \underline{I} \rightarrow \underline{Acc}$ IS A SMALL WEIGHTED (= INDEXED) 2-DIAGRAM, THEN $\varprojlim \Gamma$ (TAKEN IN CAT) IS ACCESSIBLE.



APPLICATIONS

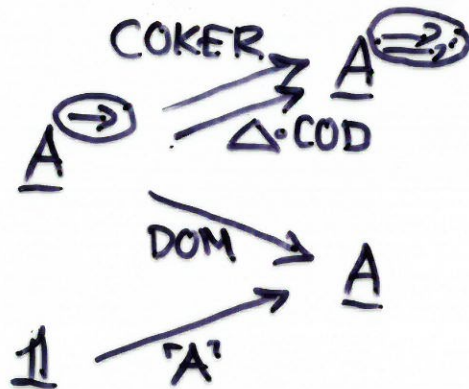
① IF A ACC. & HAS COCKER PAIRS

THEN A IS COWELL-POWERED

(SEE FREYD - AB. CATS)

PROOF: FOR $A \in \underline{A}$, EPI(A) IS

EQUIVALENT TO THE \varprojlim OF



\Rightarrow ACC POSET \Rightarrow SMALL. ■

② DEF: (GABRIEL-ULMER)

A LOCALLY PRESENTABLE

\Leftrightarrow A ACC. + ALL \varinjlim

PROP: \Leftrightarrow A ACC + ALL \varprojlim

LOC. PRES MEANS ALGEBRAIC

PROP: $\underline{A}, \underline{B}$ LOC. PRES. $F: \underline{A} \rightarrow \underline{B}$ ACC.

(i) \exists LADJ \Leftrightarrow PRES \varprojlim

(ii) \exists RADJ \Leftrightarrow PRES \varinjlim .

$R =$ COMM RING

R -MOD = CAT OF R -MODULES

$\otimes_R =$ TENSOR PROD

COALG = CAT OF COMM COALGEBRAS

(C, ε, δ) $\varepsilon: C \rightarrow R$ $\delta: C \rightarrow C \otimes_R C$

\Rightarrow COASSOC, COUNIT, COCOMM.

(BARR \approx 1970) (i) COALG HAS \varprojlim

(ii) COALG CART. CL.

(iii) \exists COFREE !

Now:

R -MOD LOC. PRES., \otimes_R ACC

COALG CAN BE CONSTRUCTED WITH \varprojlim

\Rightarrow ACC \Rightarrow LOC. PRES. \Rightarrow (i) - (iii). \square

LAX COLIMIT THEOREM

$\Phi: \underline{B}^{\text{OP}} \rightarrow \underline{\text{ACC}}$ PSEUDO-FUNCTOR \Rightarrow .

\underline{B} k -ACC & \forall k -FILT $\Gamma: \underline{I} \rightarrow \underline{B}$

$$\Phi(\varinjlim \Gamma) \xrightarrow{\cong} \varprojlim \Phi \Gamma$$

THEN THE FIBRATION ASSOC TO Φ BY GROTH. CONSTR. IS ACCESSIBLE.

EX: \underline{A} ACC. $\underline{\text{CAT}}^{\text{OP}} \xrightarrow{A(\cdot)} \underline{\text{ACC}}$
 $\underline{C} \longmapsto \underline{A^C}$

SATISFIES ABOVE HYPOTHESES.

GROTH. CONSTR. GIVES $\text{DIAG}(\underline{A})$

MORPH \rightsquigarrow

$$\begin{array}{ccc} \underline{C} & \xrightarrow{\Phi} & \underline{D} \\ & \searrow \gamma & \swarrow \Phi \\ & \underline{A} & \end{array}$$

$\text{DIAG}(\underline{A})$ IS ACC.

THERE IS A FUNCTOR $\underline{C}: \underline{A} \rightarrow \text{DIAG}(\underline{A})$

$(\underline{C}(\underline{A}) = \ulcorner \underline{A} \urcorner: \underline{1} \rightarrow \underline{A})$ WHOSE LADJ IS COLIM

DEF: \underline{A} HAS DETECTABLE COLIMITS IF THE FULL SUBCAT OF $\text{DIAG}(\underline{A})$ DETERMINED BY THE DIAGRAMS THAT HAVE COLIMS IS ACCESSIBLE.

NOTE: LOC. PRES. \Rightarrow DET^{ABLE} COLIMS.

THM: \underline{A} HAS DETECTABLE COLIMS IFF IT HAS A COMPLETION (cf. LAMBEK) TO A LOC PRES CAT.

DEF: \underline{A} IS AN ELEMENTARY CAT IF IT IS MODELS OF A THEORY IN $L_{\omega\omega}$ (ORDINARY 1ST ORD. LOGIC).

THM: \underline{A} ELEM \Rightarrow \underline{A} HAS DET^{ABLE} COLIM.

THERE ARE SMALL CATS WHICH DON'T HAVE DET^{ABLE} COLIM (MAYBE EVEN (COUNTABLE SET)^{OP}). * [IF \nexists MEAS CARDS.]

COR: $\underline{A} \text{ ELEM} \Rightarrow \exists \text{ LOC PRES COMPLETION}$
THE COMPLETION IS ALSO COCOMP.

QUESTIONS :

① IS THERE A CATEGORICAL CHAR.
OF ELEM. CAT.?

• $\text{ELEM} \not\Rightarrow \mathcal{H}_0\text{-ACC}$

• $\not\Leftarrow$

• $\text{ELEM} \Leftarrow \mathcal{H}_0\text{-LOC PRES}$

• EX: CONN. GRAPHS: MOD $\boxed{\begin{matrix} \cdot \rightrightarrows \cdot \rightarrow 1 \\ \text{COEQ} \end{matrix}}$?

② IS THERE A CATEGORICAL CHAR.
OF CATEGORIES OF POINTS
OF A GROTH. TOPOS ?

• $\exists \text{ filt COLIMS (NOT SUFF.)}$

• A ACC + FILT COLIM

FILT(A, SET) IS GROTH TOPOS.

• A \longrightarrow PT (FILT(A, SET))