FIRST ORDER THEORIES

- DATA: SET OF OPERATION SYMBOLS
  $\omega_1, \omega_2, \ldots$ WITH ARITY $m_1, m_2 \ldots$
  Ex: $+, \cdot, ()^{-1}, 0, 1 \; ; \; 2, 2, 1, 0, 0$

  SET OF PREDICATE SYMBOLS
  $P_1, P_2, \ldots$ WITH ARITY
  $< \; ; \; 2$

  VARIABLES $x, y, z, x_1, x_2, x_3, \ldots$

- TERMS: BUILT RECURSIVELY FROM VARIABLES AND OPERATIONS
  $(x+y)\cdot x \; , \; (xy)^{-1} \; , \; y^{-1}x^{-1}$

- FORMULAS (P.E.):
  ATOMIC: $t_1 \approx t_2 \; ; \; P(t_1, t_2, \ldots t_n)$
  $(xy)z \approx x(yz) \; ; \; 0 < x \cdot x + 1$

  COMPOSITE: $\Phi \wedge \Psi, \Phi \vee \Psi, \exists x \Phi, \top, \bot$

- THEORY: SET OF SEQUENTS $\Phi \vdash \Psi$
  $\mathcal{T}$ Ex: $x < y \wedge y < z \vdash x < z$
  $\top \vdash x \approx 0 \vee \exists y \, (xy \approx 1)$

- MODELS:
  - A SET M
  - FOR EACH m-ARY OP SYMBOL ω A FUNCTION $M(\omega): M^m \longrightarrow M$ (OPERATION)
  - FOR EACH m-ARY PRED SYMB P A RELATION $M(P) \subseteq M^m$.

- EXTENSION TO TERMS & FORMULAS

  $\vec{x} = (x_1, x_2, \ldots x_m)$ SEQUENCE OF VARIABLES

  IF VAR(t) ARE IN $\vec{x}$ THEN WE INTERPRET

  $M_{\vec{x}}(t): M^m \longrightarrow M$ RECURSIVELY

  $t = x_i$ (VAR) th $M_{\vec{x}}(t) = \mathrm{PROS}_i$

  IF VAR(Φ) ARE AMONG $\vec{x}$ THEN

  $M_{\vec{x}}(\Phi) \subseteq M^m$

  $M_{\vec{x}}(t_1 \approx t_2) = \left\{ (a_1, \ldots a_m) \in M^m \,\middle|\, M_{\vec{x}}(t_1(\vec{a})) = M_{\vec{x}}(t_2(\vec{a})) \right\}$

  $\wedge, \vee$ GIVEN BY $\cap$ AND $\cup$

  $\top, \perp$ " " M^m " $\emptyset$

  $\exists_x \Phi$ " " IMAGE

$\Phi \vdash \Psi$ <u>HOLDS</u> IF $M_{\vec{x}}(\Phi) \subseteq M_{\vec{x}}(\Psi)$.

GET CLASS OF MODELS $\mathcal{M}od(\mathbb{T})$

MORPHISMS OF MODELS

A FUNCTION $f : M \to N$ SUCH THAT

FOR EVERY OPERATION SYMBOL ω

$$\begin{tikzcd} M^n \arrow[r, "M(\omega)"] \arrow[d, "f^n"'] & M \arrow[d, "f"] \\ N^n \arrow[r, "N(\omega)"'] & N \end{tikzcd}$$

$f(a+b) = f(a) + f(b)$

FOR EVERY PREDICATE SYMBOL P

$$\begin{tikzcd} M(P) \arrow[r, hook] \arrow[d, dashed] & M^n \arrow[d, "f^n"] \\ N(P) \arrow[r, hook] & N^n \end{tikzcd}$$

$a < b \Rightarrow f(a) < f(b)$

GET A <u>CATEGORY</u> OF MODELS $\underline{\mathrm{Mod}}(\mathbb{T})$

<u>NOTE 1</u>: MORPHISMS PRESERVE INTERPRETATIONS $M_{\vec{x}}(t)$ AND $M_{\vec{x}}(\Phi)$ (BECAUSE WE ONLY TAKE P.E. FORMULAS.)

<u>NOTE 2</u>: RESTRICTING TO P.E. FORMULAS DOES NOT RESTRICT THE POSSIBLE CLASSES OF MODELS

<u>Ex</u>: $P \vdash Q \vee \neg R$

<u>REPLACE BY</u>

$P \vdash Q \vee \bar{R}$, $\quad R \wedge \bar{R} \vdash \bot$, $\top \vdash R \vee \bar{R}$

UNIVERSAL ALGEBRA (EQUATIONAL THYS)

- NO PREDICATES
- ONLY EQUATIONS AS AXIOMS $\quad T \vdash t_1 \approx t_2$

MODELS CALLED (UNIV.) ALGEBRAS

Ex: GROUPS, RINGS, VECT. SP.

MORPHISMS CALLED HOMOMORPHISMS

LAWVERE THEORY $\mathbb{T}$

CATEGORY WHOSE OBJECTS ARE
$[0], [1], [2], \dots$ WITH $[n] = [1] \times [1] \times \cdots \times [1]$.

$\mathbb{T}$-ALGEBRA = FUNCTOR $M : \mathbb{T} \longrightarrow \underline{Set}$
S.T. $M[n] = M[1] \times M[1] \times \cdots \times M[1]$.

HOMOMORPHISM = NATURAL TRANSF.

GIVEN $\mathcal{T}$ CONSTRUCT $\mathbb{T}$ AS FOLLOWS

$[n] \longrightarrow [m]$ $\quad$ m-TUPLE OF EQUIV. CLASSES OF TERMS IN THE VARIABLES $x_1 \dots x_n$. $\quad t_1 \sim t_2$ IFF $\mathcal{T} \vDash t_1 \approx t_2$

COMPOSITION = SUBSTITUTION

GIVEN $\mathbb{T}$ GET A $\mathcal{T}$ AS FOLLOWS

M-ARY OPS: $[m] \to [1]$

EQUATIONS: COMM DIAGS

$$
\begin{tikzcd}
{[n]} \arrow[dr] \arrow[d] & \\
{[m]} \arrow[r] & {[1]}
\end{tikzcd}
$$

MORPHISMS OF THEORIES : $G: \mathbb{T} \to \mathbb{S}$, $G[n] = [n]$.

$$\begin{tikzcd} \underline{\mathrm{THY}}^{op} \arrow[r, shift left, "\mathrm{MOD}"] & \underline{\mathrm{CAT}}/\underline{\mathrm{SET}} \arrow[l, shift left, "\mathrm{STR}"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \underline{A} \arrow[rr, "W"] \arrow[dr, "U"'] & & B \arrow[dl, "V"] \\ & \underline{\mathrm{SET}} & \end{tikzcd}$$

$$\mathbb{T} \longmapsto \left(\underline{\mathrm{MOD}}(\mathbb{T}) \xrightarrow{U} \underline{\mathrm{SET}}\right)$$

$$\mathbb{T}_U \longleftarrow\!\!\shortmid \left(\underline{A} \xrightarrow{U} \underline{\mathrm{SET}}\right)$$

$$\frac{[n] \to [m]}{t: U^n \to U^m}$$

ONLY A SET OF THESE BECAUSE U HAS A LEFT ADJOINT.

$$\mathbb{T} \longmapsto \left(\underline{\mathrm{MOD}}(\mathbb{T}) \xrightarrow{U} \underline{\mathrm{SET}}\right) \longmapsto \mathbb{T}_U = \mathbb{T}$$

LAWVERE USED THIS TO CHARACTERIZE ALGEBRAIC CATEGORIES $\underline{A} \xrightarrow{U} \underline{\mathrm{SET}}$.
(U HAS LEFT ADJ + $\underline{A}$ HAS CERTAIN QUOTIENTS)

CAN WE DO THE SAME THING FOR FIRST ORDER THEORIES? I.E. CAN WE RECOVER THE LOGIC FROM THE MODELS?

HYPERDOCTRINES (E.E.D's)

- CATEGORY $\mathbb{T}$ WITH FINITE PRODUCTS
- FOR EVERY A IN $\mathbb{T}$ AN ORDERED SET $\mathbb{P}_A$
- $\mathbb{P}_A$ HAS $\wedge, \vee, \top, \bot$ (LATTICE)
- FOR EVERY $f : A \to B$ IN $\mathbb{T}$, AN ORDER PRESERVING FUNCTION $f^* : \mathbb{P}_B \to \mathbb{P}_A$ (FUNCTOR)
- f^* PRES $\wedge, \vee, \top, \bot$
- $1_A^* = 1_{\mathbb{P}_A}$, $(gf)^* = f^* g^*$
- FOR EVERY f, f^* HAS A LEFT ADJOINT $\exists_f : \mathbb{P}_A \to \mathbb{P}_B$ $\left(\exists_f x \leq y \iff x \leq f^* y \right)$
- FROBENIUS RESIPROCITY

$$\exists_f (f^* y \wedge x) = y \wedge \exists_f x$$

__Ex:__ $\mathbb{T} = \underline{\text{Set}}$, $\mathbb{P}_A = (P(A), \subseteq)$

$f^* = f^{-1}$, $\exists_f = f(\ )$

GIVEN FIRST ORDER THY τ

$\mathbb{T}$ = LAWVERE THEORY OF τ (EQUATIONAL PART)

$\mathbb{P}_{[m]}$ = EQUIV CLASSES OF FORMULAS Φ
WITH FREE VARIABLES $x_1, x_2, \cdots x_m$

$\Phi \sim \Phi'$ IFF $\tau \models (\Phi \vdash \Phi')$ & $\tau \models (\Phi' \vdash \Phi)$

$\Phi \leq \Phi'$ IFF $\tau \models \Phi \vdash \Phi'$

$\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\top$, $\bot$ OBVIOUS

$$\exists_\omega \Phi = \exists_{y_1 \cdots y_m} \big(\omega_1(y_1 \cdots y_m) \approx x_1 \wedge \cdots \wedge \omega_m(y_1 \cdots y_m) \approx x_m \wedge \Phi(y_1, \cdots y_m) \big)$$

$[n] \xrightarrow{\omega} [m]$

<u>NOTE:</u> $\exists_\delta \top$ IS EQUALITY IN $\mathbb{P}_{[2]}$

$\delta : [1] \longrightarrow [1] \times [1] = [2] \qquad \delta = (x_1, x_1)$

$$\exists_\delta(\top) = \exists y_1 (y_1 \approx x_1 \wedge y_1 \approx x_2 \wedge \top)$$

<u>MORPHISM OF HYPERDOCTRINES</u>

$$(\mathbb{T}, \mathbb{P}) \xrightarrow{F} (\mathbb{S}, \mathbb{Q})$$

$A \longmapsto FA$ (PRES PRODS)

$\mathbb{P}_A \longrightarrow \mathbb{Q}_{FA}$

PRES $\wedge, \vee, \top, \bot, \exists$

<u>Model</u> $(\mathbb{T}, \mathbb{P}) \xrightarrow{M} (\underline{\mathrm{Set}}, P)$

$$\underline{\mathrm{HypDoc}}^{op} \underset{\mathrm{STR}}{\overset{\mathrm{MOD}}{\rightleftarrows}} \underline{\mathrm{Cat}}/\underline{\mathrm{Set}}$$

$$\frac{[n] \to [m]}{t : U^n \to U^m}$$

$$\begin{array}{c} \underline{A} \\ \downarrow U \\ \underline{\mathrm{Set}} \end{array}$$

$$\frac{x \in \mathbb{P}_{[m]}}{P \rightarrowtail U^m}$$

FOR THESE TO BE SETS
RESTRICT TO

$\underline{A}$ WITH FILT COLIM + "GENERATORS"
U PRES FILT COLIM

$\underline{\mathrm{Skolem}}/\underline{\mathrm{Set}}$