2-Monoidal Categories (à la Aguiar & Mahajan) or Duoidal Categories (à la Booker & Street)

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(April 1, 2014)

Definition

A *duoidal category* is a category $\mathbf{V}$ with two monoidal structures $(\mathbf{V}, \otimes, I, \dots)$ and $(\mathbf{V}, \boxtimes, J, \dots)$ related by interchange morphisms

$$\chi : (A \otimes B) \boxtimes (C \otimes D) \longrightarrow (A \boxtimes C) \otimes (B \boxtimes D)$$

$$\mu : I \boxtimes I \longrightarrow I$$

$$\delta : J \longrightarrow J \otimes J$$

$$\tau : J \longrightarrow I$$

such that either of the two equivalent conditions

$$\boxtimes : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V} \text{ and } J : \mathbf{1} \longrightarrow \mathbf{V}$$

are $\otimes$ comonoidal functors

or

$$\otimes : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V} \text{ and } I : \mathbf{1} \longrightarrow \mathbf{V}$$

are $\boxtimes$ monoidal functors

are satisfied

Equivalently

- $(\mathbf{V}, \otimes, I, \dots)$ is a pseudo-monoid in $\mathcal{M}on\mathcal{C}at_{comon}$ with multiplication $\boxtimes$ and unit J
  or
- $(\mathbf{V}, \boxtimes, J, \dots)$ is a pseudo-monoid in $\mathcal{M}on\mathcal{C}at_{mon}$ with multiplication $\otimes$ and unit I
- We can think of $\otimes$ as horizontal and $\boxtimes$ as vertical and write complicated expressions as matrices

Pseudo-Monoids

$\mathcal{A}$ a 2-category with products
A *pseudo-monoid* in $\mathcal{A}$ is

$$m : M \times M \longrightarrow M$$

$$e : 1 \longrightarrow M$$

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times 1} & M \times M \\ {\scriptstyle 1 \times m}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle m} \\ M \times M & \xrightarrow[m]{} & M \end{array}$$

$$\begin{array}{ccccc} 1 \times M & \xrightarrow{e \times 1} & M \times M & \xleftarrow{1 \times e} & M \times 1 \\ & \searrow \; \overset{\rho}{\Leftarrow} & \downarrow{\scriptstyle m} & \overset{\lambda}{\Rightarrow} \; \swarrow & \\ & & M & & \end{array}$$

α, λ, ρ coherent isomorphisms

Coherence Conditions

$$
\begin{array}{ccc}
(A \otimes B) \boxtimes ((C \otimes D) \boxtimes (E \otimes F)) & \xrightarrow{\alpha} & ((A \otimes B) \boxtimes (C \otimes D)) \boxtimes (E \otimes F) \\
\downarrow{\scriptstyle 1 \boxtimes \chi} & & \downarrow{\scriptstyle \chi \boxtimes 1} \\
(A \otimes B) \boxtimes ((C \boxtimes E) \otimes (D \boxtimes F)) & & ((A \boxtimes C) \otimes (B \boxtimes D)) \boxtimes (E \otimes F) \\
\downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} \\
(A \boxtimes (C \boxtimes E)) \otimes (B \boxtimes (D \boxtimes F)) & \xrightarrow[\alpha \otimes \alpha]{} & ((A \boxtimes C) \boxtimes E) \otimes ((B \boxtimes D) \boxtimes F)
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes (B \otimes C)) \boxtimes (D \otimes (E \otimes F)) & \xrightarrow{\alpha \otimes \alpha} & ((A \otimes B) \otimes C) \boxtimes ((D \otimes E) \otimes F) \\
\Big\downarrow \chi & & \Big\downarrow \chi \\
(A \boxtimes D) \otimes ((B \otimes C) \boxtimes (E \otimes F)) & & ((A \otimes B) \boxtimes (D \otimes E)) \otimes (C \boxtimes F) \\
\Big\downarrow 1 \otimes \chi & & \Big\downarrow \chi \otimes 1 \\
(A \boxtimes D) \otimes ((B \boxtimes E) \otimes (C \boxtimes F)) & \xrightarrow[\alpha]{} & ((A \boxtimes D) \otimes (B \boxtimes E)) \otimes (C \boxtimes F)
\end{array}
$$

$$
\begin{array}{ccc}
(I \otimes A) \boxtimes (I \otimes B) & \xrightarrow{\lambda \boxtimes \lambda} & A \boxtimes B \\
\Big\downarrow{\scriptstyle \chi} & & \Big\uparrow{\scriptstyle \lambda} \\
(I \boxtimes I) \otimes (A \boxtimes B) & \xrightarrow[\mu \otimes 1]{} & I \otimes (A \boxtimes B)
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes I) \boxtimes (B \otimes I) & \xrightarrow{\rho \boxtimes \rho} & A \boxtimes I \\
\Big\downarrow{\scriptstyle \chi} & & \Big\uparrow{\scriptstyle \rho} \\
(A \boxtimes B) \otimes (I \boxtimes I) & \xrightarrow[1 \otimes \mu]{} & (A \boxtimes B) \otimes I
\end{array}
$$

$$
\begin{array}{ccc}
J \boxtimes (A \otimes B) & \xrightarrow{\delta \boxtimes 1} & (J \otimes J) \boxtimes (A \otimes B) \\
\Big\downarrow{\scriptstyle \lambda'} & & \Big\downarrow{\scriptstyle \chi} \\
A \otimes B & \xleftarrow[\lambda' \otimes \lambda']{} & (J \boxtimes A) \otimes (J \boxtimes B)
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes B) \boxtimes J & \xrightarrow{1 \boxtimes \delta} & (A \otimes B) \boxtimes (J \otimes J) \\
\Big\downarrow{\scriptstyle \rho'} & & \Big\downarrow \\
A \otimes B & \xleftarrow[\rho' \otimes \rho']{} & (A \boxtimes J) \otimes (B \boxtimes J)
\end{array}
$$

$(I, \mu : I \boxtimes I \longrightarrow I, \tau : J \longrightarrow I)$ is a monoid in $\mathbf{V}_{\boxtimes}$

$(J, \delta : J \longrightarrow J \otimes J, \tau : J \longrightarrow I)$ is a comonoid in $\mathbf{V}_{\otimes}$

Morphisms

$$F : (\mathbf{W}, \otimes, \boxtimes) \longrightarrow (\mathbf{V}, \otimes, \boxtimes)$$

- *Double monoidal:* monoidal for $\otimes$, $\boxtimes$ (+ coh.)
- *Bimonoidal:* monoidal for $\boxtimes$, comonoidal for $\otimes$ (+ coh.)
- *Double comonoidal:* comonoidal for $\otimes$, $\boxtimes$ (+ coh.)

Between pseudo-monoids in a 2-category we can have monoidal or comonoidal morphisms

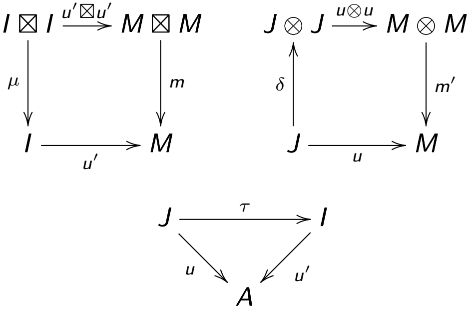
$$\begin{array}{ccc} M \times M & \xrightarrow{\;*\;} & M \\ {\scriptstyle f \times f}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ N \times N & \xrightarrow[\bullet]{} & N \end{array} \qquad \begin{array}{ccc} M \times M & \xrightarrow{\;*\;} & M \\ {\scriptstyle f \times f}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ N \times N & \xrightarrow[\bullet]{} & N \end{array}$$

In $\mathcal{M}on\mathcal{C}at_{comon}$ we get bimonoidal and double comonoidal
In $\mathcal{M}on\mathcal{C}at_{mon}$ we get double monoidal and bimonoidal

Double Monoids

Take $\mathbf{W} = \mathbf{1}$, get:
Double monoids in $\mathbf{V}$
$M \boxtimes M \xrightarrow{m} M$, $M \otimes M \xrightarrow{m'} M$
$J \xrightarrow{u} M$, $I \xrightarrow{u'} M$

$$\begin{array}{ccc}
(M \otimes M) \boxtimes (M \otimes M) & \xrightarrow{m' \boxtimes m'} & M \boxtimes M \\
\downarrow \chi & & \\
(M \boxtimes M) \otimes (M \boxtimes M) & & \downarrow m \\
\downarrow m \otimes m & & \\
M \otimes M & \xrightarrow[m']{} & M
\end{array}$$



If **V** is braided this reduces to commutative monoid
Dually we have double comonoids

Bimonoids

Also get bimonoids in **V**

$$M \boxtimes M \xrightarrow{m} M \qquad M \xrightarrow{d} M \otimes M$$

$$J \xrightarrow{u} M \qquad M \xrightarrow{e} I$$

$$\begin{array}{ccc}
(M \otimes M) \boxtimes (M \otimes M) & \xleftarrow{d \boxtimes d} & M \boxtimes M \\
\downarrow{\scriptstyle \chi} & & \\
(M \boxtimes M) \otimes (M \boxtimes M) & & \downarrow{\scriptstyle m} \\
\downarrow{\scriptstyle m \otimes m} & & \\
M \otimes M & \xleftarrow[d]{} & M
\end{array}$$

$$
\begin{CD}
I \boxtimes I @<{e \boxtimes e}<< M \boxtimes M \\
@V{\mu}VV @VV{m}V \\
I @<<{e}< M
\end{CD}
\qquad\qquad
\begin{CD}
J \otimes J @>{u \otimes u}>> M \otimes M \\
@A{\delta}AA @AA{d}A \\
J @>>{u}> M
\end{CD}
$$

$$
\begin{array}{ccc}
J & \xrightarrow{\quad \tau \quad} & I \\
& {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle e} & \\
& M &
\end{array}
$$

Tensor Product of Monoids

$\otimes : \mathbf{V}_\boxtimes \times \mathbf{V}_\boxtimes \longrightarrow \mathbf{V}_\boxtimes$ and $I : \mathbf{1} \longrightarrow \mathbf{V}_\boxtimes$ are monoidal functors, so preserve monoids

Makes $Mon(\mathbf{V}_\boxtimes)$ into a monoidal category with $\otimes$ as tensor and I as unit

Can consider monoids or comonoids in $Mon(\mathbf{V}_\boxtimes)$ and get another description of double monoids and bimonoids

That $I : \mathbf{1} \longrightarrow \mathbf{V}_\boxtimes$ preserves monoids means that I is a $\boxtimes$-monoid

Enriched Categories

Consider categories enriched in $\mathbf{V}_{\boxtimes}$
$\otimes : \mathbf{V}_{\boxtimes} \times \mathbf{V}_{\boxtimes} \longrightarrow \mathbf{V}_{\boxtimes}$ induces a 2-functor

$$(\mathbf{V}_{\boxtimes} \times \mathbf{V}_{\boxtimes})\text{-}\mathcal{C}at \longrightarrow \mathbf{V}_{\boxtimes}\text{-}\mathcal{C}at$$

For two $\mathbf{V}_{\boxtimes}$-categories $\mathbf{A}$, $\mathbf{B}$ we get a canonical $\mathbf{V}_{\boxtimes} \times \mathbf{V}_{\boxtimes}$-category $\mathbf{A} \times \mathbf{B}$
This produces $\mathbf{A} \otimes \mathbf{B}$

- Objects are pairs (A, B)
- Homs $(\mathbf{A} \otimes \mathbf{B})((A, B), (A', B')) = \mathbf{A}(A, A') \otimes \mathbf{B}(B, B')$

Makes $\mathbf{V}_{\boxtimes}$-$\mathcal{C}at$ into a monoidal category

$$
\begin{array}{c}
(\mathbf{A}(A',A'')\otimes\mathbf{B}(B',B''))\boxtimes(\mathbf{A}(A,A')\otimes\mathbf{B}(B,B')) \\
\Big\downarrow{\scriptstyle \chi} \\
(\mathbf{A}(A',A'')\boxtimes\mathbf{A}(A,A'))\otimes(\mathbf{B}(B',B'')\boxtimes\mathbf{B}(B,B')) \\
\Big\downarrow{\scriptstyle \circ_{\mathbf{A}}\otimes\circ_{\mathbf{B}}} \\
\mathbf{A}(A,A'')\otimes\mathbf{B}(B,B'')
\end{array}
\qquad
\begin{array}{c}
J \\
\Big\downarrow{\scriptstyle \delta} \\
J\otimes J \\
\Big\downarrow{\scriptstyle \mathrm{id}_A\otimes\mathrm{id}_A} \\
\mathbf{A}(A,A)\otimes\mathbf{B}(B,B)
\end{array}
$$

Examples

A braided monoidal category is duoidal with

$$\boxtimes = \otimes$$

$$J = I$$

$$\chi : A \otimes B \otimes C \otimes D \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes C \otimes B \otimes D$$

Proposition

A duoidal category with χ, μ, δ, τ isomorphisms is equivalent to a braided monoidal category

X-Graph

$$A \overset{s}{\underset{t}{\rightrightarrows}} X$$

$$A \boxtimes B = \{x \xrightarrow{a} y \xrightarrow{b} z\}$$

$$A \otimes B = \{x \overset{a}{\underset{b}{\rightrightarrows}} y\}$$

$$J = (X \overset{1}{\underset{1}{\rightrightarrows}} X) \ \text{(only loops)}$$

$$I = (X \times X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X) \ \text{(complete graph)}$$

$$(A \otimes B) \boxtimes (C \otimes D) = \left\{ x \overset{a}{\underset{b}{\rightrightarrows}} y \overset{c}{\underset{d}{\rightrightarrows}} z \right\}$$

$$(A \boxtimes C) \otimes (B \boxtimes D) = \left\{ x \begin{array}{c} \xrightarrow{a} y \xrightarrow{c} \\ \xrightarrow[b]{} y' \xrightarrow[d]{} \end{array} z \right\}$$

A monoid in X-$\mathbf{Graph}_{\boxtimes}$ is a small category. Every such monoid is uniquely a bimonoid

$$d : (x \xrightarrow{a} y) \longmapsto (x \underset{a}{\overset{a}{\rightrightarrows}} y)$$

A double monoid is a category enriched in $(\mathbf{Mon}, \times, 1)$

Products

$(\mathbf{V}, \boxtimes, J)$ monoidal category with finite products (no preservation)
If we take $\otimes = x$ and $I = 1$ we get a duoidal category with the canonical

$$\chi : (A \times B) \boxtimes (C \times D) \longrightarrow (A \boxtimes C) \times (B \boxtimes D)$$

$$\mu : 1 \boxtimes 1 \longrightarrow 1$$

$$\delta : J \longrightarrow J \times J$$

$$\tau : J \longrightarrow 1$$

- X-**Graph**
- Quantale Q, $\boxtimes = \&$, $\otimes = \wedge$
- $\mathbb{N}$-Graded sets $(A_0, A_1, A_2, \dots)$

$\boxtimes$ = Cauchy product $(A_n) \boxtimes (B_n) = (\sum_{p+q=n} A_p \times B_q)$
$\otimes$ = Hadamard product $(A_n) \otimes (B_n) = (A_n \times B_n)$

Day Convolution

- Can replace $\mathbb{N}$ by any small monoidal category $(\mathbf{M}, *, E)$
- Day convolution makes $\mathbf{Set}^{\mathbf{M}}$ into a monoidal category

$$(\Phi \boxtimes \Psi)(X) = \varinjlim_{Y*Z \to X} \Phi Y \times \Psi Z$$

$$J = \mathbf{M}(E, -)$$

- $$\frac{\Phi \boxtimes \Psi \longrightarrow \Theta}{\langle \Phi Y \times \Psi Z \longrightarrow \Theta(Y * Z) \rangle_{Y,Z}}$$
- A monoid for $\boxtimes$ is a monoidal functor

$$\Phi : (\mathbf{M}, *) \longrightarrow (\mathbf{Set}, x)$$

- A bimonoid is the same thing
- A double monoid is a monoidal functor

$$(\mathbf{M}, *) \longrightarrow (\mathbf{Mon}, x)$$

New Duoidal Categories from Old (1)

$(\mathbf{V}, \otimes, I, \boxtimes, J)$ Duoidal with coproducts preserved by $\boxtimes$, and X set
X-**Mat** is the category of $X \times X$-matrices of objects $\mathbf{V}$
$\boxtimes$ matrix multiplication

$$[V_{xy}] \boxtimes [W_{xy}] = [\sum_z V_{xz} \boxtimes W_{zy}]_{xy}$$

$$J_{xy} = \begin{cases} J & \text{if x = y} \\ 0 & \text{o.w.} \end{cases}$$

$\otimes$ Hadamard product

$$[V_{xy}] \otimes [W_{xy}] = [V_{xy} \otimes W_{xy}]$$

$$I_{xy} = I \quad \text{all } x, y$$

$$((V \otimes W) \boxtimes (R \otimes S))_{xy} = \sum_z (V_{xz} \otimes W_{xz}) \boxtimes (R_{zy} \otimes S_{zy})$$

$$((V \boxtimes R) \otimes (W \boxtimes S))_{xy} = (\sum_z V_{xz} \boxtimes R_{zy}) \otimes (\sum_{z'} W_{xz'} \boxtimes S_{z'y})$$

New Duoidal Categories from Old (2)

$\mathbb{N}$-Graded **V**-objects

$$(A_n) \boxtimes (B_n) = (\sum_{p+q=n} A_p \boxtimes B_q) \quad \text{(Cauchy)}$$

$$(A_n) \otimes (B_n) = (A_n \otimes B_n) \quad \text{(Hadamard)}$$

Can replace $\mathbb{N}$ by any small monoidal category $(\mathbf{M}, *, E)$
Assume **V** has colimits preserved by $\boxtimes$
$\mathbf{V}^{\mathbf{M}}$ (all functors) is duoidal with

$$\Phi \boxtimes \Psi(X) = \varinjlim_{Y * Z \to X} \Phi Y \boxtimes \Psi Z$$

$$J(X) = \sum_{E \to X} J$$

$$\Phi \otimes \Psi(X) = \Phi X \otimes \Psi X$$

$$I(X) = I$$

Day Convolution Redux

$(\mathbf{V}, \otimes, \boxtimes)$ a small duoidal category

$\mathbf{Set}^{\mathbf{V}^{op}}$ becomes a duoidal category with

$\otimes$ = Day convolution using $\otimes$
$\boxtimes$ = Day convolution using $\boxtimes$

$Y : \mathbf{V} \longrightarrow \mathbf{Set}^{\mathbf{V}^{op}}$ preserves $\otimes$ and $\boxtimes$

Completion of $\mathbf{V}$ (Booker/Street Theorem 4.8)

M-Objects (Actions)

Let M, N be monoids in $\mathbf{V}_{\boxtimes}$
Let (A, α) be an M-object and (B, β) an N-object in $\mathbf{V}_{\boxtimes}$
Then $A \otimes B$ becomes an $M \otimes N$-object

$$(M \otimes N) \boxtimes (A \otimes B) \xrightarrow{\chi} (M \boxtimes A) \otimes (N \boxtimes B) \xrightarrow{\alpha \otimes \beta} A \otimes B$$

Gives a functor

$$\bar{\otimes} : \mathbf{V}_{\boxtimes}^{M} \times \mathbf{V}_{\boxtimes}^{N} \longrightarrow \mathbf{V}_{\boxtimes}^{M \otimes N}$$

If M is a bimonoid in $\mathbf{V}$

$$d : M \longrightarrow M \otimes M$$

$$e : M \longrightarrow I$$

are monoid homomorphisms
Gives a functor

$$\mathbf{V}_{\boxtimes}^{M \otimes M} \longrightarrow \mathbf{V}_{\boxtimes}^{M}$$

Combine this with $\bar{\otimes}$ gives

$$\otimes : \mathbf{V}_{\boxtimes}^{M} \times \mathbf{V}_{\boxtimes}^{M} \longrightarrow \mathbf{V}_{\boxtimes}^{M}$$

So $(A, \alpha) \otimes (B, \beta)$ is given by

$$M \boxtimes (A \otimes B) \xrightarrow{d \boxtimes 1} (M \otimes M) \boxtimes (A \otimes B) \xrightarrow{\chi} (M \boxtimes A) \otimes (M \boxtimes B) \xrightarrow{\alpha \otimes \beta} A \otimes B$$

Also: $M \boxtimes I \xrightarrow{e \boxtimes 1} I \boxtimes I \xrightarrow{\mu} I$

Theorem

If M is a bimonoid then $\mathbf{V}^{M}_{\boxtimes}$ is a monoidal category with $\otimes$ and $(I, \mu \cdot e \boxtimes 1)$.

Question: If $\mathbf{V}_{\otimes}$ is closed and $\mathbf{V}$ has finite limits, is $\mathbf{V}^{M}_{\boxtimes}$ also closed?

References

- Aguiar & Mahajan, Monoidal Functors, Species and Hopf Algebras http://www.math.tamu.edu/~maguiar/a.pdf (Ch. 6)
- Booker & Street, Tannaka Duality and Convolution for Duoidal Categories, TAC Vol. 28, No. 6, 2013 (pp. 166-205)
- nLab – duoidal categories