Examples of Intercategories

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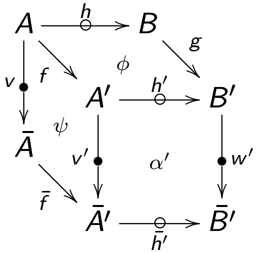
Intercategory



- Transversal composition strict
- Horizontal and vertical bicategorical (up to transversal iso)
- Interchange holds for horizontal and lateral cells
- For basic cells, $\circ$ is lax with respect to $\bullet$, i.e. we have *interchangers*

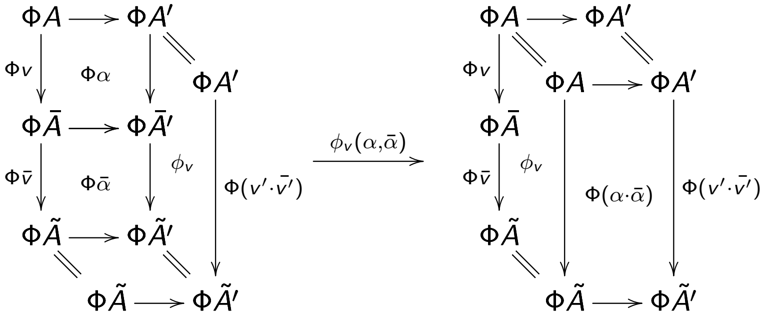
$$\chi : \frac{\alpha|\beta}{\bar{\alpha}|\bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}}\Big|\frac{\beta}{\bar{\beta}} \qquad \mu : \frac{\mathrm{id}_v}{\mathrm{id}_{\bar{v}}} \longrightarrow \mathrm{id}_{\frac{v}{\bar{v}}}$$

$$\delta : \mathrm{Id}_{h|h'} \longrightarrow \mathrm{Id}_h \,|\, \mathrm{Id}_{h'} \qquad \tau : \mathrm{Id}_{\mathrm{id}_A} \longrightarrow \mathrm{id}_{\mathrm{Id}_A}$$

- Satisfy coherence conditions

Morphisms of intercategories $\Phi : \mathbb{A} \longrightarrow \mathbb{B}$

- Three kinds – all are strict in the transversal direction
  - Lax-lax – horizontally and vertically lax
  - Colax-lax – horizontally colax and vertically lax
  - Colax-colax – horizontally and vertically colax
- (Co)laxity is for arrows and basic cells, e.g. vertical laxity



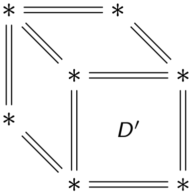
- There are cells between any two of these kinds of morphisms
- We get a *strict* triple category ICat of intercategories

Duoidal category (a.k.a. 2-monoidal category)

M. Aguiar and S. Mahajan [1]
T. Booker and R. Street [3]
G. Böhm, Y. Chen, L. Zhang [2]

$$(\mathbf{D}, \otimes, I, \boxtimes, J)$$

$\boxtimes$ is lax with respect to $\otimes$
It can be seen as an intercategory where general cubes are



Horizontal composition is $\boxtimes$, vertical is $\otimes$
χ, δ, μ, τ arbitrary

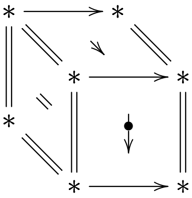
Monoidal double category

M. Shulman [8]

Double category $\mathbb{D}$ with $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ and $I : \mathbb{1} \longrightarrow \mathbb{D}$ strong

Associative and unitary up to coherent *isomorphisms*

Can be seen as an intercategory with cubes



χ, δ, μ, τ all identities.

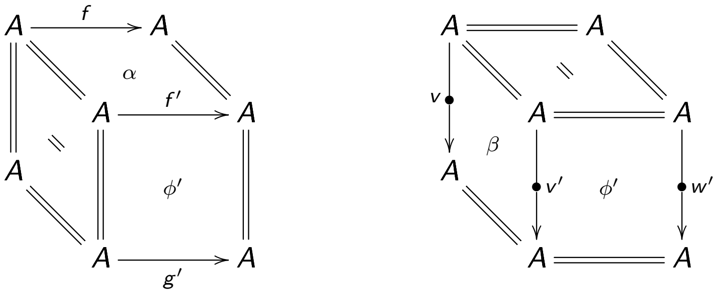
Monoidal double categories (continued)

Can relax $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ and $I : \mathbb{1} \longrightarrow \mathbb{D}$ to be lax or colax

E.g. A double category with a lax choice of finite products

For any intercategory A and object A of A we get two "monoidal double categories"

$\mathbb{HE}\mathrm{nd}(A)$ and $\mathbb{VE}\mathrm{nd}(A)$



In both cases, the χ, δ, μ, τ can be arbitrary

Locally cubical bicategory

Garner and Gurski [5]

Categories weakly enriched in the monoidal 2-category $\mathbb{D}\mathrm{bl}\mathbb{S}\mathrm{t}$:

Objects $A, B, C, \ldots$

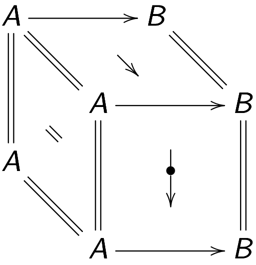
Double categories $\mathcal{A}(A, B)$

Strong functors $\qquad \mathbb{1} \longrightarrow \mathcal{A}(A, A)$

$$\mathcal{A}(A, B) \times \mathcal{A}(B, C) \xrightarrow{\bullet} \mathcal{A}(A, C)$$

Associative and unitary up to coherent *isomorphism*.

It can be seen as an intercategory where a general cube is



χ, δ, μ, τ are identities

True Gray category

A *true Gray category* is a category enriched in the monoidal category of 2-categories with the Gray tensor product
The Gray tensor product classifies *Gray functors of two variables*

$$H : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

- $H(A,-) : \mathcal{B} \longrightarrow \mathcal{C}$ 2-functor
- $H(-,B) : \mathcal{A} \longrightarrow \mathcal{C}$ 2-functor
- For every $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ a 2-cell

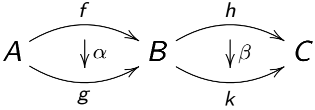
$$\begin{array}{ccc} H(A,B) & \xrightarrow{H(f,B)} & H(A',B) \\ {\scriptstyle H(A,g)}\downarrow & \overset{h(f,g)}{\Longrightarrow} & \downarrow{\scriptstyle H(A',g)} \\ H(A,B') & \xrightarrow[H(f,B')]{} & H(A',B') \end{array}$$

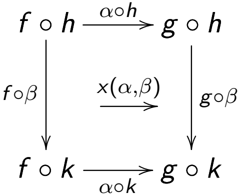
True Gray category (continued)

- Objects A, B, C
- 2-categories $\mathcal{A}(A, B)$
- Composition $\mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$
- Corresponds to a Gray functor of 2-variables

$$\circ : \mathcal{A}(A, B) \times \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$$

- Can compose arrows
- No horizontal composition of 2-cells – just whiskering

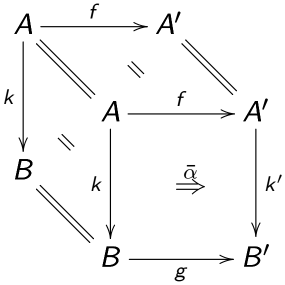




(∗)

True Gray categories as intercategories

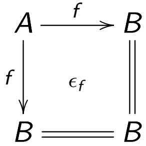
- General cube looks like (for a 3-cell $\alpha \longrightarrow \bar{\alpha}$)



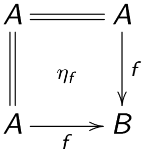
- Horizontal composition of basic cells is given by the top composite in $(*)$
- Vertical composition is given by the bottom composite
- This gives an intercategory with δ, μ, τ identities

True Gray categories as intercategories (continued)

There are connecting cells



and



$$\eta_f | \epsilon_f = \mathsf{Id}_f \quad \text{and} \quad \frac{\eta_f}{\epsilon_f} = \mathsf{id}_f$$

A cell α is a *commutativity cell* if

$$\eta | \alpha | \epsilon = \mathsf{Id}$$

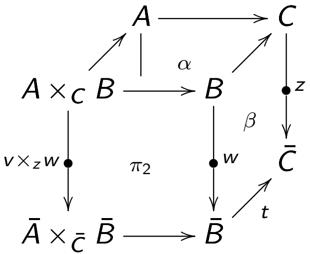
We have that

$$\chi : \frac{\alpha | \beta}{\bar{\alpha} | \bar{\beta}} \longrightarrow \frac{\alpha}{\bar{\alpha}} \Big| \frac{\beta}{\bar{\beta}}$$

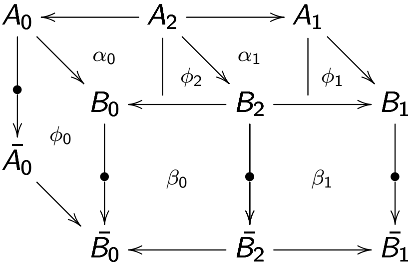
is the identity if either α or $\bar{\beta}$ is a commutativity cell

Spans in a double category

$\mathbb{A}$ a (weak) double category with a lax choice of pullbacks



We construct the intercategory Span($\mathbb{A}$). A general cube looks like



Spans in double categories (continued)

- χ and δ are not invertible in general
- μ, τ are identities

An arbitrary lax (resp. colax) functor $\mathbb{A} \longrightarrow \mathbb{B}$ induces a colax-lax (resp. colax-colax) morphism Span($\mathbb{A}$) $\longrightarrow$ Span($\mathbb{B}$)

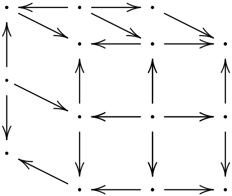
- If **A** is a category with pushouts we have a double category of cospans in **A**, $\mathbb{C}\mathrm{osp}(\mathbf{A})$
- If **A** also has pullbacks $\mathbb{C}\mathrm{osp}(\mathbf{A})$ has a colax choice of pullback. So we have Span$\mathbb{C}\mathrm{osp}(\mathbf{A})$
- We also have Cosp($\mathbb{S}\mathrm{pan}(\mathbf{A})$). Call it

$$\mathsf{C}\mathbb{S}\mathrm{pan}(\mathbf{A})$$

- Cherubini, Sabadini and Walters [4]

Also have Span($\mathbb{S}\mathrm{pan}$**A**) denoted SSpan(**A**)

- A general cube is



- Morton [7], Grandis [6]

Denote SSpan(**Set**) by Set
For arbitrary intercategory A and A in A we have the hom functor

$$\mathsf{A}(A, -) : \mathsf{A} \longrightarrow \mathsf{Set}$$

It is lax-lax

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