The Double Theory of Monads (with apologies to Lack & Street)

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for
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¡Feliz Cumpleaños!

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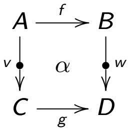
Double categories

We'll be talking about double categories $\mathbb{A}$:



Example 1: $\square\mathbf{A}$ – commutative squares in any category

Example 2: $(\square\mathbf{A})^{co}$ – horizontal part is $\mathbf{A}$, vertical part is $\mathbf{A}^{op}$

Example 3: $\mathbb{R}\mathrm{el}$ – sets, functions, relations, implications

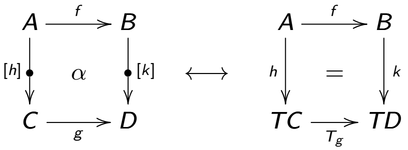
- All thin
- Category object in **Cat**

E.g. $\square\mathbf{A}$: $\mathbf{A}^3 \rightrightarrows\!\!\!\rightarrow \mathbf{A}^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathbf{A}$

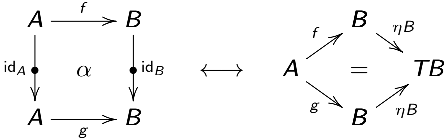
Kleisli

$\mathbb{T} = (T, \eta, \mu)$ monad on **A**

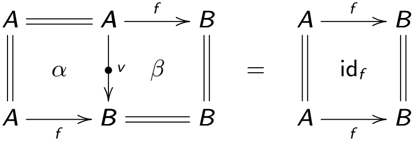
$\mathbb{K}\mathrm{l}(\mathbb{T})$:

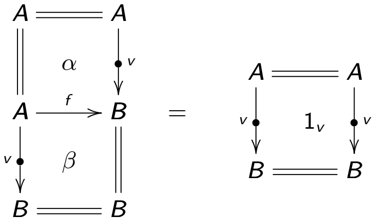


- Thin
- Horizontal 2-category $\mathcal{H}or\mathbb{K}\mathrm{l}(\mathbb{T})$ can have non identity 2-cells



$f : A \to B$ has a *companion* $v : A \to\!\!\bullet\!\!\to B$ if there are cells α and β such that





When they exist, they are unique up to globular iso – make a choice f_*

Companions in $\mathbb{K}\mathrm{l}(T)$

Proposition

Every horizontal arrow f in $\mathbb{K}\mathrm{l}(T)$ has a companion $f_* = [\eta B \cdot f]$

$$(A \xrightarrow{f} B \xrightarrow{\eta B} TB)$$

Conjoints

The (vertical) dual notion is conjoint: w is a conjoint of f if

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & = & B \\ \| & \alpha & \overset{\bullet}{\downarrow} w & \beta & \| \\ A & = & A & \xrightarrow[f]{} & B \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

$$\begin{array}{ccc} B & = & B \\ w \overset{\bullet}{\downarrow} & \beta & \| \\ A & \xrightarrow[f]{} & B \\ \| & \alpha & \overset{\bullet}{\downarrow} w \\ A & = & A \end{array} \quad = \quad \begin{array}{ccc} B & = & B \\ w \overset{\bullet}{\downarrow} & 1_w & \overset{\bullet}{\downarrow} w \\ A & = & A \end{array}$$

Write $w = f^*$ when it exists

Conjoints in $\mathbb{K}\mathrm{l}(T)$

Proposition

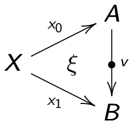
f has a conjoint in $\mathbb{K}\mathrm{l}(T)$ iff Tf is invertible

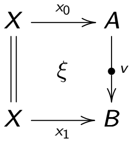
$$f^* = [(Tf)^{-1}\eta B]$$

$$B \xrightarrow{\eta B} TB \xrightarrow{(Tf)^{-1}} TA$$

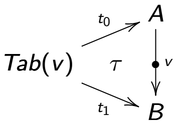
Tabulators

- $v\colon A \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow B$ has a *tabulator* if there is a universal span





- Write



- Globally $\mathrm{id}\colon \mathbf{A}_0 \longrightarrow \mathbf{A}_1$ has a right adjoint

Tabulators in $\mathbb{Kl}(T)$

Proposition

A vertical arrow $[v]\colon A \longrightarrow\!\!\!\!\!\bullet\;\; B$ in $\mathbb{Kl}(T)$ has a tabulator iff the pullback

$$\begin{array}{ccc} P & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow{\scriptstyle v} \\ B & \xrightarrow[\eta B]{} & TB \end{array}$$

exists in **A**

Proof.

$$\begin{array}{ccc} X & \xrightarrow{x_0} & A \\ \Big\| & \xi & \bullet\Big\downarrow{\scriptstyle [v]} \\ X & \xrightarrow[x_1]{} & B \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} X & \xrightarrow{x_0} & A \\ {\scriptstyle \eta X}\Big\downarrow & = & \Big\downarrow{\scriptstyle v} \\ TX & \xrightarrow[Tx_1]{} & TB \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} X & \xrightarrow{x_0} & A \\ {\scriptstyle x_1}\Big\downarrow & = & \Big\downarrow{\scriptstyle v} \\ B & \xrightarrow[\eta B]{} & TB \end{array}$$

□

Representability of vertical arrows

$$\begin{array}{c} X \\ {\scriptstyle v}\, \bullet\!\!\downarrow \\ A \end{array} \qquad \Bigg| \qquad X \xrightarrow{\ \widehat{v}\ } PA$$

Assume that $\mathbb{A}$ has functorial companions

A is *small* if there is $\epsilon A \colon PA \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> A$ such that for every $v \colon X \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> A$ there exists a unique $\widehat{v} \colon X \longrightarrow PA$ such that $\epsilon A \bullet (\widehat{v})_* = v$

$$\begin{array}{ccc} X & = \!= \!= & X \\ & & \downarrow \bullet\, {\scriptstyle (\widehat{v})_*} \\ {\scriptstyle v}\,\bullet\!\!\downarrow & = & PA \\ & & \downarrow \bullet\, {\scriptstyle \epsilon A} \\ A & = \!= \!= & A \end{array}$$

- Equivalently, the functor

$$(\mathbf{Hor}\mathbb{A})^{op} \longrightarrow \mathbf{Set}$$

$$X \longmapsto \{v \colon X \longrightarrow\!\!\bullet\!\!\longrightarrow A\}$$

is representable
- $\mathbb{A}$ has representable vertical arrows if every A is small
- Equivalently

$$(\ )_* \colon \mathbf{Hor}\mathbb{A} \longrightarrow \mathbf{Vert}\mathbb{A}$$

has a right adjoint

Proposition

$\mathbb{K}\mathrm{l}(T)$ *has representable vertical arrows, with* $P = T$

Horizontalizers

- A T-algebra $a\colon TA \longrightarrow A$ gives a uniform way of turning a vertical arrow into a horizontal one

$$\begin{array}{c} X \\ {\scriptstyle [f]}\big\downarrow\bullet \\ A \end{array} \longmapsto (X \xrightarrow{f} TA \xrightarrow{a} A)$$

- A an object of $\mathbb{A}$ (with companions). A *horizontalizer* on A is an assignment

$$\begin{array}{c} X \\ {\scriptstyle v}\big\downarrow\bullet \\ A \end{array} \rightsquigarrow h(v)\colon X \longrightarrow A$$

(1) $h(\mathrm{id}_A) = 1_x$

(2) $h(v \bullet f_*) = h(v)f$

(3) $h(v \bullet w) = h(h(v)_* \bullet w)$

Horizontalizers in $\mathbb{K}\mathrm{l}(T)$

Theorem

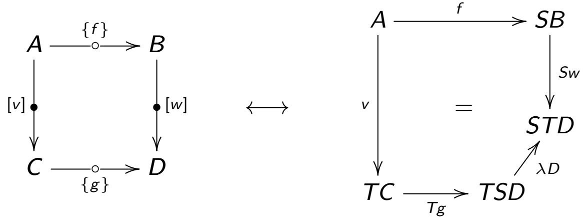
The category of horizontalizers in $\mathbb{K}\mathrm{l}(T)$ is equivalent to the category of Eilenberg-Moore algebras for T*

*A morphism of horizontalizers is $f\colon A \longrightarrow B$ such that $h(f_* \bullet v) = f\, h(v)$

Distributive laws

- $T = (T, \eta, \mu)$, $S = (S, \kappa, \nu)$ monads on **A**

  A distributive law, $\lambda\colon TS \longrightarrow ST$, of T over S gives a double category $\mathbb{K}\mathrm{l}(\lambda)$



Theorem

$\lambda\colon TS \longrightarrow ST$ a natural transformation. $\mathbb{K}\mathrm{l}(\lambda)$ is a double category iff λ is a distributive law

- $f: A \longrightarrow B$ in **A** gives a companion pair

$$f_\circ : A \overset{\circ}{\longrightarrow} B \quad \longleftrightarrow \quad A \xrightarrow{f} B \xrightarrow{\kappa B} SB$$

$$f_* : A \overset{\bullet}{\longrightarrow} B \quad \longleftrightarrow \quad A \xrightarrow{f} B \xrightarrow{\eta B} TB$$

Proposition

$\{h\}: A \overset{\circ}{\longrightarrow} B$ and $[v]: A \overset{\bullet}{\longrightarrow} B$ are companions iff

$$\begin{array}{ccc} A & \xrightarrow{h} & SB \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle S\eta B} \\ TB & \xrightarrow[\kappa TB]{} & STB \end{array}$$

commutes

Zappa-Szép

- (S, λ, T) satisfies the *Zappa-Szép condition* if

$$\begin{array}{ccc} 1_{\mathbf{A}} & \xrightarrow{\kappa} & S \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle S\eta} \\ T & \xrightarrow[\kappa T]{} & ST \end{array}$$

  is a pullback

- $ZS \Rightarrow$ companion pairs are of the form $(f_\circ, f_*)$

Tabulators in $\mathbb{K}\mathrm{l}(\lambda)$

Proposition

If the pullback

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow[\eta B]{} & TB \end{array}$$

exists in $\mathbf{A}$ *and is preserved by* S*, then* $[v]\colon A \longrightarrow\!\!\bullet\!\!\longrightarrow B$ *has a tabulator, namely* P

Horizontalizers in $\mathbb{K}\mathrm{l}(\lambda)$

- A Kleisli algebra for λ is $a\colon TA \longrightarrow SA$ such that

$$\begin{array}{ccc} & A & \\ {}^{\eta A}\swarrow & & \searrow^{\kappa A} \\ TA & \xrightarrow[a]{} & SA \end{array} \qquad \begin{array}{ccccccc} T^2A & \xrightarrow{Ta} & TSA & \xrightarrow{\lambda A} & STA & \xrightarrow{Sa} & S^2A \\ {\scriptstyle \mu A}\downarrow & & & & & & \downarrow{\scriptstyle \nu A} \\ TA & & & \xrightarrow[a]{} & & & SA \end{array}$$

- A Kleisli algebra (A, a) gives a "horizontalizer" on A

$$\begin{array}{ccc} X \xrightarrow{[v]} A & \rightsquigarrow & X \xrightarrow{\{av\}} A \\ X \xrightarrow{v} TA & & X \xrightarrow{v} TA \xrightarrow{a} SA \end{array}$$

- (1) $h(\mathrm{id}_A) = 1_A$

- (2') $h(v \bullet f_*) = h(v) \circ f_\circ$

- (3') ???

Definition: A (horizontal) *doublad* on a double category $\mathbb{A}$ consists of

(1) A horizontal monad $\mathbb{S}$ on $\mathbb{A}$

(2) A vertical monad $\mathbb{T}$ on $\mathbb{A}$

(3) A horizontal transformation $\lambda \colon TS \longrightarrow ST$

(4) A double modification

$$
\begin{array}{ccc}
S & \xrightarrow{1_S} & S \\
{\scriptstyle \eta S}\big\downarrow & \lambda_0 & \big\downarrow{\scriptstyle S\eta} \\
TS & \xrightarrow[\lambda]{} & ST
\end{array}
$$

(5) A double modification

$$
\begin{array}{ccccc}
TTS & \xrightarrow{T\lambda} & TST & \xrightarrow{\lambda T} & STT \\
{\scriptstyle \mu S}\big\downarrow & & \lambda_2 & & \big\downarrow{\scriptstyle S\mu} \\
TS & & \xrightarrow[\lambda]{} & & ST
\end{array}
$$

Doublads (cont.)

satisfying:

(6)

$$
\begin{array}{ccc}
 & T & \\
{\scriptstyle T\kappa}\swarrow & & \searrow{\scriptstyle \kappa T} \\
TS & \xrightarrow[\lambda]{} & ST
\end{array}
$$

(7)

$$
\begin{array}{ccccc}
TSS & \xrightarrow{\lambda S} & STS & \xrightarrow{S\lambda} & SST \\
{\scriptstyle T\nu}\downarrow & & & & \downarrow{\scriptstyle \nu T} \\
TS & & \xrightarrow[\lambda]{} & & ST
\end{array}
$$

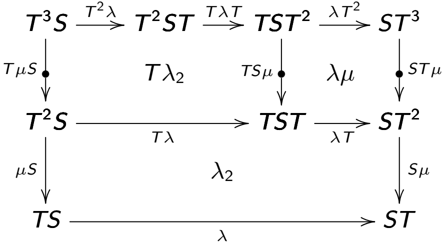
(8)

$$
\begin{array}{ccccc}
TS & \xrightarrow{1_{TS}} & TS & \xrightarrow{\lambda} & ST \\
{\scriptstyle T\eta S}\bullet\!\!\downarrow & T\lambda_0 & \bullet\!\!\downarrow{\scriptstyle TS\eta} & \lambda\eta & \bullet\!\!\downarrow{\scriptstyle ST\eta} \\
T^2S & \xrightarrow[T\lambda]{} & TST & \xrightarrow[\lambda T]{} & ST^2 \\
{\scriptstyle \mu S}\bullet\!\!\downarrow & & \lambda_2 & & \bullet\!\!\downarrow{\scriptstyle S\mu} \\
TS & & \xrightarrow[\lambda]{\qquad\qquad} & & ST
\end{array}
\quad = \quad
\begin{array}{ccc}
TS & \xrightarrow{\lambda} & ST \\
{\scriptstyle \mathrm{id}_{TS}}\bullet\!\!\downarrow & \mathrm{id}_\lambda & \bullet\!\!\downarrow{\scriptstyle \mathrm{id}_{ST}} \\
TS & \xrightarrow[\lambda]{} & ST
\end{array}
$$

(9)

$$
\begin{array}{ccccc}
TS & \xrightarrow{\lambda} & ST & \xrightarrow{1_{ST}} & ST \\
{\scriptstyle \eta TS}\bullet\!\!\downarrow & \eta\lambda & \bullet\!\!\downarrow{\scriptstyle \eta ST} & \lambda_0 T & \bullet\!\!\downarrow{\scriptstyle S\eta T} \\
T^2S & \xrightarrow[T\lambda]{} & TST & \xrightarrow[\lambda T]{} & ST^2 \\
{\scriptstyle \mu S}\bullet\!\!\downarrow & & \lambda_2 & & \bullet\!\!\downarrow{\scriptstyle S\mu} \\
TS & & \xrightarrow[\lambda]{\qquad\qquad} & & ST
\end{array}
\quad = \quad
\begin{array}{ccc}
TS & \xrightarrow{\lambda} & ST \\
{\scriptstyle \mathrm{id}_{TS}}\bullet\!\!\downarrow & \mathrm{id}_\lambda & \bullet\!\!\downarrow{\scriptstyle \mathrm{id}_{ST}} \\
TS & \xrightarrow[\lambda]{} & ST
\end{array}
$$

(10)



An ordinary distributive law gives a doublad on $\square\mathbf{A}$

A mixed distributive law (of a comonad over a monad) is one on $(\square\mathbf{A})^{co}$

Theorem

Doublads $(\mathbb{T}, \lambda, \mathbb{S})$ *are in bijection with extensions of* $\mathbb{T}$ *to vertical monads* $\widetilde{\mathbb{T}}$ *on* $\mathbb{K}\mathrm{l}(\mathbb{S})$

Define $\mathbb{K}\mathrm{l}(\mathbb{T}, \lambda, \mathbb{S})$ to be $\mathbb{K}\mathrm{l}(\widetilde{\mathbb{T}})$

- An object of $\mathbb{K}\mathrm{l}(\widetilde{\mathbb{T}})$ is just an object of $\mathbb{A}$
- A horizontal morphism $[f] \colon A \xrightarrow{\circ} B$ is given by a horizontal morphism $f \colon A \longrightarrow SB$ in $\mathbb{A}$
- A vertical morphism $[v] \colon A \xrightarrow{\bullet} C$ is given by a vertical morphism $v \colon A \xrightarrow{\bullet} TC$
- A cell

$$\begin{array}{ccc} A & \xrightarrow[\circ]{\{f\}} & B \\ {\scriptstyle [v]}\big\downarrow\bullet & [\alpha] & \bullet\big\downarrow{\scriptstyle [w]} \\ C & \xrightarrow[\{g\}]{\circ} & D \end{array} \quad \text{is} \quad \begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & SB \\ {\scriptstyle v}\big\downarrow\bullet & & \alpha & & \bullet\big\downarrow{\scriptstyle Sw} \\ TC & \xrightarrow[Tg]{} & TSD & \xrightarrow[\lambda D]{} & STD \end{array} \quad \text{in } \mathbb{A}$$

- Satisfying some "obvious" conditions

To be continued ...

¡Gracias!