Skolem Relations and Profunctors

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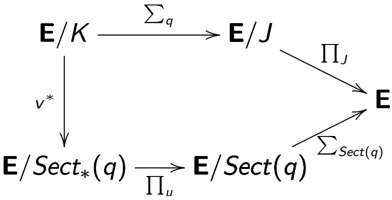
Aveiro, June 2015

Distributivity

Let $\langle K_j \rangle$ be a J-family of sets and $\langle \langle A_{j,k} \rangle_{k \in K_j} \rangle_j$ a family of families of sets

$$\prod_{j \in J} \sum_{k \in K_j} A_{j,k} \cong \sum_{s \in \prod K_j} \prod_{j \in J} A_{j,s(j)}$$

Also holds in a topos



- $q : K \longrightarrow J$ is the canonical $\sum_{j \in J} K_j \longrightarrow J$
- $Sect(q)$ is the object of sections of q, i.e. $\prod_{j \in J} K_j$
- $Sect_*(q)$ is the object of pointed sections of q, i.e. $J \times \prod_{j \in J} K_j$
- $Sect_*(q) \longrightarrow K$ is evaluation, $u : Sect_*(q) \longrightarrow Sect(q)$ forgetful

Intersection/Union

• Let $q : K \longrightarrow J$ be a morphism in a topos $\mathbf{E}$ and $\langle A_k \rangle$ a family of subobjects of A, $\bullet \rightarrowtail K \times A$

Consider

$$\bigcap_{j} \bigcup_{q(k)=j} A_k = \bigcup_{s \in Sect(q)} \bigcap_{j} A_{s(j)}$$

Holds in **Set** (and more generally in any topos satisfying IAC)

• Internalizes as

$$\begin{array}{ccccc}
\Omega^K & \xrightarrow{\bigcup_q} & \Omega^J & \xrightarrow{\bigcap_J} & \Omega \\
\downarrow{\scriptstyle v^*} & & & & \\
\Omega^{Sect_*(q)} & \xrightarrow[\bigcap_u]{} & \Omega^{Sect(q)} & \xrightarrow[\bigcup_{Sect(q)}]{} & \Omega
\end{array}$$

$$K \xleftarrow{v} Sect_*(q) \xrightarrow{u} Sect(q)$$

$$s(j) \longleftarrow\!\!\mid \quad \begin{array}{ccc} j \in J & \xrightarrow{s} & K \\ & \searrow\!\!= \quad & \swarrow{\scriptstyle q} \\ & J & \end{array} \quad \mapsto \quad \begin{array}{ccc} J & \xrightarrow{s} & K \\ & \searrow\!\!= \quad & \swarrow{\scriptstyle q} \\ & J & \end{array}$$

Skolem Relations

To prove

$$\bigcap_j \bigcup_{q(k)=j} A_k = \bigcup_{s \in Sect(q)} \bigcap_j A_{s(j)}$$

• "$\supseteq$" easy

• "$\subseteq$" $a \in \bigcap_j \bigcup_{q(k)=j} A_k$ iff for every j there is a k such that $q(k) = j$ and $a \in A_k$

• The k is not unique so you choose one (if you can) which gives a section $s : J \longrightarrow K$

• If you can't choose, take some or all of them. You get an entire relation $S : J \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow K$

Definition

A *Skolem relation* for q is a relation S such that $q_* \circ S = \mathrm{Id}_J$

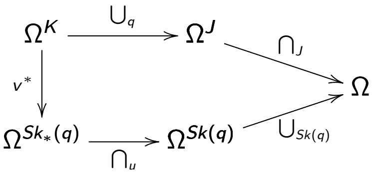
Distributivity I

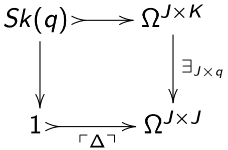
Theorem

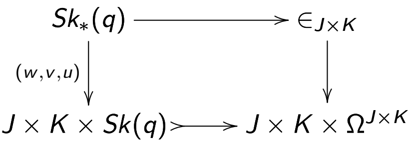
In any topos we have

$$\bigcap_{j} \bigcup_{q(k)=j} A_k = \bigcup_{S \in Sk(q)} \bigcap_{j \sim_S k} A_k$$

or







Properties of Skolem Relations

Proposition

(i) If q has a Skolem relation, then q is epi
(ii) If q is epi, then q^* is a Skolem relation
(iii) Any Skolem relation S is an entire relation
(iv) For any Skolem relation S we have $S \subseteq q^*$
(v) $Sk(q)$ is an upclosed subset of q^*
(vi) Sections of q are minimal elements of $Sk(q)$

Proposition

$$\begin{array}{ccc} & S & \\ {}^{s_1}\swarrow & & \searrow^{s_2} \\ J & & K \end{array}$$

is a Skolem relation if and only if

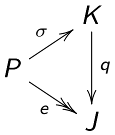
(1) s_1 is epi
(2) s_2 is mono
(3)

$$\begin{array}{ccc} & S & \\ {}^{s_1}\swarrow & & \searrow^{s_2} \\ J & \xleftarrow[q]{} & K \end{array}$$

commutes

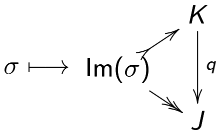
Cutting Down the Size

P is *internally projective* if $(\ )^P : \mathbf{E} \longrightarrow \mathbf{E}$ preserves epimorphisms
Let $e : P \twoheadrightarrow J$ be an internally projective cover of J. A *P-section* of q is $\sigma : P \longrightarrow K$ such that



We have a morphism

$$\phi : P\text{-}Sect(q) \longrightarrow Sk(q)$$

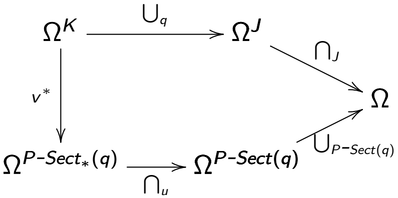


ϕ is internally initial

Cutting Down the Size

Theorem

$$\bigcap_{j} \bigcup_{q(k)=j} A_k = \bigcup_{\sigma \in P\text{-}Sect(q)} \bigcap_{p \in P} A_{\sigma(p)}$$



Corollary

If J is internally projective we have

$$\bigcap_{j} \bigcup_{q(h)=j} A_K = \bigcup_{\sigma \in \prod_q K} \bigcap_{j} A_{\sigma(j)}$$

Limit/Colimit

We would like a similar formula expressing

$$\lim_{J \in \mathbf{J}} \operatorname{colim}_{K \in \mathbf{K}_J} \Gamma_J K$$

as a colimit of limits, for diagrams

$$\Gamma_J : \mathbf{K}_J \longrightarrow \mathbf{Set}$$

Families of Diagrams

Γ_J should be functorial in J, so a functor

$$\mathbf{J} \longrightarrow \mathbf{Diag}$$

$\mathrm{colim}_{K \in \mathbf{K}_J} \Gamma_J K$ should also be functorial in J, so a functor

$$\mathbf{Diag} \xrightarrow{\mathrm{colim}} \mathbf{Set}$$

A good notion of morphism of diagram, which works well for colim is

$$\begin{array}{ccc} \mathbf{K}_1 & \xrightarrow{\Phi} & \mathbf{K}_2 \\ & \overset{\phi}{\Rightarrow} & \\ {\scriptstyle \Gamma_1} \searrow & & \swarrow {\scriptstyle \Gamma_2} \\ & \mathbf{Set} & \end{array}$$

Families of Diagrams

We can put all the $\mathbf{K}$'s together in an *opfibration*

$$Q : \mathbf{K} \longrightarrow \mathbf{J}$$

and then the Γ's and ϕ fit together to give a single diagram

$$\Gamma : \mathbf{K} \longrightarrow \mathbf{Set}$$

Notes:
(1) Our discussion leads to split opfibrations, but general opfibrations are better

(2) We could go further and take homotopy opfibrations, which are exactly the notion which makes colim functorial

(3) We could in fact take Q to be an arbitrary functor, and take Kan extension instead of colim, but we lose the "family of diagrams" intuition

Limits of Colimits

Let $Q : \mathbf{K} \longrightarrow \mathbf{J}$ be an opfibration and $\Gamma : \mathbf{K} \longrightarrow \mathbf{Set}$ a $\mathbf{J}$-family of diagrams, $\Gamma_J : \mathbf{K}_J \longrightarrow \mathbf{Set}$
An element of

$$\lim_{J} \operatorname*{colim}_{QK=J} \Gamma K$$

is a compatible family of equivalence classes

$$\langle [x_J \in \Gamma K_J]_{\mathbf{K}_J} \rangle_{\mathbf{J}}$$

• For every J we have a K_J such that $QK_J = J$
• Not unique but there is a path of K's in $\mathbf{K}_J$ connecting any two choices
• For any $j : J \longrightarrow J'$ there is a $k_j : K_J \longrightarrow j_* K_J$ and a path in $\mathbf{K}_{J'}$ connecting $\Gamma(k_j)(x_J)$ with $x_{J'}$

Profunctors

A *profunctor* $P : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{K}$ is a functor $P : \mathbf{J}^{op} \times \mathbf{K} \longrightarrow \mathbf{Set}$

An element of $P(J, K)$ is denoted $J \overset{p}{\longrightarrow\!\!\!\!\!\!\bullet}\;\; K$

Composition:

$$(R \otimes P)(J, L) = \int^{K} R(K, L) \times P(J, K)$$

An element is an equivalence class

$$[J \overset{p}{\longrightarrow\!\!\!\!\!\!\bullet}\;\; K \overset{r}{\longrightarrow\!\!\!\!\!\!\bullet}\;\; L]_K$$

Examples:

- $Q^* : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{K}$ is $Q^*(J, K) = \mathbf{J}(J, QK)$
- $Q_* : \mathbf{K} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{J}$ is $Q_*(K, J) = \mathbf{J}(QK, J)$
- $\mathrm{Id}_{\mathbf{J}} : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{J}$ is $\mathrm{Id}_{\mathbf{J}}(J, J') = \mathbf{J}(J, J')$
- $Q_* \dashv Q^*$

Prosections

A prosection for Q is a profunctor $S : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{K}$ such that $Q_* \otimes S \cong \mathsf{Id}_{\mathbf{J}}$
The isomorphism corresponds to a morphism $\sigma : S \longrightarrow Q^*$

Definition

A *prosection* for Q is a profunctor $S : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{K}$ and a morphism $\sigma : S \longrightarrow Q^*$ such that

$$Q_* \otimes S \xrightarrow{Q_* \otimes \sigma} Q_* \otimes Q^* \xrightarrow{\epsilon} \mathsf{Id}_{\mathbf{J}}$$

is an isomorphism

A *morphism of prosections* $(S, \sigma) \longrightarrow (S', \sigma')$ is $t : S \longrightarrow S'$ such that $\sigma' t = \sigma$

The category of prosections is denoted $\mathbf{Ps}(Q)$

Analysis of Prosections

In general an element of $(Q_* \otimes S)(J, J')$ is an equivalence class of pairs

$$[J \overset{s}{-\!\!\bullet\!\!\!\to} K, QK \xrightarrow{j} J']_K$$

If Q is an opfibration, $QK \xrightarrow{j} J'$ lifts to $K \xrightarrow{k_j} j_*K$ and

$[J \overset{s}{-\!\!\bullet\!\!\!\to} K, QK \xrightarrow{j} J'] = [J \overset{s}{-\!\!\bullet\!\!\!\to} K \xrightarrow{k_j} j_*K, Q(j_*K) = J']$

Proposition

For Q an opfibration and $S : \mathbf{J} -\!\!\bullet\!\!\!\to \mathbf{K}$ a profunctor, $(Q_* \otimes S)(J, J')$ consists of equivalence classes $[J \overset{s}{-\!\!\bullet\!\!\!\to} K']_{QK'=J'}$ where the equivalence relation is generated by $s \sim \bar{s}$ if there exists k such that

$$\begin{array}{ccc} J & \overset{s}{-\!\!\bullet\!\!\!\to} & K' \\ \| & & \downarrow k \\ J & \underset{\bar{s}}{-\!\!\bullet\!\!\!\to} & \bar{K}' \end{array}$$

with $Qk = 1_{J'}$

Analysis of Prosections

For a prosection (S, σ), $\sigma : S \longrightarrow Q^*$

$$(J \xrightarrow{s} K) \overset{\sigma}{\longmapsto} (J \xrightarrow{\sigma(s)} QK)$$

Induces $Q_* \otimes S \longrightarrow \mathrm{Id}_J$

$$[J \xrightarrow{s} K']_{QK'=J'} \longmapsto (J \xrightarrow{\sigma(s)} J')$$

Proposition

(S, σ) is a prosection if and only if for every J there exists $s_J : J \longrightarrow K_J$ such that
(1) $\sigma(s_J) = 1_J$ (so in particular $QK_J = J$),
(2) for every $s : J \longrightarrow K$ we have

$$[J \xrightarrow{\sigma s} QK \xrightarrow{s_{QK}} K_{QK}]_{\mathbf{K}_{QK}} = [J \xrightarrow{s} K]_{\mathbf{K}_{QK}}$$

Distributivity II

Theorem

For $Q : \mathbf{K} \longrightarrow \mathbf{J}$ an opfibration and $\Gamma : \mathbf{K} \longrightarrow \mathbf{Set}$ we have

$$\lim_{J \in \mathbf{J}} \operatorname{colim}_{K \in \mathbf{K}_J} \Gamma K \cong \operatorname{colim}_{(S,\sigma) \in \mathbf{Ps}(Q)} \lim_{s \in S(J,K)} \Gamma K$$

• We can write $\lim_{s \in S(J,K)} \Gamma K$ as an iterated limit to get an equivalent form of the isomorphism

$$\lim_{J} \operatorname{colim}_{QK = J} \Gamma K \cong \operatorname{colim}_{(S,\sigma) \in \mathbf{Ps}(Q)} \lim_{J} \lim_{s \in S(J,K)} \Gamma K$$

• If (S, σ) is representable $S = \Phi_*$, for Φ an actual section, then $\lim_{s \in S(J,K)} \cong \Gamma \Phi J$

Distributivity II (continued)

Theorem

$$\begin{array}{ccccc} \mathbf{Set}^{\mathbf{K}} & \xrightarrow{\mathrm{colim}_Q} & \mathbf{Set}^{\mathbf{J}} & \xrightarrow{\lim_J} & \mathbf{Set} \\ \downarrow{\scriptstyle V^*} & & & & \\ \mathbf{Set}^{\mathbf{Ps}_*(Q)^{op}} & \xrightarrow[\lim_U]{} & \mathbf{Set}^{\mathbf{Ps}(Q)^{op}} & \xrightarrow{\mathrm{colim}_{\mathbf{Ps}(Q)}} & \mathbf{Set} \end{array}$$

$\mathbf{Ps}_*(Q)$ the category of pointed prosections
– Objects $(S, \sigma, s), S : \mathbf{J} \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow \mathbf{K}, \sigma : S \longrightarrow Q^*, s : J \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow K, (S, \sigma)$ a prosection
– Morphisms $(t, j, k) : (S, \sigma, s) \longrightarrow (S', \sigma', s')$ $t : S \longrightarrow S'$ such that $\sigma' t = \sigma$ and

$$\begin{array}{ccc} J' & \xrightarrow{j} & J \\ {\scriptstyle s'}\downarrow\bullet & & \bullet\downarrow{\scriptstyle ts} \\ K' & \xrightarrow[k]{} & K \end{array}$$

$U : \mathbf{Ps}_*(Q) \longrightarrow \mathbf{Ps}(Q)$ and $V : \mathbf{Ps}_*(Q)^{op} \longrightarrow \mathbf{K}$ forgetful functors

Properties of Prosections

Proposition

(1) If Q has a prosection, then Q is pseudo epi ($FQ \cong GQ \Rightarrow F \cong G$)
(2) If (S, σ) is a prosection, then S is total ($\mathrm{colim}_K\, S(J, K) = 1$ for every J)
(3) $\mathbf{Ps}(Q)$ is closed under connected colimits in $\mathbf{Set}^{\mathbf{J}^{op} \times \mathbf{K}}$

Proof.

(1) $FQ \cong GQ \Rightarrow F_* \otimes Q_* \cong G_* \otimes Q_*$
$\Rightarrow F_* \otimes Q_* \otimes S \cong G_* \otimes Q_* \otimes S \Rightarrow F_* \cong G_* \Rightarrow F \cong G$

(2) S is total $\Leftrightarrow T_* \otimes S \cong T_*$ $(T : ? \longrightarrow \mathbb{1})$
$Q_* \otimes S \cong \mathrm{Id}_{\mathbf{J}} \Rightarrow T_* \otimes Q_* \otimes S \cong T_* \otimes \mathrm{Id}_{\mathbf{J}} \Rightarrow T_* \otimes S \cong T_*$

(3) $Q_* \otimes (\mathrm{colim}_\alpha S_\alpha) \cong \mathrm{colim}_\alpha (Q_* \otimes S_\alpha) \cong \mathrm{colim}_\alpha \mathrm{Id}_{\mathbf{J}} \cong \mathrm{Id}_{\mathbf{J}}$

□

Properties of Prosections (continued)

Proposition

$\mathbf{Ps}(Q)$ *is accessible*

Proof.

$$\begin{array}{ccc} \mathbf{Ps}(Q) & \longrightarrow & \mathbf{Set}^{\mathbf{J}^{op}\times\mathbf{K}} \\ \downarrow & \cong & \downarrow{\scriptstyle Lan_{(\mathbf{J}^{op}\times Q)}} \\ \mathbb{1} & \xrightarrow[\ulcorner \mathbf{J}(-,-) \urcorner]{} & \mathbf{Set}^{\mathbf{J}^{op}\times\mathbf{J}} \end{array}$$

is a pseudo pullback □

Remark

$\mathbf{Ps}(Q)$ *is models of a colimit-terminal object sketch. It is κ-accessible for any infinite $\kappa > \#\mathbf{K}_J$, all J*

Cutting Down the Size

Corollary

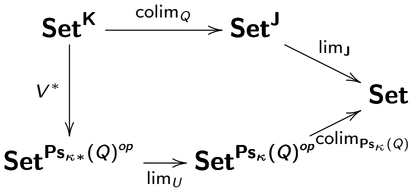
$\mathbf{Ps}(Q)$ *has a small initial subcategory*

Proof.

If $\mathbf{Ps}(Q)$ is κ-accessible, then the full subcategory of the κ-presentable objects $\mathbf{Ps}_\kappa(Q)$ is initial

□

So we have



Example (Q Discrete Opfibration)

Proposition

If Q is a discrete opfibration then any prosection is represented by an actual section

This gives the distributive law

$$\lim_{J} \sum_{QK=J} \Gamma_J K \cong \sum_{S \in Sect(Q)} \lim_{J} \Gamma_J S(J)$$

If $\mathbf{J}$ is discrete we recover distributivity of $\prod$ over $\sum$

Example ($\mathbf{J} = \mathbb{1}$)

A prosection $S : \mathbb{1} \longrightarrow\!\!\!\!\!\bullet\,\, \mathbf{K}$ of $Q : \mathbf{K} \longrightarrow \mathbb{1}$ is a functor $S : \mathbf{K} \longrightarrow \mathbf{Set}$ such that $\operatorname{colim} S = 1$
So $\mathbf{Ps}(Q) \simeq \mathit{Conn}^*(\mathbf{Set}^{\mathbf{K}})$

The representables $\mathbf{K}^{op} \longrightarrow \mathit{Conn}^*(\mathbf{Set}^{\mathbf{K}})$ form an initial subcategory, so our distributive law

$$\lim_{J} \operatorname*{colim}_{QK=J} \Gamma K \cong \operatorname*{colim}_{S \in \mathbf{Ps}(Q)} \lim_{J} \lim_{s \in S(J,K)} \Gamma K$$

reduces to

$$\operatorname*{colim}_{K \in \mathbf{K}} \Gamma K \cong \operatorname*{colim}_{K \in \mathbf{K}} \lim_{k \in K/\mathbf{K}} \Gamma(\operatorname{cod} k)$$

$$\cong \operatorname*{colim}_{K \in \mathbf{K}} \Gamma K$$

Example ($\mathbf{J}$ Discrete)

$Q : \mathbf{K} \longrightarrow \mathbf{J}$ is just a $\mathbf{J}$-family of categories $\mathbf{K}_J$
A prosection $S : \mathbf{J} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{K}$ of Q is equivalent to a family of functors $S_J : \mathbf{K}_J \longrightarrow \mathbf{Set}$ such that $\operatorname{colim} S_J = 1$
$\mathbf{Ps}(Q) \simeq \prod_J Conn^*(\mathbf{Set}^{\mathbf{K}_J})$

Initial Functors

• $\Phi : \mathbf{X} \longrightarrow \mathbf{Y}$ is initial if
(1) for every $Y \in \mathbf{Y}$ there are $X \in \mathbf{X}$ and $y : \Phi X \longrightarrow Y$
(2) for any other $y' : \Phi X' \longrightarrow Y$ there exists a path

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots \longleftarrow X_n = X'$$

$$\begin{array}{ccccccccccccc}
\Phi X & = & \Phi X_0 & \to & \Phi X_1 & \leftarrow & \Phi X_2 & \to & \cdots & \leftarrow & \Phi X_n & = & \Phi X' \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y_0} & & \downarrow{\scriptstyle y_1} & & \downarrow{\scriptstyle y_2} & & & & \downarrow{\scriptstyle y_n} & & \downarrow{\scriptstyle y'} \\
Y & = & Y & = & Y & = & Y & = & \cdots & = & Y & = & Y
\end{array}$$

• A finite product of initial functors $\prod \Phi_\alpha : \mathbf{X}_\alpha \longrightarrow \mathbf{Y}_\alpha$ is again initial
• Does not hold for infinite products
• Say that $\Phi : \mathbf{X} \longrightarrow \mathbf{Y}$ is *very initial* if for every y and y' there exists a path of length 2 joining them
• An infinite product of very initial functors is very initial

Example (**J** Discrete)

$Q : \mathbf{K} \longrightarrow \mathbf{J}$ is just a **J**-family of categories $\mathbf{K}_J$
A prosection $S : \mathbf{J} \longrightarrow\!\!\!\!\bullet\;\; \mathbf{K}$ of Q is equivalent to a family of functors
$S_J : \mathbf{K}_J \longrightarrow \mathbf{Set}$ such that colim $S_J = 1$
$\mathbf{Ps}(Q) \simeq \prod_J Conn^*(\mathbf{Set}^{\mathbf{K}_J})$

If **J** is finite, then families of representables are initial in $\mathbf{Ps}(Q)$

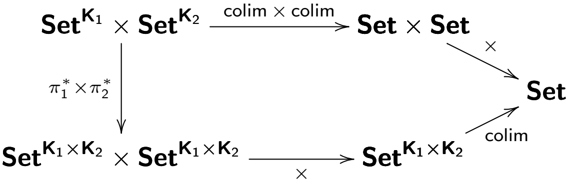
$$\prod_J \mathbf{K}_J^{op} \longrightarrow \prod_J Conn^*(\mathbf{Set}^{\mathbf{K}_J})$$

so our distributive law becomes

$$\prod_J \operatorname*{colim}_{K \in \mathbf{K}_J} \Gamma_J K \cong \operatorname*{colim}_{\langle K_J \rangle \in \prod \mathbf{K}_J} \prod_J \Gamma_J K_J$$

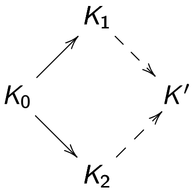
For example, if $\mathbf{J}$ is $\mathbb{1} + \mathbb{1}$

$$\Big(\operatorname*{colim}_{K_1 \in \mathbf{K}_1} \Gamma_1 K_1\Big) \times \Big(\operatorname*{colim}_{K_2 \in \mathbf{K}_2} \Gamma_2 K_2\Big) \cong \operatorname*{colim}_{(K_1, K_2)} \big(\Gamma_1 K_1 \times \Gamma_2 K_2\big)$$



Example

If the categories $\mathbf{K}_J$ have the property that every span can be completed to a commutative square



then the representables $\mathbf{K}_J^{op} \longrightarrow Conn^*(\mathbf{Set}^{\mathbf{K}_J})$ are very initial so we get a distributive law

$$\prod_J \operatorname*{colim}_{K \in \mathbf{K}_J} \Gamma_J K \cong \operatorname*{colim}_{\langle K_J \rangle \in \prod \mathbf{K}_J} \prod_J \Gamma_J K_J$$

If the $\mathbf{K}_J$ are discrete we recover distributivity of $\prod$ over $\sum$ again

Example

For general $\mathbf{K}_J$ and $\mathbf{J}$ infinite, the representables are no longer initial
We can take finite nonempty connected colimits of representables

Let $\mathbb{G}$ be a finite, nonempty, connected graph
For any diagram $D : \mathbb{G} \longrightarrow \mathbf{K}$, let

$$H_D = \operatorname*{colim}_{v \in \mathbb{G}} \mathbf{K}(D(v), -)$$

We have $\operatorname{colim} H_D \cong 1$

Let $Diag_0\mathbf{K}$ be the category of such diagrams $D : \mathbb{G} \longrightarrow \mathbf{K}$

Proposition

$H_{(\ )} : Diag_0\mathbf{K} \longrightarrow Conn^*(\mathbf{Set}^{\mathbf{K}})$ is very initial

This gives the distributive law

$$\prod_J \operatorname*{colim}_{K \in \mathbf{K}_J} \Gamma_J K \cong \operatorname*{colim}_{\langle D_J \rangle} \prod_J \lim_{v \in \mathbb{G}_J} \Gamma_J D_J v$$