

# FIRST ORDER THEORIES

• DATA: SET OF OPERATION SYMBOLS  
 $w_1, w_2, \dots$  WITH ARITY  $m_1, m_2, \dots$

Ex:  $+, \cdot, ()^{-1}, 0, 1; 2, 2, 1, 0, 0$

SET OF PREDICATE SYMBOLS

$P_1, P_2, \dots$  WITH ARITY  
 $< ; 2$

VARIABLES  $x, y, z, x_1, x_2, x_3, \dots$

• TERMS: BUILT RECURSIVELY FROM  
 VARIABLES AND OPERATIONS  
 $(x+y) \cdot x, (xy)^{-1}, y^{-1}x^{-1}$

• FORMULAS (P.E.):

ATOMIC:  $t_1 \approx t_2; P(t_1, t_2, \dots, t_n)$   
 $(xy)z \approx x(yz); 0 < x \cdot x + 1$

COMPOSITE:  $\Phi \wedge \Psi, \Phi \vee \Psi, \exists x \Phi, \top, \perp$

• THEORY: SET OF SEQUENTS  $\Phi \vdash \Psi$

$\tau$  Ex:  $x < y \wedge y < z \vdash x < z$

$\top \vdash x \approx 0 \vee \exists y (xy \approx 1)$

(2)

- MODELS:  $\rightarrow$  A SET  $M$ 
  - FOR EACH  $n$ -ARY OP SYMBOL  $w$   
A FUNCTION  $M(w): M^n \rightarrow M$  (OPERATION)
  - $\rightarrow$  FOR EACH  $n$ -ARY PRED SYMB  $P$   
A RELATION  $M(P) \subseteq M^n$ .

- EXTENSION TO TERMS & FORMULAS

$\vec{x} = (x_1, x_2, \dots, x_n)$  SEQUENCE OF VARIABLES

IF VAR ( $t$ ) ARE IN  $\vec{x}$  THEN WE INTERPRET

$M_{\vec{x}}(t): M^n \rightarrow M$  RECURSIVELY

$t = x_i$  (VAR)  $\text{th } M_{\vec{x}}(t) = \text{PROJ}_i$

IF VAR ( $\Phi$ ) ARE AMONG  $\vec{x}$  THEN

$M_{\vec{x}}(\Phi) \subseteq M^n$

$M_{\vec{x}}(t_1 \approx t_2) = \left\{ (a_1, \dots, a_n) \in M^n \mid M_{\vec{x}}(t_1) = M_{\vec{x}}(t_2) \right\}$

$\wedge, \vee$  GIVEN BY  $\cap$  AND  $\cup$

$\top, \perp$  " "  $M^n$  "  $\emptyset$

$\exists_x \Phi$  " " IMAGE

$\Phi \vdash \Psi$  HOLDS IF  $M_{\vec{x}}(\Phi) \subseteq M_{\vec{x}}(\Psi)$ .

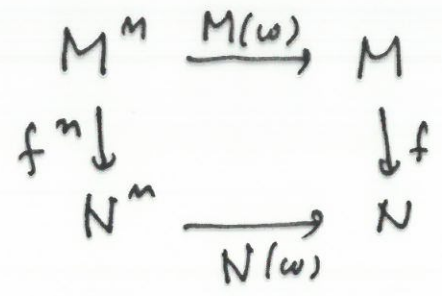
GET CLASS OF MODELS  $\text{Mod}(\mathcal{T})$

# MORPHISMS OF MODELS

A FUNCTION  $f: M \rightarrow N$  SUCH THAT

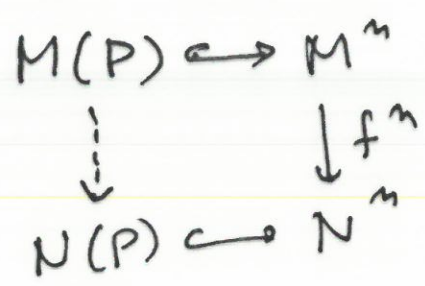
FOR EVERY OPERATION SYMBOL  $w$

$$f(a+b) = f(a) + f(b)$$



FOR EVERY PREDICATE SYMBOL  $P$

$$a < b \Rightarrow f(a) < f(b)$$



GET A CATEGORY OF MODELS MOD( $\mathcal{T}$ )

NOTE 1: MORPHISMS PRESERVE INTERPRETATIONS  $M_{\mathcal{X}}(t)$  AND  $M_{\mathcal{X}}(\Phi)$  (BECAUSE WE ONLY TAKE P.E. FORMULAS.)

NOTE 2: RESTRICTING TO P.E. FORMULAS DOES NOT RESTRICT THE POSSIBLE CLASSES OF MODELS

Ex:  $P \vdash Q \vee \neg R$

REPLACE BY

$$P \vdash Q \vee \bar{R}, \quad R \wedge \bar{R} \vdash \perp, \quad \neg R \vee \bar{R}$$

# UNIVERSAL ALGEBRA (EQUATIONAL THEORY) (4)

- NO PREDICATES
- ONLY EQUATIONS AS AXIOMS  $\vdash t_1 \approx t_2$

MODELS CALLED (UNIV.) ALGEBRAS

Ex: GROUPS, RINGS, VECT. SP.

MORPHISMS CALLED HOMOMORPHISMS

## LAWVERE THEORY $\mathbb{T}$

CATEGORY WHOSE OBJECTS ARE  
 $[0], [1], [2], \dots$  WITH  $[m] = [1] \times [1] \times \dots \times [1]$ .

$\mathbb{T}$ -ALGEBRA = FUNCTOR  $M: \mathbb{T} \rightarrow \underline{\text{SET}}$   
S.T.  $M[m] = M[1] \times M[1] \times \dots \times M[1]$ .

HOMOMORPHISM = NATURAL TRANSF.

GIVEN  $\mathcal{T}$  CONSTRUCT  $\mathbb{T}$  AS FOLLOWS

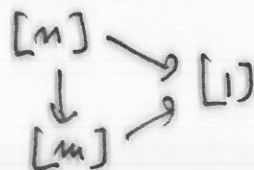
$[m] \rightarrow [m]$   $m$ -TUPLE OF EQUIV.  
CLASSES OF TERMS IN THE VARIABLES  
 $x_1 \dots x_m$ .  $t_1 \sim t_2$  IFF  $\mathcal{T} \models t_1 \approx t_2$

COMPOSITION = SUBSTITUTION

GIVEN  $\mathbb{T}$  GET A  $\mathcal{T}$  AS FOLLOWS

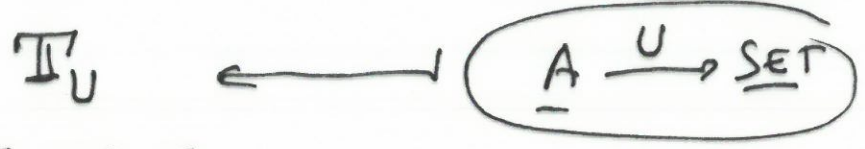
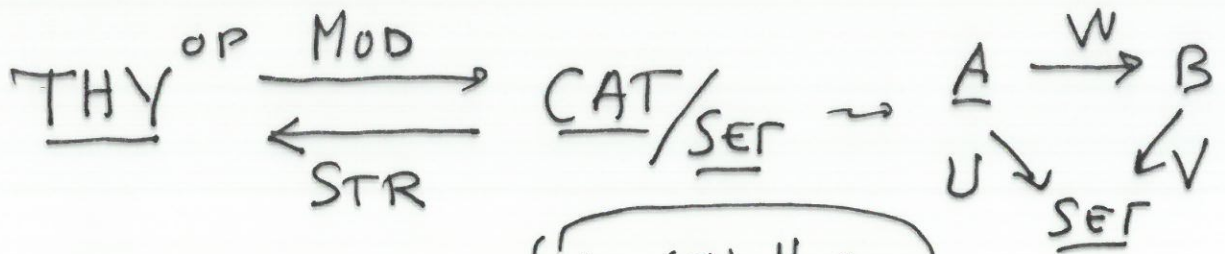
$m$ -ARY OPS:  $[m] \rightarrow [1]$

EQUATIONS: COMM DIAGS





MORPHISMS OF THEORIES:  $G: \mathbb{T} \rightarrow \mathbb{S}$ ,  $G[m] = [m]$ .



$[n] \rightarrow [m]$   
 $t: U^m \rightarrow U^n$

ONLY A SET OF THESE BECAUSE U HAS A LEFT ADJOINT.

$\mathbb{T} \mapsto (\text{MOD}(\mathbb{T}) \xrightarrow{U} \text{SET}) \mapsto \mathbb{T}_U = \mathbb{T}$

LAWRENTZ USED THIS TO CHARACTERIZE ALGEBRAIC CATEGORIES  $A \xrightarrow{U} \text{SET}$ .  
 (U HAS LEFT ADJ + A HAS CERTAIN QUOTIENTS)

CAN WE DO THE SAME THING FOR FIRST ORDER THEORIES? I.E. CAN WE RECOVER THE LOGIC FROM THE MODELS?

⑥

## HYPERDOCTRINES (E.E.D's)

- CATEGORY  $\mathbb{T}$  WITH FINITE PRODUCTS
- FOR EVERY  $A$  IN  $\mathbb{T}$  AN ORDERED SET  $\mathbb{P}_A$
- $\mathbb{P}_A$  HAS  $\wedge, \vee, \top, \perp$  (LATTICE)
- FOR EVERY  $f: A \rightarrow B$  IN  $\mathbb{T}$ , AN ORDER PRESERVING FUNCTION  $f^*: \mathbb{P}_B \rightarrow \mathbb{P}_A$  (FUNCTOR)
- $f^*$  PRES  $\wedge, \vee, \top, \perp$
- $1_A^* = 1_{\mathbb{P}_A}$ ,  $(g \circ f)^* = f^* \circ g^*$
- FOR EVERY  $f$ ,  $f^*$  HAS A LEFT ADJOINT  $\exists_f: \mathbb{P}_A \rightarrow \mathbb{P}_B$  ( $\exists_f x \leq y \iff x \leq f^* y$ )
- FROBENIUS RECIPROCITY

$$\exists_f (f^* y \wedge x) = y \wedge \exists_f x$$

Ex:  $\mathbb{T} = \text{SET}$ ,  $\mathbb{P}_A = (P(A), \subseteq)$

$$f^* = f^{-1}, \exists_f = f(\_)$$

GIVEN FIRST ORDER THY  $\mathcal{T}$

$\mathbb{T}$  = LAWVERE THEORY OF  $\mathcal{T}$  (EQUATIONAL PART)

$\mathbb{P}_{[m]}$  = EQUIV CLASSES OF FORMULAS  $\Phi$

WITH FREE VARIABLES  $x_1, x_2, \dots, x_m$

$\Phi \sim \Phi'$  IFF  $\mathcal{T} \vDash (\Phi \vdash \Phi')$  &  $\mathcal{T} \vDash (\Phi' \vdash \Phi)$

$\Phi \leq \Phi'$  IFF  $\mathcal{T} \vDash \Phi \vdash \Phi'$

$\Phi \wedge \Psi, \Phi \vee \Psi, \top, \perp$  OBVIOUS

$\exists_{\omega} \Phi = \exists_{y_1, \dots, y_m} (\omega_1(y_1, \dots, y_m) \approx x_1 \wedge \dots \wedge \omega_m(y_1, \dots, y_m) \approx x_m \wedge \Phi(y_1, \dots, y_m))$

$[m] \xrightarrow{\omega} [m]$

NOTE:  $\exists_{\delta} \top$  IS EQUALITY IN  $\mathbb{P}_{[2]}$

$\delta: [1] \rightarrow [1] \times [1] = [2] \quad \delta = (x_1, x_1)$

$\exists_{\delta}(\top) = \exists y_1 (y_1 \approx x_1 \wedge y_1 \approx x_2 \wedge \top)$

MORPHISM OF HYPERDOCTRINES

$(\mathbb{T}, \mathbb{P}) \xrightarrow{F} (\mathbb{S}, \mathbb{Q})$

$A \longmapsto FA \quad (\text{PRES PRODS})$

$\mathbb{P}_A \longrightarrow \mathbb{Q}_{FA}$

PRES  $\wedge, \vee, \top, \perp, \exists$



MODEL  $(\mathbb{T}, \mathbb{P}) \xrightarrow{M} (\underline{\text{SET}}, \mathbb{P})$

HYPDOC<sup>OP</sup>  $\xrightarrow{\text{MOD}}$  CAT/SET  
 $\xleftarrow{\text{STR}}$

$[m] \rightarrow [m]$   
 $t: U^n \rightarrow U^m$

A  
 $\downarrow U$   
SET

$x \in \mathbb{P}_m$   

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 $\mathbb{P} \rightsquigarrow U^m$

FOR THESE TO BE SETS  
RESTRICT TO

A WITH FILT COLIM + "GENERATORS"

U PRES FILT COLIM

SKOLEM / SET