EMBEDDING THE AFFINE GROUP IN THE PROJECTIVE PLANE

by

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Preface

In a paper by Professor H. Schwerdtfeger [3], the one-dimensional affine group was studied by embedding it in a projective plane. Geometrical concepts were interpreted within the group and constructions were found for group-theoretical operations. Thus, geometrical properties led to relations within the group and group-theoretical proofs were found for theorems of the projective plane.

The first seven sections of this thesis present Professor Schwerdtfeger's work but from a different point of view; the main difference lies in the introduction of theorem 1 of paragraph 6.1. This theorem plays a fundamental rôle in subsequent sections. Section 8 gives a new proof of Pappus' theorem. Finally, in section 9 two classes of conics are investigated from the group-theoretical point of view. Thus, section 9 partially solves the problem of describing all conics from within the group.

§ 1. Definitions.

Let $\mathbb{R}$ be the field of real numbers. We shall define the one-dimensional affine group over $\mathbb{R}$ to be the set of functions, $\mathcal{G} = \{f:\mathbb{R} \to \mathbb{R} \mid f(x) = ax + \alpha, \ \ a, \alpha \in \mathbb{R}, \ a \neq 0\}$.

We shall represent an element $f \in \mathcal{G}$ by the ordered pair (a, α) if $f(x) = ax + \alpha$. We shall also denote elements of $\mathcal{G}$ by capital letters, e.g. A, B, C, ..., X, Y, ...

For A, B $\in \mathcal{G}$ the product AB is defined as being the composition of A and B as functions. If $A = (a, \alpha)$ and $B = (b, \beta)$ then $AB = (ab, a\beta + \alpha)$. We see immediately that this operation is not commutative.

The unit element corresponds to the identity mapping $f(x) = x$ and will be represented by $I = (1, 0)$.

The inverse of $A = (a, \alpha)$ will be $A^{-1} = (a^{-1}, -a^{-1}\alpha)$.

Since the associative law holds for functions, we see that $\mathcal{G}$ forms a group with respect to the operation defined above.

Let us identify the element $A = (a, \alpha)$ of $\mathcal{G}$ with the point (a, α) in the cartesian plane.

All real values can be taken by α, and all real values, except zero, can be taken by a; therefore all the points of the plane, except those on the y-axis, correspond to group elements. For this reason, we shall consider the y-axis as an exceptional line with respect to our group. We shall denote this line by ℓ_0.

It will be to our advantage if we turn our plane into a projective plane by adding a straight line at infinity, which will be considered as a second exceptional line with respect to our group $\mathcal{G}$. This line we shall denote by ℓ_∞.

The plane minus these two exceptional lines will be called the $\mathcal{G}$-plane. Let ℓ be a line in the projective plane, such that $\ell \neq \ell_0$ and $\ell \neq \ell_\infty$; we shall call $\ell' = \ell \cap \mathcal{G}$ the $\mathcal{G}$-line carrying ℓ. ℓ is said to be the support of ℓ'. In general it will be clear from the context whether we mean a $\mathcal{G}$-line or a line in the projective plane; in this case, we shall identify the two and refer to them by the same symbol.

If two lines have no common point in the $\mathcal{G}$-plane, there are two possibilities: either their common point lies on ℓ_0 or on ℓ_∞. In the first case we say that the lines are 0-parallel, and in the second case, that they are ∞-parallel. We shall denote this $\ell_1 \parallel_0 \ell_2$ and

$\mathcal{L}_1 \parallel_\infty \mathcal{L}_2$ respectively.

The intersection of $\mathcal{L}_0$ and $\mathcal{L}_\infty$ will be called $\mathcal{U}$.

§ 2. Elementary properties.

The following propositions are easily proved:

2.1. Any two points in the $\mathcal{G}$-plane can be joined by a unique straight line in the $\mathcal{G}$-plane.

2.2. Any two lines in the $\mathcal{G}$-plane are either 0-parallel, ∞-parallel or meet in one point of the $\mathcal{G}$-plane.

2.3. Each of the relations $\parallel_0$ and $\parallel_\infty$ is reflexive, symmetric, and transitive, i.e. they are equivalence relations.

2.4. The lines $\mathcal{L}_0$ and $\mathcal{L}_\infty$ divide the plane into two regions.

2.5. To any line $\mathcal{L}$ not through $\mathcal{U}$ there is exactly one 0-parallel and exactly one ∞-parallel through every given point $A \in \mathcal{G}$; if the point A is on $\mathcal{L}$ these two parallels coincide, otherwise they are different.

2.6. To any line through $\mathfrak{U}$, there is one and only one parallel line through each point of the $\mathfrak{G}$-plane, i.e. both parallelisms coincide.

2.7. To any two lines $\mathfrak{L}_1$ and $\mathfrak{L}_2$ not passing through $\mathfrak{U}$, there exists a unique line $\mathfrak{L}$ such that $\mathfrak{L} \parallel_\infty \mathfrak{L}_1$ and $\mathfrak{L} \parallel_0 \mathfrak{L}_2$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ coincide, then $\mathfrak{L} = \mathfrak{L}_1 = \mathfrak{L}_2$.

§ 3. Algebraic definition of straight lines and parallelisms.

3.1. Consider the normalizer, $\mathfrak{N}_A$, of an arbitrary element $A \in \mathfrak{G}$ where $A \neq I$. $\mathfrak{N}_A = \{X \mid XA = AX\}$ is a subgroup of $\mathfrak{G}$. Let $A = (a, \alpha)$ and $X = (x, y)$, then $XA = (xa, x\alpha + y)$ and $AX = (ax, ay + \alpha)$; therefore $XA = AX$ if and only if the following relation holds:

$$(a - 1)y = \alpha x - \alpha \qquad (1).$$

But $A \neq I \Rightarrow a - 1$ and α cannot vanish simultaneously and therefore (1) is a linear equation and represents a straight line. Obviously $I, A \in \mathfrak{N}_A$; therefore $\mathfrak{N}_A$ is the straight line through I and A.

Now let $\mathfrak{L}$ be any line through I and let A be any point on $\mathfrak{L}$, such that $A \neq I$. $\mathfrak{N}_A$ is the line through I and A, therefore $\mathfrak{L} = \mathfrak{N}_A$. Hence, all lines through I are of the form $\mathfrak{N}_A$.

3.2. Consider $B\mathfrak{N}_A = \{BY \mid Y \in \mathfrak{N}_A\}$, a coset of $\mathfrak{N}_A$ in $\mathcal{G}$. $B\mathfrak{N}_A = \{X \mid B^{-1}X \in \mathfrak{N}_A\}$ and $B^{-1}X = (b^{-1}x,\ b^{-1}y - b^{-1}\beta)$ and if $B^{-1}X \in \mathfrak{N}_A$, then $(a - 1)(b^{-1}y - b^{-1}\beta) = \alpha(b^{-1}x) - \alpha$, i.e.

$$(a - 1)y = \alpha x + (a - 1)\beta - b\alpha \qquad (2).$$

Since $A \neq I$, (2) contains x or y or both linearly and therefore represents a straight line. Thus $B\mathfrak{N}_A$ is the straight line through B and BA. Similarly we show that $\mathfrak{N}_A B$ is the straight line through B and AB.

3.3. Notice that if $B \notin \mathfrak{N}_A$, then $\mathfrak{N}_A \cap B\mathfrak{N}_A = \emptyset$; thus $\mathfrak{N}_A$ and $B\mathfrak{N}_A$ are either 0-parallel or ∞-parallel. If $a = 1$, (1) and (2) represent two vertical lines, therefore two ∞-parallel lines. Now comparing (1) and (2) we see that for $a \neq 1$, the slopes of $\mathfrak{N}_A$ and $B\mathfrak{N}_A$ are both $\alpha/(a - 1)$, and therefore $\mathfrak{N}_A \parallel_\infty B\mathfrak{N}_A$. We also find that $\mathfrak{N}_A \parallel_0 \mathfrak{N}_A B$.

3.4. $B\mathfrak{N}_A$ is ∞-parallel to $\mathfrak{N}_A$ and passes through B. By §§ 2.5 and 2.6, there is only one such line; therefore if $\mathcal{L} \parallel_\infty \mathfrak{N}_A$ and if $B \in \mathcal{L}$, then $\mathcal{L} = B\mathfrak{N}_A$.

Let $\mathcal{L}$ be any line in the $\mathcal{G}$-plane. By §§ 2.5 and 2.6, there is a line which is ∞-parallel to $\mathcal{L}$ and passes through I. This line must be of the form $\mathfrak{N}_A$, for some $A \neq I$. Suppose $B \in \mathcal{L}$, then $\mathcal{L} = B\mathfrak{N}_A$. From this, we see that all lines of $\mathcal{G}$ are of the form $B\mathfrak{N}_A$ with $A \neq I$.

$B\mathfrak{N}_A \parallel_\infty \mathfrak{N}_A$ and $\mathfrak{N}_A \parallel_\infty B'\mathfrak{N}_A$ for any $B, B' \in \mathcal{G}$, therefore

$B\mathfrak{N}_A \parallel_\infty B'\mathfrak{N}_A$. Conversely if $\mathfrak{L} \parallel_\infty \mathfrak{L}'$ and $\mathfrak{L} = B\mathfrak{N}_A$ then $\mathfrak{L} \parallel_\infty \mathfrak{N}_A$ and so $\mathfrak{L}' \parallel_\infty \mathfrak{N}_A$. If $B' \in \mathfrak{L}'$ then $\mathfrak{L}' = B'\mathfrak{N}_A$. Two ∞-parallel lines, $\mathfrak{L}$ and $\mathfrak{L}'$, can therefore be written in the form $B\mathfrak{N}_A$ and $B'\mathfrak{N}_A$, where $B \in \mathfrak{L}$, $B' \in \mathfrak{L}'$ for some $A \neq I$.

Similarly we show $\mathfrak{N}_A B \parallel_0 \mathfrak{N}_A$, and if $\mathfrak{L} \parallel_0 \mathfrak{N}_A$ and if $B \in \mathfrak{L}$ then $\mathfrak{L} = \mathfrak{N}_A B$. All lines can also be written in the form $\mathfrak{N}_A B$. $\mathfrak{N}_A B \parallel_0 \mathfrak{N}_A B'$ and if $\mathfrak{L} \parallel_0 \mathfrak{L}'$ and $\mathfrak{L} = \mathfrak{N}_A B$ then $\mathfrak{L}' = \mathfrak{N}_A B'$ for some $B' \in \mathfrak{L}'$.

3.5. Define $\mathfrak{h} = \mathfrak{N}_{(1,\ 1)}$. $\mathfrak{h}$ is a straight line through $I = (1, 0)$ and $(1, 1)$, and therefore is a vertical line. We see that $\mathfrak{h}$ cuts $\mathfrak{L}_0$ on $\mathfrak{L}_\infty$ and therefore $\mathfrak{h}$ passes through $\mathfrak{U}$. $\mathfrak{h} = \{(1, \alpha) \mid \alpha \in \mathbb{R}\}$.

3.6. By § 2.6 the lines through A which are 0-parallel and those which are ∞-parallel to $\mathfrak{h}$ coincide. Therefore $A\mathfrak{h} = \mathfrak{h}A$ which shows us that $\mathfrak{h}$ is a normal subgroup of $\mathfrak{G}$; $\mathfrak{h} \lhd \mathfrak{G}$.

3.7. Let $A = (a, \alpha)$ and $B = (b, \beta)$ then we see that

$$BAB^{-1} = (a,\ (1 - a)\beta + b\alpha) \qquad (3).$$

Note that A and BAB^{-1} belong to the same vertical line, i.e. BAB^{-1} lies on the line which is parallel to $\mathfrak{h}$ and which passes through A. $BAB^{-1} \in A\mathfrak{h} = \mathfrak{h}A$.

3.8. Conversely let $A \notin \mathfrak{h}$ and let $C \in A\mathfrak{h}$. $A = (a, \alpha)$ and $C = (a, \gamma)$. Since $a \neq 1$, set $\beta = (\gamma - \alpha)/(1 - a)$. Setting $B = (1, \beta)$ we obtain $BAB^{-1} = (a, (1 - a)\beta + \alpha)$ thus $BAB^{-1} = (a, \gamma) = C$. Therefore if A and C are two elements of a proper coset of $\mathfrak{h}$, there exists a unique $B \in \mathfrak{h}$ such that $BAB^{-1} = C$. $A\mathfrak{h}$ is the conjugate class of A. Note also that for any $A, B \in \mathcal{G}$, $A \notin \mathfrak{h}$ there is a unique $H \in \mathfrak{h}$ such that $BAB^{-1} = HAH^{-1}$.

3.9. Let $A, C \in \mathfrak{h}$, $A = (1, \alpha)$ and $C = (1, \gamma)$ then $BAB^{-1} = (1, b\alpha)$ where $B = (b, \beta)$. If $\alpha \neq 0$ and $\gamma \neq 0$ choose $b = \gamma/\alpha$ and then $BAB^{-1} = (1, \gamma) = C$. Therefore if $A \neq I$ and $C \neq I$ there is a B such that $BAB^{-1} = C$. Thus $\mathfrak{h} - \{1\}$ is a conjugate class, $\{1\}$ being a conjugate class of its own.

Note that if $B \in \mathfrak{h}$ then $BAB^{-1} = (1, \alpha) = A$ and therefore $\mathfrak{h}$ is commutative. In fact it is easily seen that $\mathfrak{h}$ is isomorphic to the additive group of the real numbers, since $(1, \alpha)(1, \beta) = (1, \alpha + \beta)$.

§ 4. Consequences of section 3.

4.1. If $T \in \mathfrak{N}_A$ and $T \neq I$, then $\mathfrak{N}_T = \mathfrak{N}_A$. This is obvious since $\mathfrak{N}_A$ and $\mathfrak{N}_T$ are both straight lines and

$I, T \in \mathfrak{N}_A \cap \mathfrak{N}_T$; therefore $\mathfrak{N}_A = \mathfrak{N}_T$.

4.2. We shall now prove a group-theoretical relation which we shall often use in the sequel. Let $A, B \in \mathfrak{G}$, $A \neq I$. $B\mathfrak{N}_A$ is a straight line passing through B and BA. Now $\mathfrak{N}_{BAB^{-1}}B$ is also a straight line passing through B and $(BAB^{-1})B = BA$. We conclude that $B\mathfrak{N}_A = \mathfrak{N}_{BAB^{-1}}B$ or $B\mathfrak{N}_A B^{-1} = \mathfrak{N}_{BAB^{-1}}$.

4.3. For $A, B \in \mathfrak{G}$, $A \notin \mathfrak{h}$, $\mathfrak{N}_A \neq \mathfrak{h}$ and therefore $B\mathfrak{N}_A$ is not parallel to $\mathfrak{h}$. Let $H \in \mathfrak{h} \cap B\mathfrak{N}_A$. $H \in B\mathfrak{N}_A \Rightarrow B\mathfrak{N}_A = H\mathfrak{N}_A$. There is obviously only one such H. Similarly there exists a unique $H' \in \mathfrak{h}$ such that $\mathfrak{N}_A B = \mathfrak{N}_A H'$.

4.4. All lines through I, except $\mathfrak{h}$, are conjugate to one another. Let one of these lines be $\mathfrak{N}_A$ and another be $\mathfrak{N}'$. Let $B \in A\mathfrak{h} \cap \mathfrak{N}'$. Then by § 4.1, $\mathfrak{N}' = \mathfrak{N}_B$. But since $B \in A\mathfrak{h}$, by § 3.8 there exists $H \in \mathfrak{h}$ such that $HAH^{-1} = B$. But now $H\mathfrak{N}_A H^{-1} = \mathfrak{N}_{HAH^{-1}} = \mathfrak{N}_B$ by § 4.2, which proves our assertion.

4.5. For $B \notin \mathfrak{h}$, $K \in \mathfrak{h}$, K can be represented as a commutator $[H, B] = K$, $H \in \mathfrak{h}$. Indeed, $KB \in \mathfrak{h}B$ and therefore by § 3.8 there exists $H \in \mathfrak{h}$ such that $KB = HBH^{-1}$. Post-multiplying by B^{-1}, we obtain $K = HBH^{-1}B^{-1}$; therefore $K = [H, B]$ as was claimed.

4.6. $\mathfrak{H}$ is the commutator subgroup of $\mathfrak{G}$. Indeed, let $A, B \in \mathfrak{G}$, $[A, B] = ABA^{-1}B^{-1}$.

(i) If $B \in \mathfrak{H}$ then $ABA^{-1} \in \mathfrak{H}$ and it follows that $(ABA^{-1})B^{-1} = [A, B] \in \mathfrak{H}$.

(ii) If $B \notin \mathfrak{H}$ then by § 3.8 there is an element $H \in \mathfrak{H}$ such that $ABA^{-1} = HBH^{-1}$ and therefore $[A, B] = HBH^{-1}B^{-1}$ but $BH^{-1}B^{-1} \in \mathfrak{H}$ and thus $[A, B] \in \mathfrak{H}$.

Therefore all commutators lie in $\mathfrak{H}$, and by § 4.5 all elements of $\mathfrak{H}$ are commutators. $\mathfrak{H}$ is consequently the commutator subgroup of $\mathfrak{G}$.

4.7. Because $\mathfrak{H}$ is the commutator subgroup of $\mathfrak{G}$, the factor group $\mathfrak{G}/\mathfrak{H}$ is abelian and from this we conclude that $\mathfrak{H}AB = \mathfrak{H}BA$ and that $AB\mathfrak{H} = BA\mathfrak{H}$.

§ 5. Collineations.

Definition: A collineation is a one-to-one mapping of the plane onto itself, in which the image of every line is a line.

5.1. Consider the function $f_T : \mathfrak{G} \to \mathfrak{G}$. $f_T(X) = TX$ where T is a fixed element of $\mathfrak{G}$. f_T is one-to-one and onto. Now any straight line in $\mathfrak{G}$ can be written $A\mathfrak{M}_B$. $f_T(A\mathfrak{M}_B) = TA\mathfrak{M}_B$ which is another straight line. f_T is therefore a collineation of the $\mathfrak{G}$-plane. f_T is even an

affine transformation in the sense that ∞-parallels are transformed into ∞-parallels and O-parallels are transformed into O-parallels. Indeed, if $A\mathcal{N}_B$ and $A'\mathcal{N}_B$ are two ∞-parallel lines then $f_T(A\mathcal{N}_B) = TA\mathcal{N}_B$ and $f_T(A'\mathcal{N}_B) = TA'\mathcal{N}_B$ which are ∞-parallel, i.e. $\mathcal{L} \parallel_{\infty} \mathcal{L}' \Rightarrow f_T(\mathcal{L}) \parallel_{\infty} f_T(\mathcal{L}')$.
Also let $\mathcal{N}_B A$ and $\mathcal{N}_B A'$ be any two O-parallel lines. Then $f_T(\mathcal{N}_B A) = T\mathcal{N}_B A$ and $f_T(\mathcal{N}_B A') = T\mathcal{N}_B A'$.
But $T\mathcal{N}_B A = T\mathcal{N}_B T^{-1}TA = \mathcal{N}_{TBT^{-1}}TA$ and $T\mathcal{N}_B A' = \mathcal{N}_{TBT^{-1}}TA'$ which is O-parallel to $\mathcal{N}_{TBT^{-1}}TA$.
Therefore $\mathcal{L} \parallel_0 \mathcal{L}' \Rightarrow f_T(\mathcal{L}) \parallel_0 f_T(\mathcal{L}')$.

We may also note that for any line $\mathcal{L}$ of the $\mathcal{G}$-plane, $f_T(\mathcal{L}) \parallel_{\infty} \mathcal{L}$ and $f_T(\mathcal{L}) \parallel_0 T\mathcal{L}T^{-1}$.

5.2. Consider the function $g_T : \mathcal{G} \longrightarrow \mathcal{G}$ defined by $g_T(X) = XT$ for a fixed $T \in \mathcal{G}$. g_T is one-to-one and onto, and since every line of $\mathcal{G}$ can be written in the form $\mathcal{N}_B A$, then by the same reasoning as in § 5.1, g_T is a collineation of the $\mathcal{G}$-plane. g_T is also an affine transformation in the same sense as above. Furthermore $g_T(\mathcal{L}) \parallel_0 \mathcal{L}$ and $g_T(\mathcal{L}) \parallel_{\infty} T^{-1}\mathcal{L}T$, for any line $\mathcal{L}$ in the $\mathcal{G}$-plane.

5.3. Let $\varphi : \mathcal{G} \rightarrow \mathcal{G}$ be any automorphism of $\mathcal{G}$. φ is, of course, one-to-one and onto. $B' \in \mathcal{N}_B \Leftrightarrow BB' = B'B$ $\Leftrightarrow \varphi(BB') = \varphi(B'B) \Leftrightarrow \varphi(B)\varphi(B') = \varphi(B')\varphi(B)$ $\Leftrightarrow \varphi(B') \in \mathcal{N}_{\varphi(B)}$; therefore $\varphi(\mathcal{N}_B) = \mathcal{N}_{\varphi(B)}$. If $A\mathcal{N}_B$ is any straight line then $\varphi(A\mathfrak{N}_B) = \varphi(A)\mathfrak{N}_{\varphi(B)}$ which is another straight line. φ is then a collineation. φ is also an affine transformation. Indeed, let $A\mathfrak{N}_B$ and $A'\mathfrak{N}_B$ be two ∞-parallel lines. Then $\varphi(A\mathfrak{N}_B) = \varphi(A)\mathfrak{N}_{\varphi(B)}$ and $\varphi(A'\mathfrak{N}_B) = \varphi(A')\mathfrak{N}_{\varphi(B)}$, therefore $\varphi(A\mathfrak{N}_B) \parallel_\infty \varphi(A'\mathfrak{N}_B)$. Similarly if $\mathfrak{N}_B A$ and $\mathfrak{N}_B A'$ are two 0-parallel lines then $\varphi(\mathfrak{N}_B A) = \mathfrak{N}_{\varphi(B)}\varphi(A)$ and $\varphi(\mathfrak{N}_B A') = \mathfrak{N}_{\varphi(B)}\varphi(A')$, therefore $\mathfrak{L} \parallel_0 \mathfrak{L}' \Rightarrow \varphi(\mathfrak{L}) \parallel_0 \varphi(\mathfrak{L}')$.

In particular the mapping $\varphi(X) = AXA^{-1}$ (conjugation by A) is a collineation.

5.4. Consider the mapping $h: \mathfrak{G} \to \mathfrak{G}$ defined by $h(X) = X^{-1}$. If $A\mathfrak{N}_B$ is an arbitrary line of the $\mathfrak{G}$-plane, $h(A\mathfrak{N}_B) = h(\mathfrak{N}_B)h(A) = \mathfrak{N}_B A^{-1}$ which is a straight line. h is one-to-one and onto, therefore it is a collineation of $\mathfrak{G}$.

h is not an affine transformation, but it maps ∞-parallel lines into 0-parallel lines and conversely; we shall say that h is anti-affine. Indeed, if $A\mathfrak{N}_B$ and $A'\mathfrak{N}_B$ are any two ∞-parallel lines then $h(A\mathfrak{N}_B) = \mathfrak{N}_B A^{-1}$ and $h(A'\mathfrak{N}_B) = \mathfrak{N}_B A'^{-1}$ and therefore $h(A'\mathfrak{N}_B) \parallel_0 h(A\mathfrak{N}_B)$. Similarly if $\mathfrak{L} \parallel_0 \mathfrak{L}'$ then $h(\mathfrak{L}) \parallel_\infty h(\mathfrak{L}')$.

5.5. Extension of f_T, g_T, φ, and h to collineations of the projective plane π.

Let P be any point of the projective plane. We shall denote the set of all $\mathcal{G}$-lines whose support passes through P by [P]; this is the pencil of lines with centre P. Since f_T, g_T, and φ are affine transformations and h is anti-affine, to any pencil of lines having its centre, $\mathcal{M}$, on $\mathcal{L}_0$ or $\mathcal{L}_\infty$ there corresponds a pencil with centre, $\mathcal{N}$, on $\mathcal{L}_0$ or $\mathcal{L}_\infty$. Hence we define $\psi(\mathcal{M}) = \mathcal{N}$ where ψ is any of the four types of collineations. In this way we have extended ψ to a collineation of the projective plane π. f_T, g_T, and φ, all map $\mathcal{L}_0$ onto $\mathcal{L}_0$ and $\mathcal{L}_\infty$ onto $\mathcal{L}_\infty$, whereas h maps $\mathcal{L}_0$ onto $\mathcal{L}_\infty$ and $\mathcal{L}_\infty$ onto $\mathcal{L}_0$. f_T fixes $\mathcal{L}_\infty$ pointwise and g_T fixes $\mathcal{L}_0$ pointwise. If $\mathfrak{N}$ is any normalizer meeting $\mathcal{L}_0$ and $\mathcal{L}_\infty$ in $\mathcal{N}$ and $\mathcal{N}'$ then $h(\mathcal{N}) = \mathcal{N}'$ and $h(\mathcal{N}') = \mathcal{N}$.

Obviously all finite combinations of f_T, g_T, φ and h, as functions, are collineations, but in all cases $\mathfrak{U}$ is a fixed point.

5.6. Application.

φ, being an automorphism, fixes I. $\mathfrak{z}$ is the straight line through $\mathfrak{U}$ and I, two fixed points, therefore $\varphi(\mathfrak{z}) = \mathfrak{z}$ for all automorphisms of $\mathcal{G}$, i.e. $\mathfrak{z}$ is a characteristic subgroup of $\mathcal{G}$.

5.7. We shall now introduce a special kind of collineation which we shall use in section 9. (cf. [1] § 2.3.)

Definition 1: If a collineation fixes every line through a given point C, we say that the collineation is central. The point C is called the centre of the collineation.

The following properties of central collineations are easily proved:

1. A centre of a collineation is an invariant point of this collineation.

2. A collineation with two different centres is the identity.

3. If in a central collineation a line not through the centre is invariant, then every point of this line is invariant.

4. A collineation with two different lines of invariant points is the identity.

We now state an important theorem concerning central collineations.

Theorem 1: Every central collineation which is not the identity has one and only one line of invariant points.

Definition 2: The line of invariant points of a central collineation is called its axis.

Note that the axis of a central collineation may or may not pass through the centre.

We shall now state a fundamental theorem on the existence of central collineations.

Theorem 2: Given three collinear points O, P, P' such that $O \neq P$ and $O \neq P'$ and a line $\mathfrak{L}$ not passing through P or P', there exists a unique central collineation with centre O, axis $\mathfrak{L}$, and transforming P into P'.

Examples:

(i) f_T is a central collineation with axis $\mathfrak{L}_\infty$ whose centre is the intersection of $\mathfrak{L}_0$ with $\mathfrak{N}_T$. Indeed, all lines through the intersection of $\mathfrak{L}_0$ with $\mathfrak{N}_T$ are O-parallel to $\mathfrak{N}_T$ and are therefore of the form $\mathfrak{N}_T A$. $f_T(\mathfrak{N}_T A) = T\mathfrak{N}_T A = \mathfrak{N}_T A$ which proves our assertion.

(ii) Similarly g_T is a central collineation whose centre is the intersection of $\mathfrak{L}_\infty$ with $\mathfrak{N}_T$. The axis is $\mathfrak{L}_0$.

(iii) As we saw in §§ 3.8 and 3.9 the conjugate classes in $\mathfrak{G}$ are the proper cosets of $\mathfrak{H}$, $\mathfrak{H} - \{I\}$ and $\{I\}$. Since, by conjugation, these classes are transformed into themselves, the collineation $\varphi_A : X \to AXA^{-1}$ leaves all lines through $\mathfrak{U}$ fixed. It is therefore a central collineation with centre $\mathfrak{U}$. The axis of this collineation is the line $\mathfrak{N}_A$. By properties 3 and 4 of central collineations, if φ_A leaves a line, not through $\mathfrak{U}$ and different from $\mathfrak{N}_A$, fixed, then φ_A is the identity mapping. If so, A is in the centrum of $\mathfrak{G}$ but the centrum of $\mathfrak{G}$ is $\{I\}$, therefore A = I.

(iv) Consider the collineation $h: X \to X^{-1}$. If $\mathfrak{N}_A$ is an arbitrary line through I, then $h(\mathfrak{N}_A) = \mathfrak{N}_A$, therefore h is a central collineation whose centre is I. Now from theorem 1 we know that there exists one and only one line of invariant points: the axis. $\mathfrak{U}$ is invariant and therefore the axis passes through $\mathfrak{U}$. If V is a point on the axis, then $V = V^{-1}$ or $V^2 = I$. The axis is therefore the set of all involutory elements of $\mathfrak{G}$. From what we have seen, this line is of the form $\mathfrak{h}V$.

We shall return to this line in section 7.10.

§ 6. Group-theoretical interpretation of geometrical constructions.

6.1. Let A, B $\in \mathfrak{G}$ with A $\neq$ B, then $A\mathfrak{N}_{A^{-1}B}$ is the straight line passing through A and B. $B\mathfrak{N}_{B^{-1}A}$, $\mathfrak{N}_{AB^{-1}}B$ and $\mathfrak{N}_{BA^{-1}}A$ also represent the same line and are therefore equal to $A\mathfrak{N}_{A^{-1}B}$.

Note: In the sequel, we shall denote the $\mathcal{G}$-line whose support passes through P and Q by $\overline{PQ}$, where P and Q are two different points of the projective plane.

Proposition 1: Let $\mathcal{A}_1, \mathcal{A}_3 \in \mathcal{L}_0$ and $\mathcal{A}_2, \mathcal{A}_4 \in \mathcal{L}_\infty$ where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \neq \mathcal{U}$. Let A_1 and A_2 be any two points of the $\mathcal{G}$-plane. Suppose that $A_1 \in \overline{\mathcal{A}_1\mathcal{A}_2}$, $I \in \overline{\mathcal{A}_2\mathcal{A}_3}$ and $A_2 \in \overline{\mathcal{A}_3\mathcal{A}_4}$, then the product $A_1A_2 \in \overline{\mathcal{A}_4\mathcal{A}_1}$.

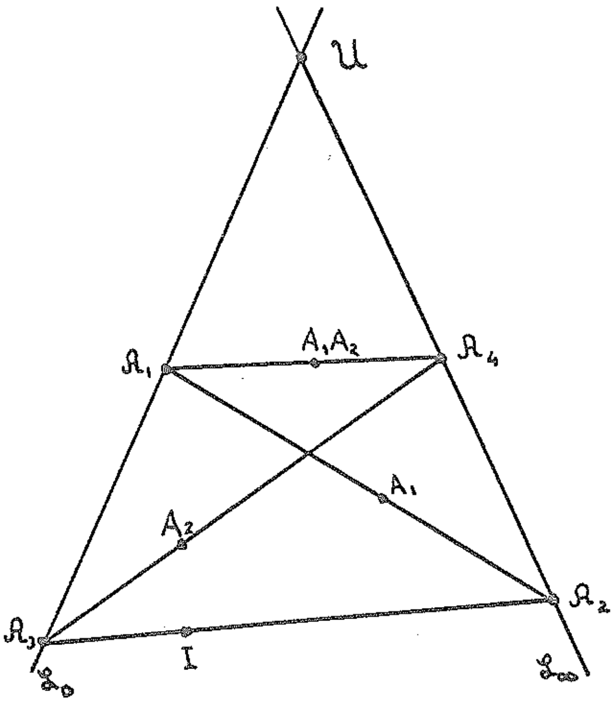

Figure 1.

Proof: Clearly the line $\overline{\mathcal{A}_2\mathcal{A}_3} = \mathcal{N}$ is a normalizer; $\overline{\mathcal{A}_1\mathcal{A}_2}$ is ∞-parallel to $\mathcal{N}$ therefore $\overline{\mathcal{A}_1\mathcal{A}_2} = A_1\mathcal{N}$; $\overline{\mathcal{A}_3\mathcal{A}_4}$ is 0-parallel to $\mathcal{N}$ therefore $\overline{\mathcal{A}_3\mathcal{A}_4} = \mathcal{N}A_2$. Consider $A_1\mathcal{N}A_2$. $A_1\mathcal{N}A_2 = f_{A_1}(\mathcal{N}A_2)$ and therefore by § 5.1 $A_1\mathcal{N}A_2 \parallel_\infty \mathcal{N}A_2$. Also $A_1\mathcal{N}A_2 = g_{A_2}(A_1\mathcal{N})$ and by § 5.2

$A_1\mathfrak{N}A_2 \parallel_0 A_1\mathfrak{N}$. But $\overline{\mathfrak{A}_1\mathfrak{A}_4} \parallel_\infty \mathfrak{N}A_2$ and $\overline{\mathfrak{A}_1\mathfrak{A}_4} \parallel_0 A_1\mathfrak{N}$ and by § 2.7 there is only one such line, therefore $\overline{\mathfrak{A}_1\mathfrak{A}_4} = A_1\mathfrak{N}A_2$; hence we conclude that $A_1A_2 \in \overline{\mathfrak{A}_1\mathfrak{A}_4}$.

Q.E.D.

Let us apply the collineation f_{B_2} to the plane π, where B_2 is an arbitrary element of $\mathcal{G}$. f_{B_2} is affine, therefore $\mathfrak{A}_1$ and $\mathfrak{A}_3$ are transformed into points of the line $\mathcal{L}_0$; $\mathfrak{A}_2$ and $\mathfrak{A}_4$ are transformed into points of $\mathcal{L}_\infty$. $\mathfrak{U}$ is a fixed point of f_{B_2}, therefore the images of $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ and $\mathfrak{A}_4$ will be different from $\mathfrak{U}$. A_1 is mapped into B_2A_1, I into B_2, A_2 into B_2A_2 and A_1A_2 into $B_2A_1A_2$. Set $B_2A_1 = B_1$ and $B_2A_2 = B_3$; then $B_2A_1A_2 = B_1B_2^{-1}B_3$.

We are now in a position to state the fundamental theorem of this exposition.

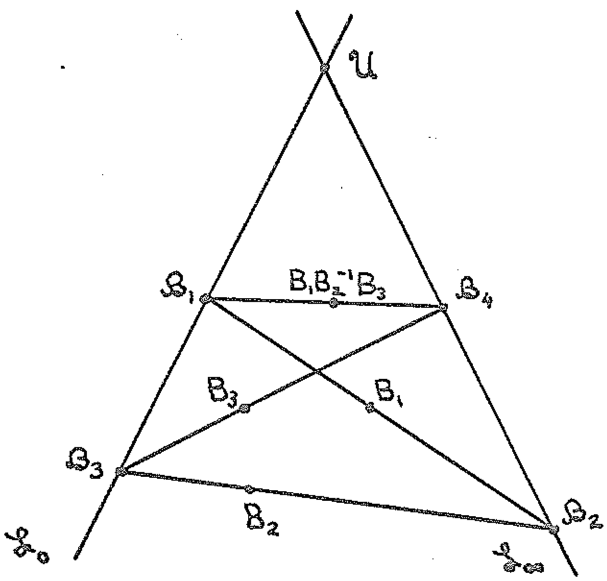

Figure 2.

Theorem 1: Let $\mathcal{B}_1, \mathcal{B}_3 \in \mathcal{L}_0$ and $\mathcal{B}_2, \mathcal{B}_4 \in \mathcal{L}_\infty$ where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \neq \mathcal{U}$. Let $B_1, B_2, B_3 \in \mathcal{G}$. Suppose that $B_1 \in \overline{\mathcal{B}_1\mathcal{B}_2}$, $B_2 \in \overline{\mathcal{B}_2\mathcal{B}_3}$ and $B_3 \in \overline{\mathcal{B}_3\mathcal{B}_4}$; then the product $B_1B_2^{-1}B_3 \in \overline{\mathcal{B}_4\mathcal{B}_1}$. (See figure 2.)

We can generalize the result of theorem 1 to a general 2n-gon inscribed in two lines ($n \geqslant 2$).

Theorem 2: Let $\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5, \ldots, \mathcal{A}_{2n-1} \in \mathcal{L}_0$ and $\mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_6, \ldots, \mathcal{A}_{2n} \in \mathcal{L}_\infty$ where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \ldots, \mathcal{A}_{2n} \neq \mathcal{U}$. Also let $A_1, A_2, \ldots, A_{2n-1} \in \mathcal{G}$. Suppose that $A_1 \in \overline{\mathcal{A}_1\mathcal{A}_2}$, $A_2 \in \overline{\mathcal{A}_2\mathcal{A}_3}$, $A_3 \in \overline{\mathcal{A}_3\mathcal{A}_4}$, $A_4 \in \overline{\mathcal{A}_4\mathcal{A}_5}$, $\ldots$, $A_{2n-1} \in \overline{\mathcal{A}_{2n-1}\mathcal{A}_{2n}}$; then the product $A_1A_2^{-1}A_3A_4^{-1}\ldots A_{2n-2}^{-1}A_{2n-1} \in \overline{\mathcal{A}_{2n}\mathcal{A}_1}$.

Proof: By theorem 1, this is true for $n = 2$. Suppose that it is true for $n = k$. Let $\mathcal{A}_1, \mathcal{A}_3, \ldots, \mathcal{A}_{2k-1}, \mathcal{A}_{2k+1} \in \mathcal{L}_0$ and $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2k+2} \in \mathcal{L}_\infty$ with $A_1 \in \overline{\mathcal{A}_1\mathcal{A}_2}$, $A_2 \in \overline{\mathcal{A}_2\mathcal{A}_3}$, $\ldots$, $A_{2k+1} \in \overline{\mathcal{A}_{2k+1}\mathcal{A}_{2k+2}}$. By the induction hypothesis, $A_1A_2^{-1}\ldots A_{2k-1} \in \overline{\mathcal{A}_{2k}\mathcal{A}_1}$. Applying theorem 1 to the quadrilateral with vertices $\mathcal{A}_1, \mathcal{A}_{2k}, \mathcal{A}_{2k+1}$ and $\mathcal{A}_{2k+2}$, we see that

$$(A_1A_2^{-1}\ldots A_{2k-1})A_{2k}^{-1}A_{2k+1} = A_1A_2^{-1}\ldots A_{2k+1} \in \overline{\mathcal{A}_{2k+2}\mathcal{A}_1}$$

and by induction we see that our theorem is true for all $n \geqslant 2$. For $n = 1$, we have a degenerate case which is trivial.

Q.E.D.

If one of the first 2n - 1 sides is known in the 2n-gon of the preceding theorem, then the last side is uniquely determined as is expressed in the following theorem.

Theorem 3: Using the notation of theorem 2, suppose that $\overline{\mathcal{R}_i \mathcal{R}_{i+1}} = \mathcal{L}_i$, $\mathcal{L}_i$ not passing through $\mathcal{U}$, then
$\overline{\mathcal{R}_{2n} \mathcal{R}_1} = A_1 A_2^{-1} \dots \mathcal{L}_i \dots A_{2n-1}$ if i is odd and
$\overline{\mathcal{R}_{2n} \mathcal{R}_1} = A_1 A_2^{-1} \dots \mathcal{L}_i^{-1} \dots A_{2n-1}$ if i is even.

Proof: We shall prove our assertion for i odd, the case where i is even being similar.
Let $A_i, A_i' \in \mathcal{L}_i$, $A_i \neq A_i'$. By theorem 2, the following relations hold:

$$A_1 A_2^{-1} \dots A_i \dots A_{2n-1} \in \overline{\mathcal{R}_{2n} \mathcal{R}_1},$$

$$A_1 A_2^{-1} \dots A_i' \dots A_{2n-1} \in \overline{\mathcal{R}_{2n} \mathcal{R}_1}.$$

The only line through these two points is obviously

$$A_1 A_2^{-1} \dots \mathcal{L}_i \dots A_{2n-1}.$$

Therefore $\qquad \overline{\mathcal{R}_{2n} \mathcal{R}_1} = A_1 A_2^{-1} \dots \mathcal{L}_i \dots A_{2n-1}.$

Q.E.D.

§ 7. The geometrical construction of operations in the group.

7.1. Given A and B, construct AB. Choose $\mathfrak{N}$ to be any normalizer $\neq \mathfrak{h}$. Let $\mathfrak{N}$ cut $\mathcal{L}_0$ and $\mathcal{L}_\infty$ in $\mathcal{R}_3$ and $\mathcal{R}_2$

respectively. Construct the line through $\mathcal{A}_2$ and A meeting $\mathcal{L}_0$ in $\mathcal{A}_1$ and construct the line through $\mathcal{A}_3$ and B meeting $\mathcal{L}_\infty$ in $\mathcal{A}_4$. Then by proposition 6.2.1, AB $\in \overline{\mathcal{A}_1\mathcal{A}_4}$. Choose another normalizer $\mathcal{N}'$, $\mathcal{N}' \neq \mathcal{N}$ and $\mathcal{N}' \neq \mathcal{I}$. Let $\mathcal{N}'$ meet $\mathcal{L}_\infty$ in $\mathcal{A}_2'$. $\mathcal{N}' \neq \mathcal{N} \Rightarrow \mathcal{A}_2' \neq \mathcal{A}_2$. Let the line through $\mathcal{A}_2'$ and A meet $\mathcal{L}_0$ in $\mathcal{A}_1'$. Since $\mathcal{A}_2' \neq \mathcal{A}_2$, then $\mathcal{A}_1' \neq \mathcal{A}_1$; therefore $\overline{\mathcal{A}_1'\mathcal{A}_4'} \neq \overline{\mathcal{A}_1\mathcal{A}_4}$.
AB $= \overline{\mathcal{A}_1'\mathcal{A}_4'} \cap \overline{\mathcal{A}_1\mathcal{A}_4}$ determines AB uniquely. This construction is illustrated in figure 3.

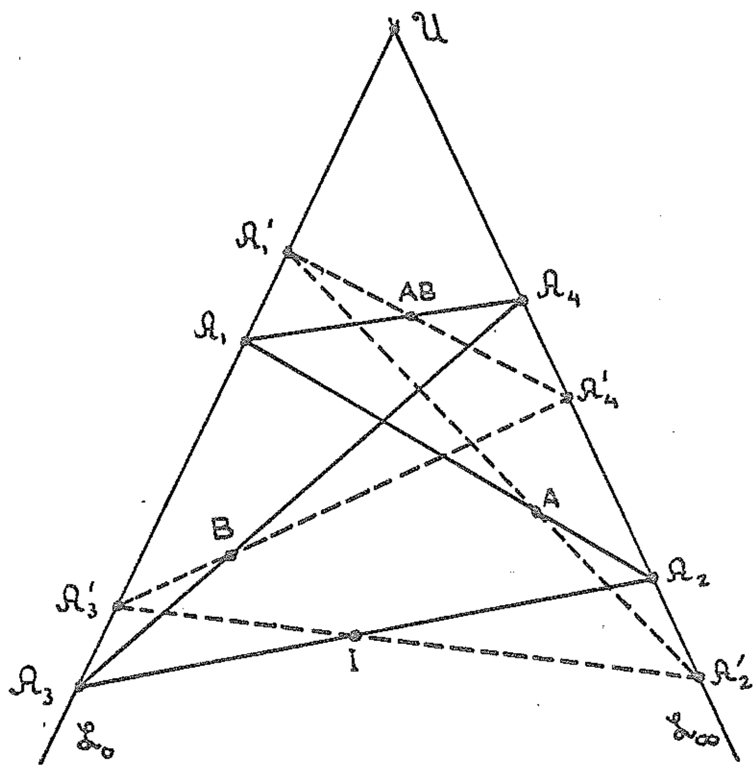

Figure 3.

Note that we could have arrived at the same results by observing that by theorem 6.2.3, $\overline{\mathcal{A}_1\mathcal{A}_4} = \text{A}\mathcal{N}\text{B}$ and $\overline{\mathcal{A}_1'\mathcal{A}_4'} = \text{A}\mathcal{N}'\text{B}$. AB $= \text{A}\mathcal{N}\text{B} \cap \text{A}\mathcal{N}'\text{B}$.

In special cases we can simplify our construction for the product of A and B, by appropriate choices of $\mathfrak{N}$ and $\mathfrak{N}'$.

(i) If A, B $\notin \mathfrak{g}$ and A $\notin \mathfrak{N}_B$, choose $\mathfrak{N} = \mathfrak{N}_A$ and $\mathfrak{N}' = \mathfrak{N}_B$. Since A, B $\notin \mathfrak{g}$, it follows that $\mathfrak{N}, \mathfrak{N}' \neq \mathfrak{g}$ and since A $\notin \mathfrak{N}_B$, it follows that $\mathfrak{N} \neq \mathfrak{N}'$. $A\mathfrak{N}B = A\mathfrak{N}_A B = \mathfrak{N}_A B$. $A\mathfrak{N}'B = A\mathfrak{N}_B B = A\mathfrak{N}_B$. Therefore from the preceding considerations $AB = \mathfrak{N}_A B \cap A\mathfrak{N}_B$ determines AB uniquely. See figure 4.

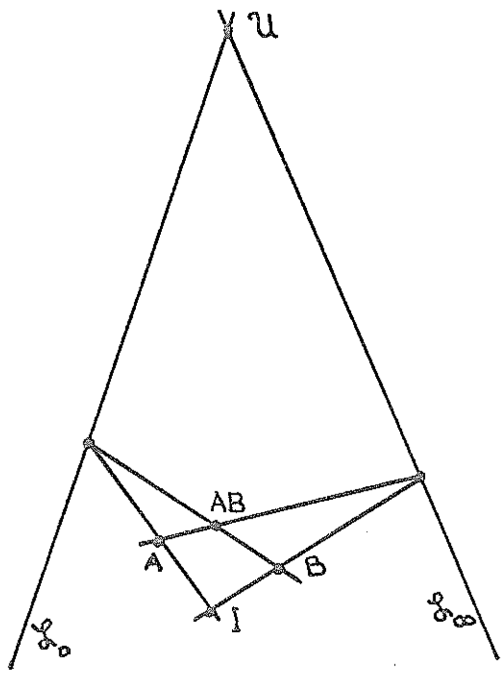

Figure 4.

(ii) If A, B $\notin \mathfrak{g}$ and A $\in \mathfrak{N}_B$, then we choose $\mathfrak{N}' = \mathfrak{N}_A$ and take $\mathfrak{N}$ to be different from $\mathfrak{N}_A$ and $\mathfrak{g}$. $\overline{\mathfrak{R}_1'\mathfrak{R}_4'} = A\mathfrak{N}'B = \mathfrak{N}_A$. $AB = \mathfrak{N}_A \cap A\mathfrak{N}B$ determines AB uniquely.

(iii) If $A \in \mathfrak{g}$ but $B \notin \mathfrak{g}$, choose $\mathfrak{N}' = \mathfrak{N}_B$; then $\overline{\mathfrak{R}_1'\mathfrak{R}_4'} = A\mathfrak{N}'B = A\mathfrak{N}_B$. $AB = A\mathfrak{N}_B \cap A\mathfrak{N}B$ determines AB uniquely as long as $\mathfrak{N} \neq \mathfrak{N}_B$ and $\mathfrak{N} \neq \mathfrak{g}$. Note that if we take $\mathfrak{N} = \mathfrak{g}$, then $AB \in A\mathfrak{N}B$ but the construction described above for $A\mathfrak{N}B$ is invalid. In this case, $A\mathfrak{N}B = \mathfrak{g}B$ which is the straight line through $\mathfrak{U}$ and B. Therefore $AB = A\mathfrak{N}_B \cap \mathfrak{g}B$ determines AB uniquely and is easily constructed.

(iv) If A and $B \in \mathfrak{g}$, then $AB \in \mathfrak{g}$ also. Choose an arbitrary $\mathfrak{N} \neq \mathfrak{g}$; then $AB = \mathfrak{g} \cap A\mathfrak{N}B$.

7.2. Given any line $\mathfrak{M} \neq A\mathfrak{g}$, construct $h(\mathfrak{M}) = \mathfrak{M}^{-1}$.

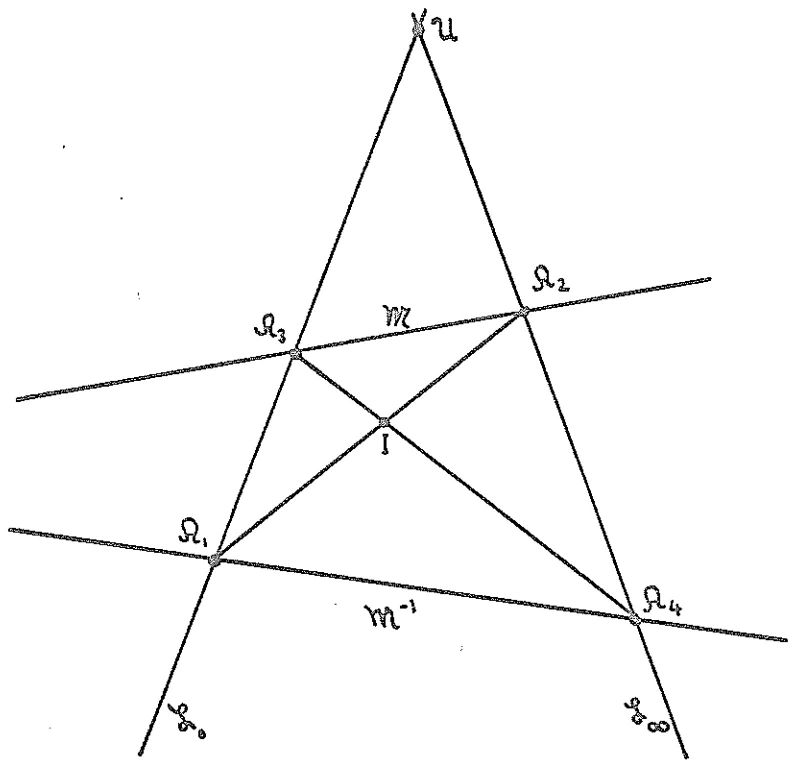

Figure 5.

Let $\mathfrak{M}$ cut $\mathcal{L}_0$ and $\mathcal{L}_\infty$ in $\mathfrak{R}_3$ and $\mathfrak{R}_2$ respectively. Construct the line through $\mathfrak{R}_2$ and I meeting $\mathcal{L}_0$ in $\mathfrak{R}_1$. Also construct the line through $\mathfrak{R}_3$ and I meeting $\mathcal{L}_\infty$ in $\mathfrak{R}_4$. By theorem 6.2.3, $\overline{\mathfrak{R}_1\mathfrak{R}_4} = I\mathfrak{M}^{-1}I = \mathfrak{M}^{-1}$. (This construction is illustrated in figure 5.)

Note that if $\mathfrak{M}$ is a normalizer, all four lines coincide and $\mathfrak{M}^{-1} = \mathfrak{M}$ as it should be.

7.3. Given $A \in \mathcal{G}$, construct A^{-1}.

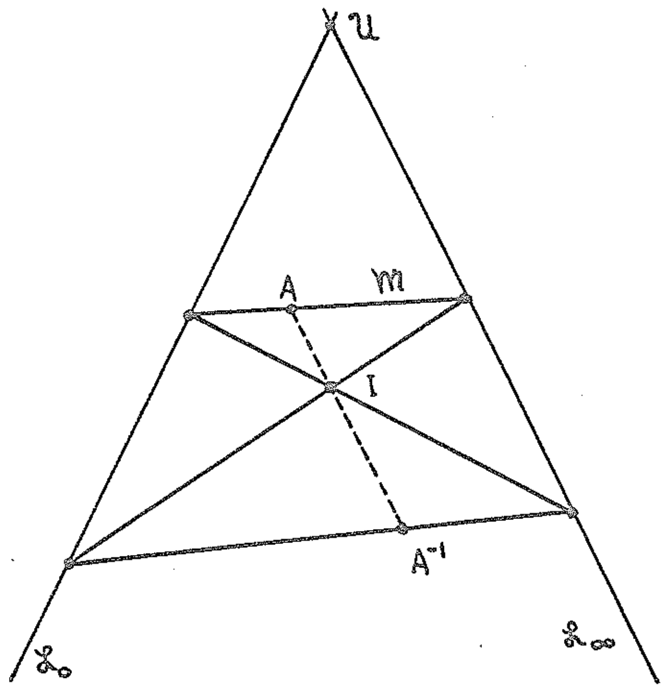

Figure 6.

If $\mathfrak{M}$ and $\mathfrak{M}'$ are any two different lines, then $\mathfrak{M}^{-1} \neq \mathfrak{M}'^{-1}$. Choose $\mathfrak{M} \neq \mathfrak{M}'$, both passing through A but not through $\mathfrak{U}$. Construct $\mathfrak{M}^{-1}$ and $\mathfrak{M}'^{-1}$ as indicated in § 7.2. $A^{-1} = \mathfrak{M}^{-1} \cap \mathfrak{M}'^{-1}$.

We can simplify the construction by choosing $\mathfrak{M}' = \mathfrak{N}_A$. Then $\mathfrak{M}'^{-1} = \mathfrak{N}_A^{-1} = \mathfrak{N}_A$ and $A^{-1} = \mathfrak{M}^{-1} \cap \mathfrak{N}_A$. (See figure 6.)

We note that if $\mathfrak{M} = A\mathfrak{h}$, then $\mathfrak{M}^{-1} = \mathfrak{h}A^{-1} = A^{-1}\mathfrak{h}$ which is the straight line through $\mathfrak{U}$ and A^{-1}. In view of the preceding argument and § 7.2, we can now construct the inverse of any point or line in the $\mathfrak{G}$-plane.

7.4. Given $A \in \mathfrak{G}$ and $\mathfrak{N}$ any normalizer, construct $A\mathfrak{N}A^{-1}$.

(i) If $\mathfrak{N} = \mathfrak{h}$, then by § 3.6, $A\mathfrak{N}A^{-1} = \mathfrak{N}$.

(ii)

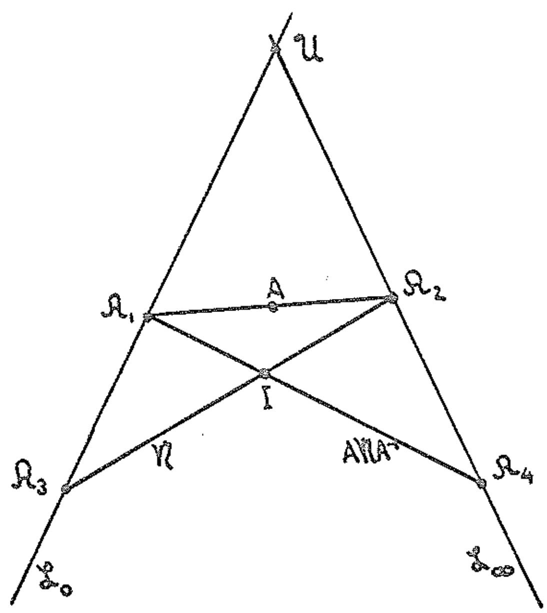

Figure 7.

If $\mathfrak{N} \neq \mathfrak{h}$, let $\mathfrak{N}$ meet $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ in $\mathfrak{R}_3$ and $\mathfrak{R}_2$ respectively. Construct the line through A and $\mathfrak{R}_2$ meeting $\mathfrak{L}_0$ in $\mathfrak{R}_1$. Also construct the line through $\mathfrak{R}_1$ and I meeting $\mathfrak{L}_\infty$ in $\mathfrak{R}_4$. (See figure 7.) Then $\overline{\mathfrak{R}_1\mathfrak{R}_4} = A\mathfrak{N}A^{-1}$. Indeed, by § 7.3, $A^{-1} \in \overline{\mathfrak{R}_3\mathfrak{R}_4}$ and by theorem 6.2.3, $\overline{\mathfrak{R}_1\mathfrak{R}_4} = A\mathfrak{N}^{-1}A^{-1} = A\mathfrak{N}A^{-1}$.

7.5. Given two normalizers $\mathfrak{N}$, $\mathfrak{N}' \neq \mathfrak{h}$, find all A such that $\mathfrak{N}' = A\mathfrak{N}A^{-1}$.

By § 4.4, there exists at least one such A. Let $\mathfrak{N}'$ meet $\mathfrak{L}_0$ in $\mathfrak{R}_1$ and $\mathfrak{N}$ meet $\mathfrak{L}_\infty$ in $\mathfrak{R}_2$. Let $A \in \overline{\mathfrak{R}_1\mathfrak{R}_2}$. By § 7.4, $\mathfrak{N}' = A\mathfrak{N}A^{-1}$. Conversely, let $A' \notin \overline{\mathfrak{R}_1\mathfrak{R}_2}$. Construct the line through $\mathfrak{R}_2$ and A' meeting $\mathfrak{L}_0$ in $\mathfrak{R}_1'$. $A' \notin \overline{\mathfrak{R}_1\mathfrak{R}_2}$ therefore $\mathfrak{R}_1' \neq \mathfrak{R}_1$. $A'\mathfrak{N}A'^{-1} = \overline{\mathfrak{R}_1'I} \neq \overline{\mathfrak{R}_1 I}$ but $\mathfrak{N}' = \overline{\mathfrak{R}_1 I}$, therefore $A'\mathfrak{N}A'^{-1} \neq \mathfrak{N}'$.
Hence $\{A \mid A\mathfrak{N}A^{-1} = \mathfrak{N}'\} = \overline{\mathfrak{R}_1\mathfrak{R}_2}$.

7.6. Given $A, B \in \mathfrak{G}$ and $\mathfrak{L}$ an arbitrary line not through $\mathfrak{U}$, construct $A\mathfrak{L}B^{-1}$.

Let $\mathfrak{L}$ cut $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ in $\mathfrak{R}_3$ and $\mathfrak{R}_4$ respectively. Construct $\overline{\mathfrak{R}_3 I}$ cutting $\mathfrak{L}_\infty$ in $\mathfrak{R}_2$, and then construct $\overline{\mathfrak{R}_2 A}$ cutting $\mathfrak{L}_0$ in $\mathfrak{R}_1$. Also construct $\overline{\mathfrak{R}_4 B}$ cutting $\mathfrak{L}_0$ in $\mathfrak{R}_5$, and then construct $\overline{\mathfrak{R}_5 I}$ cutting $\mathfrak{L}_\infty$ in $\mathfrak{R}_6$. (See figure 8.) Then $\overline{\mathfrak{R}_1\mathfrak{R}_6} = A\mathfrak{L}B^{-1}$. Indeed, according to theorem 6.2.3, $\overline{\mathfrak{R}_1\mathfrak{R}_6} = AI^{-1}\mathfrak{L}B^{-1}I = A\mathfrak{L}B^{-1}$.

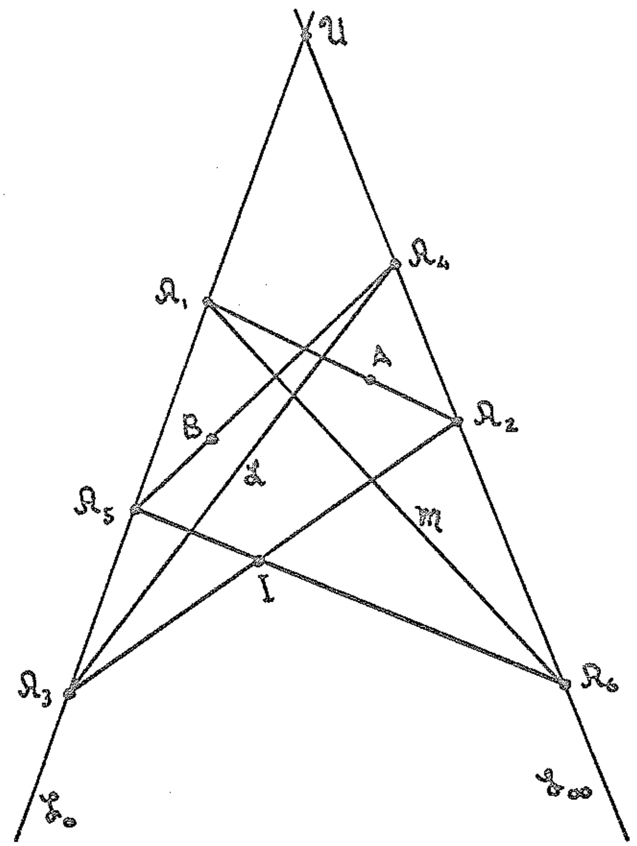

Figure 8.

In particular, if we set A = B we obtain a construction for $A\mathfrak{L}A^{-1}$, the conjugate of $\mathfrak{L}$ by A.

7.7. If $\mathfrak{L}$ and $\mathfrak{M}$ are two lines not through $\mathfrak{U}$, find all couples (A, B) such that $A\mathfrak{L}B^{-1} = \mathfrak{M}$.

Referring to figure 8, let $\mathfrak{L}$ cut $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ in $\mathfrak{R}_3$ and $\mathfrak{R}_4$ respectively and $\mathfrak{M}$ cut $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ in $\mathfrak{R}_1$ and $\mathfrak{R}_6$ respectively. Construct $\overline{\mathfrak{R}_3 I}$ cutting $\mathfrak{L}_\infty$ in $\mathfrak{R}_2$ and $\overline{\mathfrak{R}_6 I}$ cutting $\mathfrak{L}_0$ in $\mathfrak{R}_5$. By § 7.6, if $A \in \overline{\mathfrak{R}_1\mathfrak{R}_2}$ and $B \in \overline{\mathfrak{R}_4\mathfrak{R}_5}$, then $\mathfrak{M} = A\mathfrak{L}B^{-1}$. Conversely, suppose $A' \notin \overline{\mathfrak{R}_1\mathfrak{R}_2}$,

then $\overline{\mathfrak{R}_2 A'}$ will meet $\mathfrak{L}_0$ in $\mathfrak{R}_1' \neq \mathfrak{R}_1$ and therefore $A'\mathfrak{L}T \neq \mathfrak{M}$ for all $T \in \mathcal{G}$. Similarly, if $B' \notin \overline{\mathfrak{R}_4\mathfrak{R}_5}$ then $T\mathfrak{L}B'^{-1} \neq \mathfrak{M}$ for all $T \in \mathcal{G}$.
Consequently, $\{(A, B) \mid A\mathfrak{L}B^{-1} = \mathfrak{M}\} = \overline{\mathfrak{R}_1\mathfrak{R}_2} \times \overline{\mathfrak{R}_4\mathfrak{R}_5}$ (the cartesian product of $\overline{\mathfrak{R}_1\mathfrak{R}_2}$ and $\overline{\mathfrak{R}_4\mathfrak{R}_5}$).

We can now prove the following theorem.

Theorem 1: If $\mathfrak{L}$ and $\mathfrak{M}$ are two lines, not passing through $\mathfrak{U}$ or I, then there exists a unique $A \in \mathcal{G}$ such that $A\mathfrak{L}A^{-1} = \mathfrak{M}$.

Proof: Let $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6$ be the same as above (see figure 8). Suppose that $\overline{\mathfrak{R}_1\mathfrak{R}_2} \parallel_0 \overline{\mathfrak{R}_4\mathfrak{R}_5}$, then $\mathfrak{R}_1 = \mathfrak{R}_5$. But $\mathfrak{M} = \overline{\mathfrak{R}_1\mathfrak{R}_6} = \overline{\mathfrak{R}_5\mathfrak{R}_6}$ and by construction $I \in \overline{\mathfrak{R}_5\mathfrak{R}_6}$, which contradicts the fact that $\mathfrak{M}$ does not pass through I. Therefore, $\overline{\mathfrak{R}_1\mathfrak{R}_2} \nparallel_0 \overline{\mathfrak{R}_4\mathfrak{R}_5}$. Suppose, now, that $\overline{\mathfrak{R}_1\mathfrak{R}_2} \parallel_\infty \overline{\mathfrak{R}_4\mathfrak{R}_5}$; then $\mathfrak{R}_2 = \mathfrak{R}_4$ but $\mathfrak{L} = \overline{\mathfrak{R}_3\mathfrak{R}_4} = \overline{\mathfrak{R}_3\mathfrak{R}_2}$ and by construction $I \in \overline{\mathfrak{R}_3\mathfrak{R}_2}$, contradicting the hypothesis that $\mathfrak{L}$ does not pass through I. Therefore $\overline{\mathfrak{R}_1\mathfrak{R}_2} \nparallel_\infty \overline{\mathfrak{R}_4\mathfrak{R}_5}$. Let $A = \overline{\mathfrak{R}_1\mathfrak{R}_2} \cap \overline{\mathfrak{R}_4\mathfrak{R}_5}$, then $A\mathfrak{L}A^{-1} = \mathfrak{M}$. Since $\overline{\mathfrak{R}_1\mathfrak{R}_2}$ and $\overline{\mathfrak{R}_4\mathfrak{R}_5}$ are not parallel, they are not equal and there is obviously only one such A.

Q.E.D.

In particular, if $\mathfrak{L} = \mathfrak{M}$ then $A = I$ which agrees with § 5.7, example 3.

We note the principal results of these considerations:

(i) All lines through $\mathfrak{U}$ are invariant under conjugation. Each such line, except $\mathfrak{g}$, is a complete class of conjugate elements.

(ii) All lines through I, except $\mathfrak{g}$ which is invariant, are conjugate to one another; any such line is invariant under conjugacy only by its own elements.

(iii) Any two lines, not through $\mathfrak{U}$ or I, are conjugate to each other with a uniquely defined conjugating element.

7.8. Given A and $B \in \mathcal{G}$, construct ABA^{-1}.

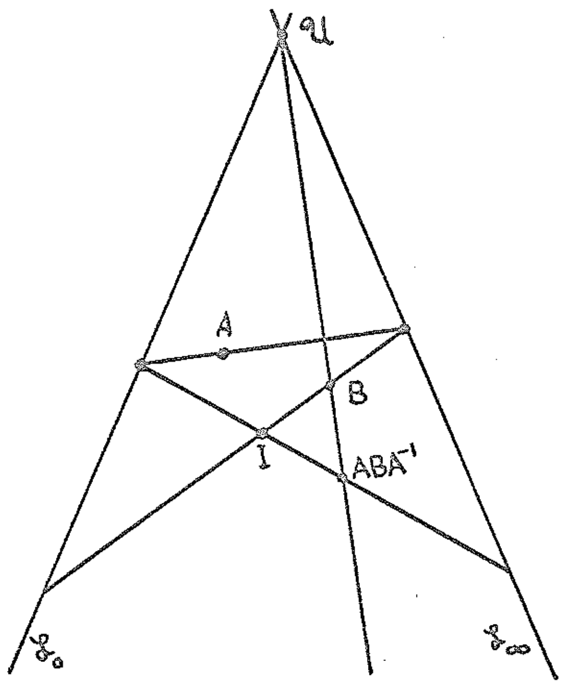

Figure 9.

Let $\mathfrak{g}$ be any line through B, but not through $\mathfrak{U}$. Construct $A\mathfrak{g}A^{-1}$ as described in § 7.6. $ABA^{-1} \in A\mathfrak{g}A^{-1}$ and also $ABA^{-1} \in \mathfrak{h}B$. ABA^{-1} is uniquely determined by the relation $ABA^{-1} = A\mathfrak{g}A^{-1} \cap \mathfrak{h}B$.

If $B \notin \mathfrak{h}$ we can simplify our construction by choosing $\mathfrak{g} = \mathfrak{N}_B$. Then, construct $A\mathfrak{N}_BA^{-1}$ as indicated in § 7.4 and $B = A\mathfrak{N}_BA^{-1} \cap \mathfrak{h}B$. This construction is illustrated in figure 9.

7.9. Given $A, B \in \mathfrak{G}$, construct $[A, B] = ABA^{-1}B^{-1}$.

(i)

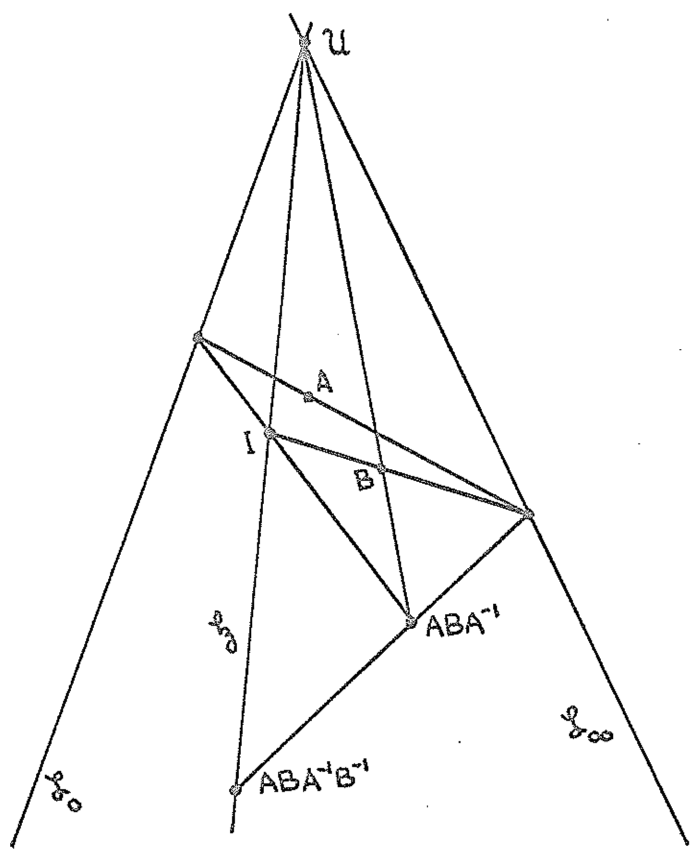

Figure 10.

Note that $[A, B] \in \mathcal{H}$. If $B \notin \mathcal{H}$, then $\mathfrak{N}_B \neq \mathcal{H}$ and $[A, B] \in ABA^{-1}\mathfrak{N}_B$. Construct ABA^{-1} as described in § 7.8 and then $ABA^{-1}\mathfrak{N}_B$ is ∞-parallel to $\mathfrak{N}_B$ and passes through ABA^{-1}. $[A, B] = ABA^{-1}\mathfrak{N}_B \cap \mathcal{H}$. (See figure 10.)

(ii) Suppose that $B' \in B\mathfrak{N}_A$, then $B' = BA'$ where $A' \in \mathfrak{N}_A$. $B'A^{-1}B'^{-1} = BA'A^{-1}A'^{-1}B^{-1} = BA^{-1}B^{-1}$. Therefore if $B \in \mathcal{H}$ but $A \notin \mathcal{H}$, choose $B' \in B\mathfrak{N}_A$ but $B' \notin \mathcal{H}$. It follows that $[A, B'] = AB'A^{-1}B'^{-1} = ABA^{-1}B^{-1} = [A, B]$, and now we can use the construction described in (i).

(iii) If $A, B \in \mathcal{H}$ then $AB = BA$ and therefore $[A, B] = I$.

7.10. Construct the involutory line $\mathcal{H}V$, i.e. the set of all $T \neq I$ such that $T^2 = I$.

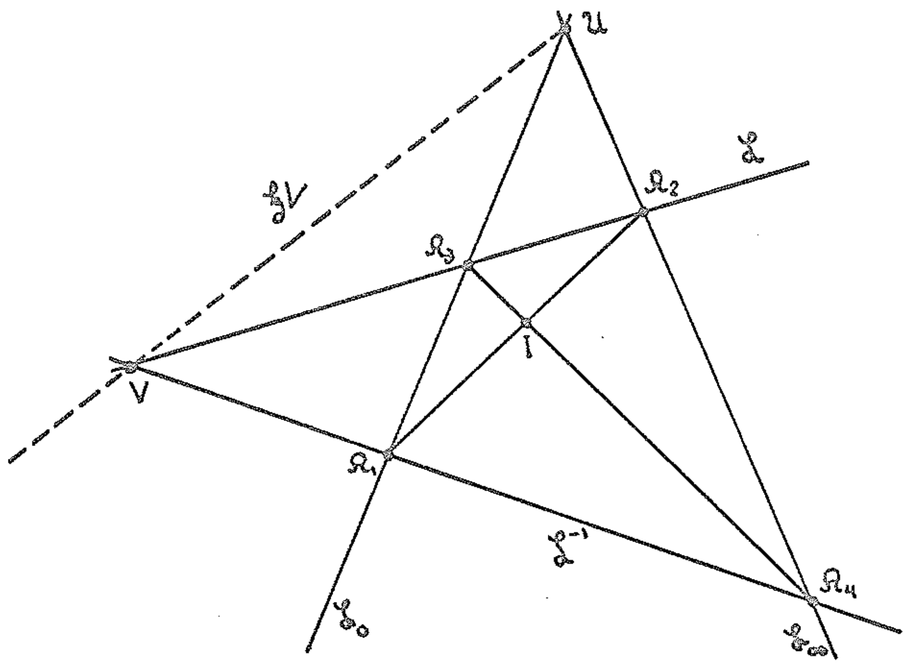

Figure 11.

Let $\mathfrak{L}$ be any line not passing through $\mathfrak{U}$ or I. Construct $\mathfrak{L}^{-1}$ as indicated in § 7.2. Let $\mathfrak{L}$ cut $\mathfrak{L}_\infty$ and $\mathfrak{L}_0$ in $\mathfrak{R}_2$ and $\mathfrak{R}_3$ respectively. Construct $\overline{\mathfrak{R}_2 I}$ cutting $\mathfrak{L}_0$ in $\mathfrak{R}_1$ and $\overline{\mathfrak{R}_3 I}$ cutting $\mathfrak{L}_\infty$ in $\mathfrak{R}_4$; then $\overline{\mathfrak{R}_1\mathfrak{R}_4} = \mathfrak{L}^{-1}$. If $\mathfrak{L} \parallel_0 \mathfrak{L}^{-1}$, then $\mathfrak{R}_1 = \mathfrak{R}_3$ and $\mathfrak{L} = \overline{\mathfrak{R}_3\mathfrak{R}_2} = \overline{\mathfrak{R}_1\mathfrak{R}_2}$, but $I \in \overline{\mathfrak{R}_1\mathfrak{R}_2}$ contradicting the fact that $\mathfrak{L}$ does not pass through I. We conclude that $\mathfrak{L} \nparallel_0 \mathfrak{L}^{-1}$. Similarly, we show that $\mathfrak{L} \nparallel_\infty \mathfrak{L}^{-1}$. Let $V = \mathfrak{L} \cap \mathfrak{L}^{-1}$; $V = \mathfrak{L} \cap \mathfrak{L}^{-1} \Rightarrow V^{-1} = \mathfrak{L}^{-1} \cap \mathfrak{L} \Rightarrow V = V^{-1}$, therefore $V^2 = I$. Taking into account the considerations of § 5.7 example 4, we see that $\{T \mid T^2 = I,\ T \neq I\} = \mathfrak{g}V$. (See figure 11.)

§ 8. A group-theoretical proof for Pappus' theorem.

Theorem (Pappus): If the vertices of a simple hexagon lie alternately on a pair of intersecting lines, then the three pairs of opposite sides will intersect in collinear points.

Proof: Since there exist collineations which will send any two different lines into any other two different lines, there will be no loss in generality if we suppose that our pair of intersecting lines are $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$. Let the hexagon be $\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3\mathfrak{R}_4\mathfrak{R}_5\mathfrak{R}_6$ and let $\overline{\mathfrak{R}_1\mathfrak{R}_2}$ and $\overline{\mathfrak{R}_4\mathfrak{R}_5}$ cut in A, $\overline{\mathfrak{R}_2\mathfrak{R}_3}$ and $\overline{\mathfrak{R}_5\mathfrak{R}_6}$ cut in B, $\overline{\mathfrak{R}_3\mathfrak{R}_4}$ and $\overline{\mathfrak{R}_6\mathfrak{R}_1}$ cut in C. (See figure 12.)

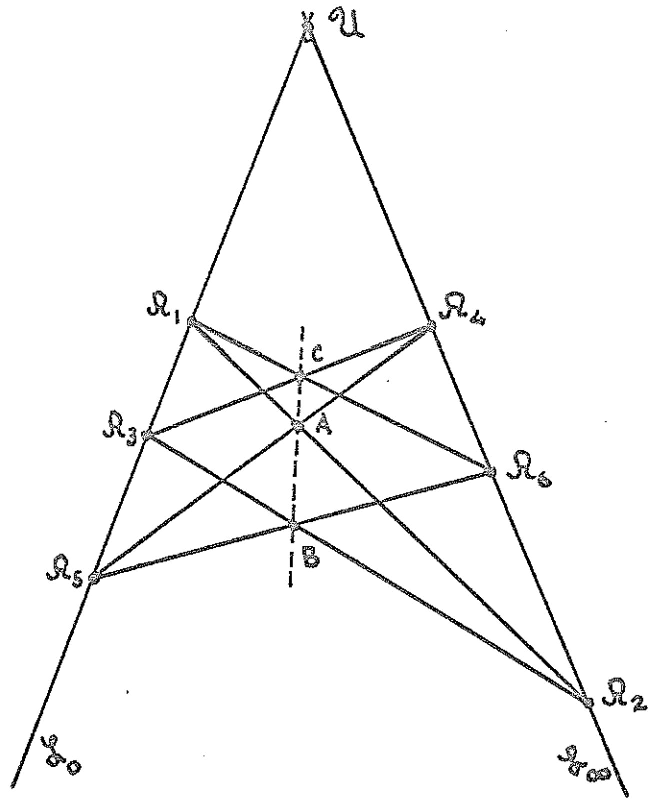

Figure 12.

If one of the $\mathcal{R}_i$ is equal to $\mathcal{U}$, two of the points A, B, C coincide and the theorem is obvious.

Suppose that no $\mathcal{R}_i$ is equal to $\mathcal{U}$. Since $A \in \overline{\mathcal{R}_1\mathcal{R}_2}$, $B \in \overline{\mathcal{R}_2\mathcal{R}_3}$, $C \in \overline{\mathcal{R}_3\mathcal{R}_4}$, $A \in \overline{\mathcal{R}_4\mathcal{R}_5}$ and $B \in \overline{\mathcal{R}_5\mathcal{R}_6}$, theorem 6.2.2 shows that $AB^{-1}CA^{-1}B \in \overline{\mathcal{R}_1\mathcal{R}_6}$. By § 4.7, $AB^{-1}CA^{-1}B\mathfrak{g} = CAA^{-1}B^{-1}B\mathfrak{g} = C\mathfrak{g}$; therefore $AB^{-1}CA^{-1}B = \overline{\mathcal{R}_1\mathcal{R}_6} \cap C\mathfrak{g}$. But $C = \overline{\mathcal{R}_1\mathcal{R}_6} \cap C\mathfrak{g}$, therefore $AB^{-1}CA^{-1}B = C \Rightarrow AB^{-1}C = CB^{-1}A \Rightarrow AB^{-1}CB^{-1} = CB^{-1}AB^{-1}$ $\Rightarrow CB^{-1} \in \mathfrak{N}_{AB^{-1}}$. We conclude that $C \in \mathfrak{N}_{AB^{-1}}B$ and thus A, B and C are collinear.

Q.E.D.

§ 9. Conic sections.

9.1. Definition 1: The set of all lines in the plane passing through a point, P, is called a pencil of lines, and will be denoted by [P].

Definition 2: Two different pencils, [P] and [Q], are said to be perspective if there is a one-to-one correspondence $\sigma: [P] \to [Q]$ such that corresponding lines intersect on the same line, i.e. if there exists a line $\mathfrak{L}$ such that for all $\mathfrak{M} \in [P]$, $\mathfrak{M} \cap \sigma(\mathfrak{M}) \cap \mathfrak{L} \neq \emptyset$. We shall denote this by $[P] \overline{\overline{\wedge}} [Q]$. (See figure 13.)

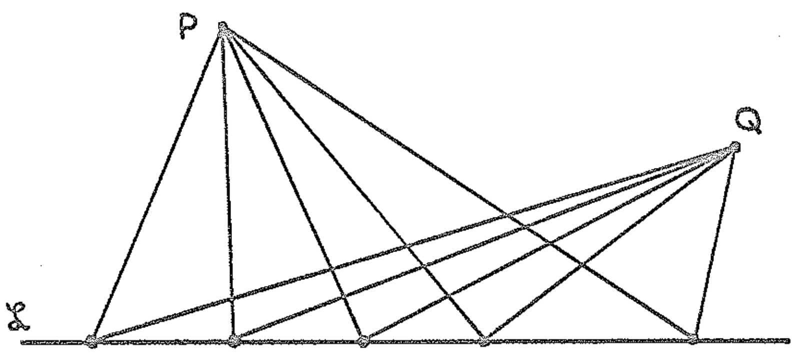

Figure 13.

Definition 3: Two pencils, [P] and [Q], in the same plane are said to be projective, $[P] \overline{\wedge} [Q]$, provided that there exists a one-to-one correspondence between the lines of [P] and those of [Q], and a finite number of pencils $[R_1]$, $[R_2]$, ..., $[R_n]$ such that $[P] = [R_1]$, $[Q] = [R_n]$

and $[R_i] \,\bar{\wedge}\, [R_{i+1}]$ for $i = 1, 2, \ldots, n - 1$; where the one-to-one correspondence between $[P]$ and $[Q]$ is the composition of the correspondences between the successive pencils.

Definition 4: The set of all points of intersection of corresponding lines of two projective, non-perspective pencils $[P]$ and $[Q]$, such that $P \neq Q$, is called a conic (point conic).

We shall now state an important theorem due to Pascal (cf. [4]).

Theorem 1: The points P_1, P_2, P_3, P_4, P_5, P_6, no three of which are collinear, lie on a conic if and only if the three pairs of opposite sides of the simple hexagon $P_1P_2P_3P_4P_5P_6$ meet in collinear points. (See figure 14.)

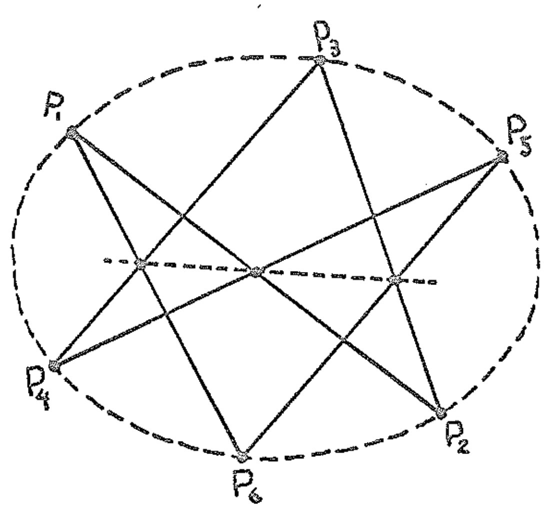

Figure 14.

Corollary: A conic is completely determined by five of its points.

Note: This theorem leads to a construction of the conic passing through five given points, no three of which are collinear.

Definition 5: A line which meets a conic in one and only one point is called a tangent to that conic.

If in Pascal's theorem (theorem 1), one of the sides of the hexagon cuts the conic in one point only, then two of the vertices of the hexagon coincide; from this, we get the following theorem.

Theorem 2: If P_1, P_2, P_3, P_4, P_5 are points such that no three are collinear, and if ℓ_1 is a line passing through P_1 but not through any of the other four points, then there is a conic through P_1, P_2, P_3, P_4, P_5 and tangent to ℓ_1, if and only if ℓ_1 meets $\overline{P_3P_4}$ in a point collinear with the points of intersection of $\overline{P_2P_3}$ with $\overline{P_1P_5}$ and $\overline{P_1P_2}$ with $\overline{P_4P_5}$. (See figure 15.)

Corollary: There is one and only one conic through four given points, no three of which are collinear, and tangent to a given line through one and only one of these four points.

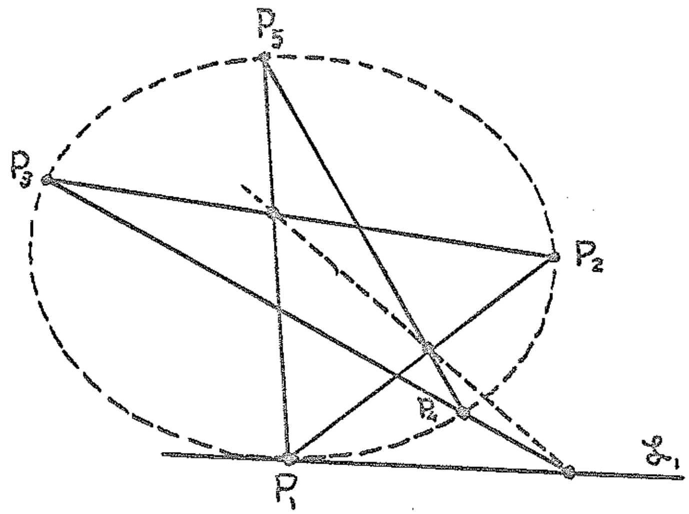

Figure 15.

Note: This theorem leads to a construction of the conic passing through four points, no three of which are collinear, and tangent to a given line through one and only one of these points.

It also leads to a construction of the tangent to a conic through a point on that conic.

By further specialization of Pascal's theorem, we obtain the two following theorems.

Theorem 3: If P_1, P_2, P_3, P_4 are points such that no three are collinear, and ℓ_1 is a line through P_1 but not through P_2, P_3 or P_4, and ℓ_2 is a line through P_2 but not through P_1, P_3 or P_4, then P_1, P_2, P_3, P_4 lie on a conic tangent to ℓ_1 and ℓ_2 if and only if ℓ_2 meets $\overline{P_1P_4}$, ℓ_1 meets $\overline{P_2P_3}$, and $\overline{P_1P_2}$ meets $\overline{P_3P_4}$ in three collinear points. (See figure 16.)

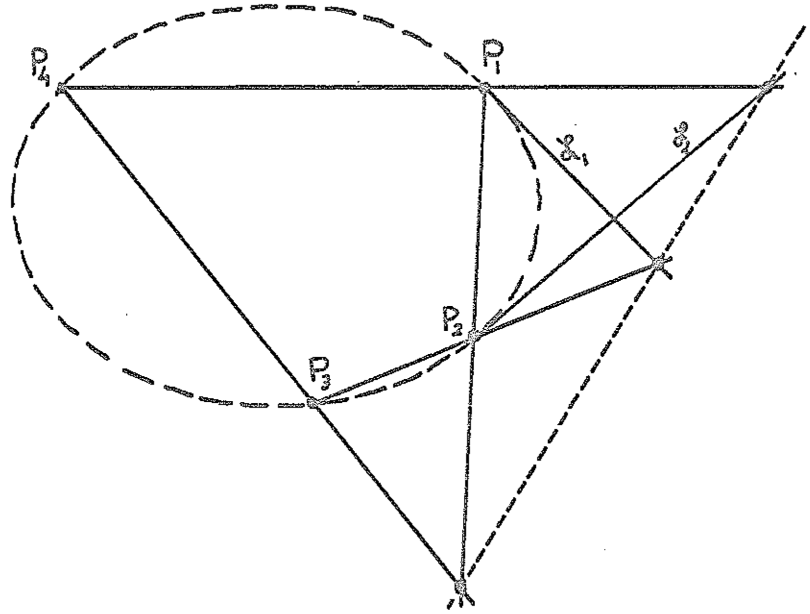

Figure 16.

Theorem 4: Let P_1, P_2, P_3, P_4, ℓ_1, ℓ_2 be the same as in theorem 3. P_1, P_2, P_3, P_4 lie on a conic tangent to ℓ_1 and ℓ_2 if and only if ℓ_1 and ℓ_2, $\overline{P_1P_4}$ and $\overline{P_2P_3}$, $\overline{P_1P_3}$ and $\overline{P_2P_4}$ cut in three collinear points. (See figure 17.)

Corollary: A conic is determined by three of its points and the tangents at two of these points.

Theorems 3 and 4 lead to constructions for a conic, given three of its points and the tangents at two of them.

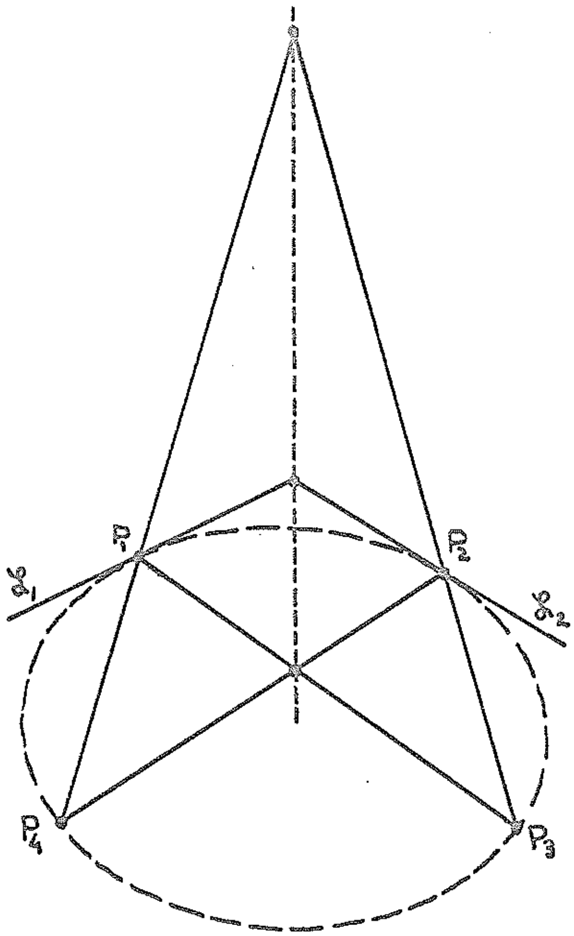

Figure 17.

9.2. Let $\mathfrak{X} \subseteq \mathcal{G}$, define $S(\mathfrak{X}) = \{X^2 \mid X \in \mathfrak{X}\}$. Note that $S(\mathfrak{N}_A) \subseteq \mathfrak{N}_A$ and that $S(A\mathfrak{z}) \subseteq (A\mathfrak{z})(A\mathfrak{z}) = A^2\mathfrak{z}$.

We shall now prove a theorem which will permit us to determine a certain class of conics.

Theorem 1: If $\mathfrak{L}$ is any line not passing through $\mathfrak{U}$ or I, then the points of $S(\mathfrak{L})$ lie on a conic passing through I and tangent to $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ at their points of intersection with $\mathfrak{L}$.

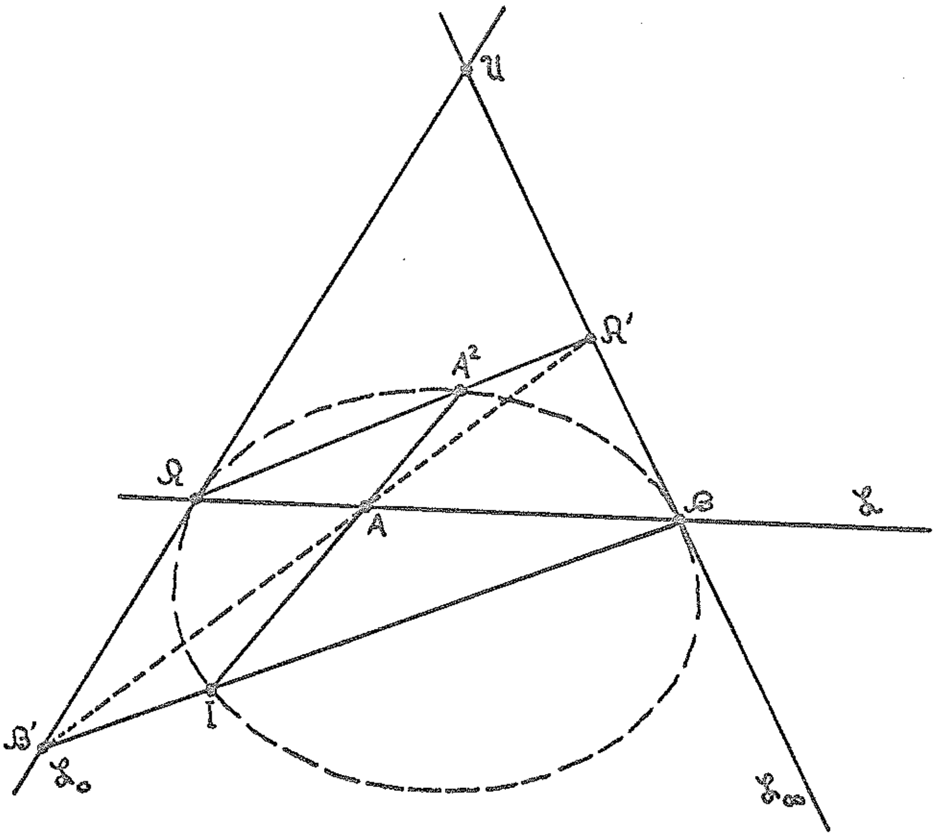

Figure 18.

Proof: Let $\mathcal{R}$ and $\mathcal{B}$ be the points of intersection of $\mathcal{L}$ with $\mathcal{L}_0$ and $\mathcal{L}_\infty$ respectively. Let $A \in \mathcal{G}$ be any point on $\mathcal{L}$. Then $A^2 \in S(\mathcal{L})$. Construct $\overline{\mathcal{R}A^2}$ meeting $\mathcal{L}_\infty$ in $\mathcal{R}'$. Construct also $\overline{\mathcal{B}I}$ meeting $\mathcal{L}_0$ in $\mathcal{B}'$. (See figure 18.) $I \in \overline{\mathcal{B}'\mathcal{B}}$, $A \in \overline{\mathcal{B}\mathcal{R}}$, $A^2 \in \overline{\mathcal{R}\mathcal{R}'}$ and by theorem 6.2.1, $IA^{-1}A^2 = A \in \overline{\mathcal{R}'\mathcal{B}'}$. Therefore $\mathcal{R}'$, $\mathcal{B}'$ and A are collinear; since the line through I and A^2 is η_A, which meets $\mathcal{L}$ in A, then by theorem 9.1.3, A^2 lies on the conic through $\mathcal{R}$, $\mathcal{B}$, I and tangent to $\mathcal{L}_0$ and $\mathcal{L}_\infty$ at $\mathcal{R}$ and $\mathcal{B}$ respectively.

Q.E.D.

Let us denote the conic, passing through I and tangent to ℓ_0 and ℓ_∞ at $\mathfrak{A}$ and $\mathfrak{B}$ respectively, by $\mathfrak{C}$.

Consider the tangent to the conic $\mathfrak{C}$ at the point I. This tangent does not pass through $\mathfrak{A}$ or $\mathfrak{B}$ since it touches the conic only in I, and is therefore not parallel to ℓ. Let D be the intersection of ℓ with this tangent. The tangent is therefore $\mathfrak{N}_D$. $D \in \ell \Rightarrow D^2 \in \mathfrak{C}$, but $D^2 \in \mathfrak{N}_D$ and therefore $D^2 = I$, i.e. D belongs to the involutory line $\mathfrak{J}V$.

Now let B be any point of $\mathfrak{C}$, $B \neq I, \mathfrak{A}, \mathfrak{B}$. $\mathfrak{N}_B$ cuts $\mathfrak{C}$ in I and B; therefore $\mathfrak{N}_B$ does not pass through $\mathfrak{A}$ or $\mathfrak{B}$, i.e. $\mathfrak{N}_B$ is not parallel to ℓ. Let A be the point of intersection of $\mathfrak{N}_B$ with ℓ. By theorem 1, $A^2 \in \mathfrak{C}$, and also $A^2 \in \mathfrak{N}_B$; therefore $A^2 = I$ or B. But on every line ℓ not through $\mathfrak{U}$ or I, there is one and only one point D such that $D^2 = I$; since $\mathfrak{N}_B$ is not tangent to $\mathfrak{C}$, then $A \neq D$ and thus $A^2 = B$.

We have therefore shown that all points of $\mathfrak{C}$, except $\mathfrak{A}$ and $\mathfrak{B}$, belong to $S(\ell)$, i.e. $\mathfrak{C} = S(\ell) \cup \{\mathfrak{A}, \mathfrak{B}\}$. We shall say that $S(\ell)$ is a $\mathfrak{G}$-conic or, when there is no confusion, simply a conic.

Let us apply the collineation $f_A : X \rightarrow AX$ to our conic. We know that collineations send conics into conics, and tangents to a conic into tangents to the corresponding conic. Let $\mathcal{L}$ be any line not passing through $\mathcal{U}$ or I; then $f_A(S(\mathcal{L}))$ is a conic passing through A and tangent to $\mathcal{L}_0$ and $\mathcal{L}_\infty$ at their points of intersection with $f_A(\mathcal{L}) = A\mathcal{L}$.

$$f_A(S(\mathcal{L})) = \{AX^2 \mid X \in \mathcal{L}\} = \{A(A^{-1}Y)^2 \mid A^{-1}Y \in \mathcal{L}\}$$
$$= \{YA^{-1}Y \mid Y \in A\mathcal{L}\}.$$

Definition: $S_A(\mathcal{L}) = \{XA^{-1}X \mid X \in \mathcal{L}\}$.

From the preceding argument, we may state the following theorem.

Theorem 2: If $\mathcal{L}$ is any line not passing through $\mathcal{U}$ or A, then $S_A(\mathcal{L})$ is a conic passing through A and tangent to $\mathcal{L}_0$ and $\mathcal{L}_\infty$ at their points of intersection with $\mathcal{L}$.

Note: $S_I(\mathcal{L}) = S(\mathcal{L})$.

Since a conic is determined by three points and the tangents at two of them, and since $\mathcal{L}$ and A are arbitrary, we may conclude that all conics tangent to $\mathcal{L}_0$ and $\mathcal{L}_\infty$ are of the form $S_A(\mathcal{L})$.

9.3. Properties of $S_A(\mathcal{L})$.

(i) A conic being uniquely determined by three of its points and the tangents at two of them, we conclude that if $B \in S_A(\mathfrak{X})$, then $S_A(\mathfrak{X}) = S_B(\mathfrak{X})$ and conversely.

(ii) $$f_B(S_A(\mathfrak{X})) = \{BXA^{-1}X \mid X \in \mathfrak{X}\}$$
$$= \{YA^{-1}B^{-1}Y \mid B^{-1}Y \in \mathfrak{X}\}$$
$$= \{Y(BA)^{-1}Y \mid Y \in B\mathfrak{X}\}$$
$$= S_{BA}(B\mathfrak{X}).$$

(iii) Similarly, $g_B(S_A(\mathfrak{X})) = S_{AB}(\mathfrak{X}B)$.

(iv) Let φ be any automorphism of $\mathfrak{G}$, then
$$\varphi(S_A(\mathfrak{X})) = \{\varphi(X)\varphi(A^{-1})\varphi(X) \mid X \in \mathfrak{X}\}$$
$$= \{\varphi(X)\varphi(A)^{-1}\varphi(X) \mid \varphi(X) \in \varphi(\mathfrak{X})\}$$
$$= \{Y\varphi(A)^{-1}Y \mid Y \in \varphi(\mathfrak{X})\}$$
$$= S_{\varphi(A)}(\varphi(\mathfrak{X})).$$

(v) $$h(S_A(\mathfrak{X})) = \{(XA^{-1}X)^{-1} \mid X \in \mathfrak{X}\}$$
$$= \{X^{-1}AX^{-1} \mid X^{-1} \in \mathfrak{X}^{-1}\}$$
$$= \{YAY \mid Y \in \mathfrak{X}^{-1}\}$$
$$= S_{A^{-1}}(\mathfrak{X}^{-1}).$$

(vi) We have seen that the tangent to $S(\mathfrak{X})$, at the point I, cuts $\mathfrak{X}$ on $\mathfrak{g}V$. Apply the collineation $f_A: X \to AX$ to this conic. $f_A(S(\mathfrak{X})) = S_A(A\mathfrak{X})$ and $f_A(\mathfrak{g}V) = \mathfrak{g}AV$; therefore the tangent to $S_A(A\mathfrak{X})$ at the point A cuts $A\mathfrak{X}$ on $\mathfrak{g}AV$. Replace $A\mathfrak{X}$ by $\mathfrak{X}'$ and note that if $A \in S_B(\mathfrak{X}')$, then $S_B(\mathfrak{X}') = S_A(\mathfrak{X}')$ and we can now state the following theorem.

Theorem 1: If A is a point of the conic $S_B(\mathfrak{x}')$, then the tangent to $S_B(\mathfrak{x}')$ at the point A cuts $\mathfrak{x}'$ in a point of $\mathfrak{h}AV$, where $V^2 = I$, $V \neq I$.

(vii) From (vi), we know that the tangent to $S_A(\mathfrak{x})$ at a point B of $S_A(\mathfrak{x})$ cuts $\mathfrak{x}$ on $\mathfrak{h}BV$. Since two conics are tangent at a point if and only if their tangents, at this point, coincide, we have the following theorem.

Theorem 2: Two conics, $S_A(\mathfrak{x})$ and $S_{A'}(\mathfrak{x}')$ cutting in B, are tangent at B, if and only if $\mathfrak{x}$ and $\mathfrak{x}'$ cut in a point of $\mathfrak{h}BV$.

We saw that if we apply the collineation $h: X \to X^{-1}$ to the conic $S(\mathfrak{x})$, we obtain $S(\mathfrak{x}^{-1})$. $I \in S(\mathfrak{x}) \cap S(\mathfrak{x}^{-1})$ and $\mathfrak{x}$ and $\mathfrak{x}^{-1}$ meet on $\mathfrak{h}V$; consequently, we have the following corollary.

Corollary: The conic $h(S(\mathfrak{x})) = S(\mathfrak{x})^{-1}$ is tangent to the conic $S(\mathfrak{x})$ at the point I.

(viii) Suppose we are given a conic of the form $S(\mathfrak{x})$ and two points A and $B \in \mathfrak{G}$ such that $A, B \notin \mathfrak{h}$, and such that the line through A and B does not pass through $\mathfrak{U}$ or I. Suppose, furthermore, that A, B and I are in the same region of the plane, as determined by $\mathfrak{L}_0$ and $\mathfrak{L}_\infty$ in § 2.4. We shall construct a line $\mathfrak{L}'$ such that $S(\mathfrak{L}')$ is a conic passing through A and B.

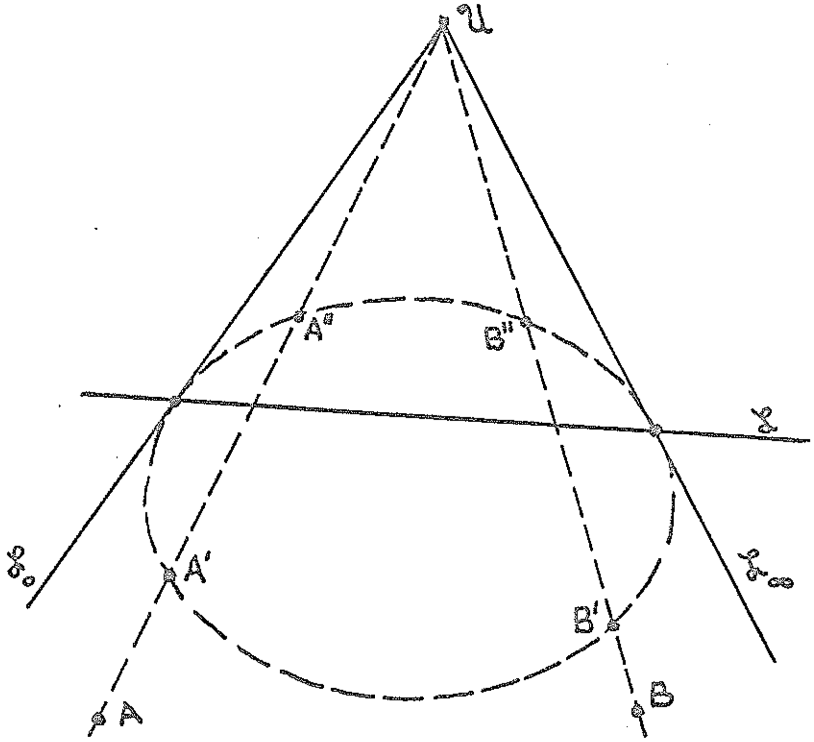

Figure 19.

Let the intersections of $\mathfrak{U}A$ with $S(\mathfrak{L})$ be A' and A", and let the intersections of $\mathfrak{U}B$ with $S(\mathfrak{L})$ be B' and B". $\overline{A'B'}$ does not pass through I because, then, $\overline{A'B'}$ would cut $S(\mathfrak{L})$ in three points, which is impossible. $\overline{A'B'}$ does not pass through $\mathfrak{U}$, since then $\overline{AB}$ would also pass through $\mathfrak{U}$, which is contrary to hypothesis. By theorem 7.7.1, there exists a unique C such that $C(\overline{A'B'})C^{-1} = \overline{AB}$. The construction for this C is given in § 7.7. $CA'C^{-1} \in \mathfrak{U}A$ and $CA'C^{-1} \in \overline{AB}$; therefore, $CA'C^{-1} = A$. Similarly, we see that $CB'C^{-1} = B$. Define $\mathfrak{g}' = C\mathfrak{g}C^{-1}$.

$$S(\mathfrak{g}') = S(C\mathfrak{g}C^{-1}) = C(S(\mathfrak{g}))C^{-1}.$$

$A', B' \in S(\mathfrak{g})$, therefore $A, B \in S(\mathfrak{g}')$; hence $\mathfrak{g}'$ is our required line.

If in the preceding consideration we had taken A" instead of A' or B" instead of B', we would have obtained different conics. There are four such conics.

If we have three non-collinear points A, B, C in the same region of the plane, determined by $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$, and if no two are collinear with $\mathfrak{U}$, then we can construct a line $\mathfrak{g}$ such that $S_A(\mathfrak{g})$ is a conic through A, B, C and tangent to $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$. We find, first of all, the conic through I, $A^{-1}B$ and $A^{-1}C$ as above; then, we apply the collineation $f_A: X \rightarrow AX$.

9.4. We shall now study another class of conics which, in a certain sense, are inverse to the class just studied.

For $\mathfrak{X} \subseteq \mathfrak{G}$, define $R(\mathfrak{X}) = \{X \mid X^2 \in \mathfrak{X}\}$.

Note that $R(\mathfrak{N}_A) \supseteq \mathfrak{N}_A$ and $R(A^2\mathfrak{h}) \supseteq A\mathfrak{h}$.

Theorem 1: If $\mathfrak{g}$ is any line not passing through $\mathfrak{U}$ or I, then $R(\mathfrak{g})$ is the conic passing through $\mathfrak{U}$ and the two points of intersection of $\mathcal{L}$ with $\mathcal{L}_0$ and $\mathcal{L}_\infty$, and tangent to the normalizers through these two points.

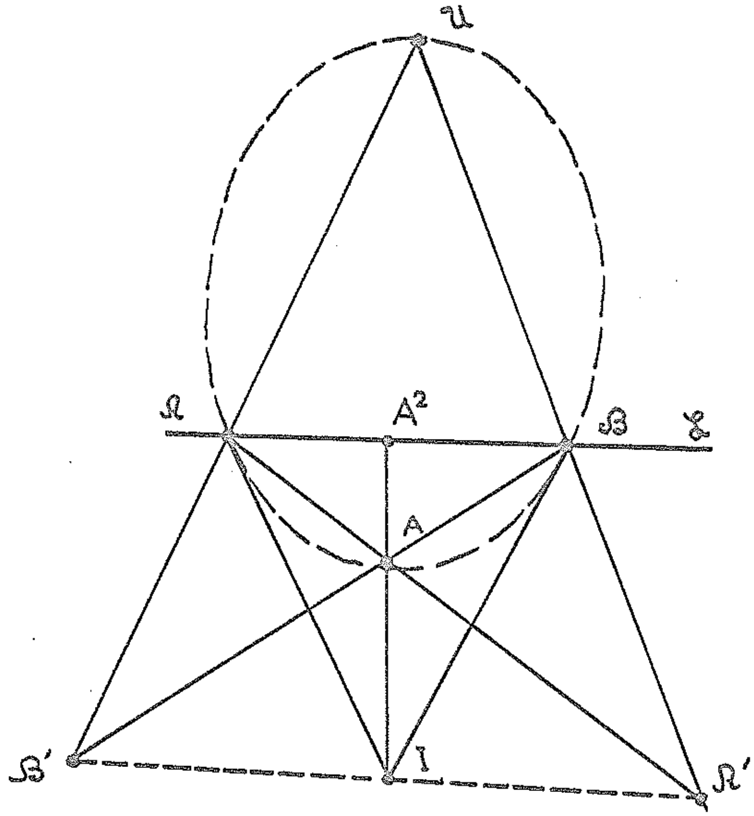

Figure 20.

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be the intersection of $\mathcal{L}$ with $\mathcal{L}_0$ and $\mathcal{L}_\infty$ respectively. Let $A \in R(\mathcal{L})$, then $A^2 \in \mathcal{L}$. Construct $\overline{\mathcal{A}A}$ cutting $\mathcal{L}_\infty$ in $\mathcal{A}'$, and construct $\overline{\mathcal{B}A}$ cutting $\mathcal{L}_0$ in $\mathcal{B}'$. $A \in \overline{\mathcal{B}'\mathcal{B}}$, $A^2 \in \overline{\mathcal{B}\mathcal{A}}$ and $A \in \overline{\mathcal{A}\mathcal{A}'}$; therefore, by theorem 6.2.1, $AA^{-2}A = I \in \overline{\mathcal{A}'\mathcal{B}'}$. By theorem 9.1.4, A lies on the conic passing through $\mathfrak{U}$, $\mathcal{A}$, $\mathcal{B}$ and tangent to the normalizers through $\mathcal{A}$ and $\mathcal{B}$. (See figure 20.)

Conversely, let A be any point, $A \neq \mathfrak{U}, \mathcal{A}, \mathcal{B}$, on the conic through $\mathfrak{U}$, $\mathcal{R}$, $\mathcal{B}$ and tangent to the normalizers through $\mathcal{R}$ and $\mathcal{B}$. Construct $\overline{\mathcal{R}A}$ meeting $\mathcal{L}_\infty$ in $\mathcal{R}'$ and construct $\overline{\mathcal{B}A}$ meeting $\mathcal{L}_0$ in $\mathcal{B}'$. By theorem 9.1.4, we know that $I \in \overline{\mathcal{R}'\mathcal{B}'}$. $A \in \overline{\mathcal{R}\mathcal{R}'}$, $I \in \overline{\mathcal{R}'\mathcal{B}'}$ and $A \in \overline{\mathcal{B}'\mathcal{B}}$; therefore, by theorem 6.2.1, $AI^{-1}A = A^2 \in \overline{\mathcal{R}\mathcal{B}} = \mathcal{L}$ and consequently $A \in R(\mathcal{L})$.

$R(\mathcal{L})$ is therefore the required conic.

Q.E.D.

Let us apply the collineation $f_A : X \to AX$ to our conic $R(\mathcal{L})$. $f_A(R(\mathcal{L}))$ will be a conic passing through $\mathfrak{U}$ and the points of intersection of $A\mathcal{L}$ with $\mathcal{L}_0$ and $\mathcal{L}_\infty$; the tangents to the conic at these last two points will meet in A.

$$\begin{aligned} f_A(R(\mathcal{L})) &= \{AX \mid X^2 \in \mathcal{L}\} \\ &= \{Y \mid (A^{-1}Y)^2 \in \mathcal{L}\} \\ &= \{Y \mid YA^{-1}Y \in A\mathcal{L}\}. \end{aligned}$$

Defining $R_A(\mathcal{L}) = \{X \mid XA^{-1}X \in \mathcal{L}\}$, we can now state the following theorem.

Theorem 2: If $\mathcal{L}$ is any line not passing through $\mathfrak{U}$ or A, then $R_A(\mathcal{L})$ is a conic passing through $\mathfrak{U}$ and the two points of intersection of $\mathcal{L}$ with $\mathcal{L}_0$ and $\mathcal{L}_\infty$, and the tangents at these last two points meet in A.

Note: $R(\mathfrak{X}) = R_I(\mathfrak{X})$.

A conic is entirely determined by three points and the tangents at two of them. We may choose A arbitrarily in $\mathfrak{G}$ and $\mathfrak{X}$ arbitrarily, as long as it does not pass through $\mathfrak{U}$ or A. We conclude that we obtain all conics passing through $\mathfrak{U}$, but not tangent to $\mathfrak{X}_0$ or $\mathfrak{X}_\infty$.

Note that the class of conics $R_A(\mathfrak{X})$ is inverse to the class $S_A(\mathfrak{X})$ in the sense that $\mathfrak{X} \subseteq R_A(S_A(\mathfrak{X}))$ and $S_A(R_A(\mathfrak{X})) \subseteq \mathfrak{X}$.

9.5. Properties of $R_A(\mathfrak{X})$.

(i) If $\mathfrak{X} \neq \mathfrak{X}'$ or $A \neq A'$, then $R_A(\mathfrak{X}) \neq R_{A'}(\mathfrak{X}')$.

(ii)
$$\begin{aligned} f_B(R_A(\mathfrak{X})) &= \{BX \mid XA^{-1}X \in \mathfrak{X}\} \\ &= \{Y \mid B^{-1}YA^{-1}B^{-1}Y \in \mathfrak{X}\} \\ &= \{Y \mid Y(BA)^{-1}Y \in B\mathfrak{X}\} \\ &= R_{BA}(B\mathfrak{X}). \end{aligned}$$

(iii) Similarly, $g_B(R_A(\mathfrak{X})) = R_{AB}(\mathfrak{X}B)$.

(iv) Let φ be any automorphism of $\mathfrak{G}$.
$$\begin{aligned} \varphi(R_A(\mathfrak{X})) &= \{\varphi(X) \mid XA^{-1}X \in \mathfrak{X}\} \\ &= \{\varphi(X) \mid \varphi(X)\varphi(A^{-1})\varphi(X) \in \varphi(\mathfrak{X})\} \end{aligned}$$

$$= \{Y \mid Y\varphi(A)^{-1}Y \in \varphi(\mathcal{L})\}$$
$$= R_{\varphi(A)}(\varphi(\mathcal{L})).$$

(v) $h(R_A(\mathcal{L})) = \{X^{-1} \mid XA^{-1}X \in \mathcal{L}\}$
$$= \{X^{-1} \mid X^{-1}AX^{-1} \in \mathcal{L}^{-1}\}$$
$$= \{Y \mid YAY \in \mathcal{L}^{-1}\}$$
$$= R_{A^{-1}}(\mathcal{L}^{-1}).$$

(vi) Consider the conic $R(\mathcal{L})$ for some $\mathcal{L}$ not passing through $\mathfrak{U}$ or I. $R(\mathcal{L})$ passes through $\mathfrak{U}$ and so does the line $\mathfrak{g}V$, $V^2 = I$. We have the following theorem.

Theorem 1: The line $\mathfrak{g}V$ is tangent to the conic $R(\mathcal{L})$ at the point $\mathfrak{U}$.

Proof: Suppose $R(\mathcal{L})$ and $\mathfrak{g}V$ meet in a point other than $\mathfrak{U}$, say A. Then $A^2 \in \mathcal{L}$, but $A \in \mathfrak{g}V \Rightarrow A^2 = I$ and $I \notin \mathcal{L}$; consequently, $\mathfrak{g}V$ is tangent to $R(\mathcal{L})$ at $\mathfrak{U}$.

Q.E.D.

Note that theorem 1 is independent of $\mathcal{L}$ and now apply the collineation $f_A: X \to AX$. We obtain the following, more general, theorem.

Theorem 2: If $\mathcal{L}$ is any line not passing through $\mathfrak{U}$ or A, then the tangent to the conic $R_A(\mathcal{L})$ at the point $\mathfrak{U}$

is the line $\mathfrak{g}$AV.

Corollary: If $\mathcal{L}$ and $\mathcal{L}'$ are two lines not passing through $\mathfrak{U}$ or A, then the conics $R_A(\mathcal{L})$ and $R_A(\mathcal{L}')$ are tangent to each other at $\mathfrak{U}$.

9.6. Relations between $S_A(\mathcal{L})$ and $R_A(\mathcal{L})$.

We may notice that the properties of $S_A(\mathcal{L})$ and $R_A(\mathcal{L})$ are somehow related. We shall now investigate this relation.

Let $\mathcal{L}$ be a line not passing through $\mathfrak{U}$ or A. Let the intersection of $\mathcal{L}$ with $\mathfrak{g}$A be HA. (See figure 21.)

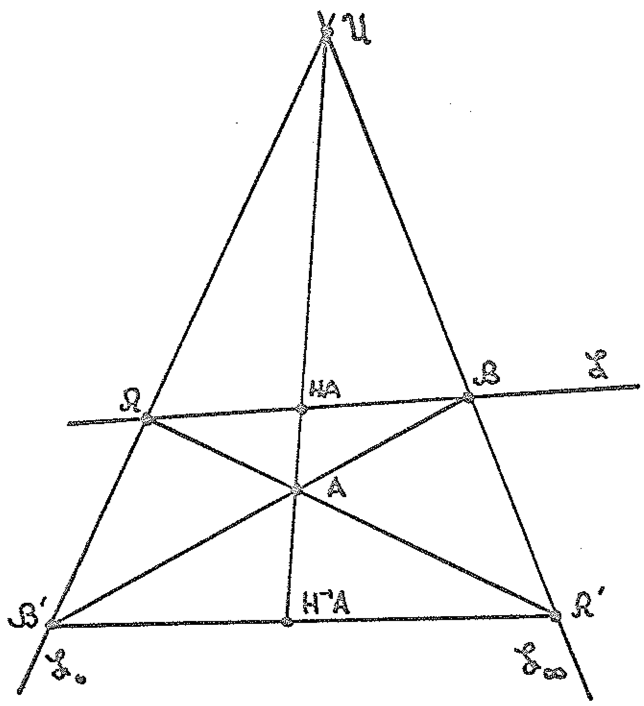

Figure 21.

Let $\mathfrak{g}$ cut $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$ in $\mathcal{R}$ and $\mathcal{B}$ respectively. Construct $\overline{\mathcal{R}A}$ cutting $\mathfrak{g}_\infty$ in $\mathcal{R}'$, and $\overline{\mathcal{B}A}$ cutting $\mathfrak{g}_0$ in $\mathcal{B}'$. $A \in \overline{\mathcal{B}'\mathcal{B}}$, $HA \in \overline{\mathcal{B}\mathcal{R}}$ and $A \in \overline{\mathcal{R}\mathcal{R}'}$; therefore, by theorem 6.2.1, $A(HA)^{-1}A = H^{-1}A \in \overline{\mathcal{B}'\mathcal{R}'}$. $H^{-1}A \in \mathfrak{g}A$ and therefore, by theorem 5.7.2, there exists a unique central collineation, σ, with centre $H^{-1}A$ and axis $\mathfrak{g}$, which transforms A into $\mathcal{U}$, i.e. $\sigma(A) = \mathcal{U}$. $\overline{\mathcal{R}'\mathcal{B}'}$ passes through $H^{-1}A$, the centre, and consequently $\sigma(\overline{\mathcal{R}'\mathcal{B}'}) = \overline{\mathcal{R}'\mathcal{B}'}$. $\sigma(\overline{\mathcal{R}A}) = \overline{\mathcal{R}\mathcal{U}} = \mathfrak{g}_0$ and $\sigma(\overline{\mathcal{B}A}) = \overline{\mathcal{B}\mathcal{U}} = \mathfrak{g}_\infty$.

$\mathcal{R}' \in \overline{\mathcal{R}'\mathcal{B}'} \cap \overline{\mathcal{R}A} \Rightarrow \sigma(\mathcal{R}') \in \overline{\mathcal{R}'\mathcal{B}'} \cap \mathfrak{g}_0 \Rightarrow \sigma(\mathcal{R}') = \mathcal{B}'$.

$\mathcal{B}' \in \overline{\mathcal{R}'\mathcal{B}'} \cap \overline{\mathcal{B}A} \Rightarrow \sigma(\mathcal{B}') \in \overline{\mathcal{R}'\mathcal{B}'} \cap \mathfrak{g}_\infty \Rightarrow \sigma(\mathcal{B}') = \mathcal{R}'$.

Now $\overline{\mathcal{R}\mathcal{B}'} = \mathfrak{g}_0$, therefore $\sigma(\mathfrak{g}_0) = \sigma(\overline{\mathcal{R}\mathcal{B}'}) = \overline{\mathcal{R}\mathcal{R}'}$.

Also $\overline{\mathcal{B}\mathcal{R}'} = \mathfrak{g}_\infty$, therefore $\sigma(\mathfrak{g}_\infty) = \sigma(\overline{\mathcal{B}\mathcal{R}'}) = \overline{\mathcal{B}\mathcal{B}'}$.

$\mathcal{U} \in \mathfrak{g}_0 \cap \mathfrak{g}_\infty \Rightarrow \sigma(\mathcal{U}) \in \sigma(\mathfrak{g}_0) \cap \sigma(\mathfrak{g}_\infty) = \overline{\mathcal{R}\mathcal{R}'} \cap \overline{\mathcal{B}\mathcal{B}'}$.

Therefore, $\sigma(\mathcal{U}) = A$.

Since $S_A(\mathfrak{g})$ is the conic through $\mathcal{R}$, $\mathcal{B}$, A and tangent to $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$, then $\sigma(S_A(\mathfrak{g}))$ is the conic through $\mathcal{R}$, $\mathcal{B}$, $\mathcal{U}$ and tangent to $\overline{\mathcal{R}\mathcal{R}'}$ and $\overline{\mathcal{B}\mathcal{B}'}$, i.e.

$$\sigma(S_A(\mathfrak{g})) = R_A(\mathfrak{g}).$$

Also $R_A(\mathfrak{g})$ is the conic through $\mathcal{R}$, $\mathcal{B}$, $\mathcal{U}$ and tangent to $\overline{\mathcal{R}\mathcal{R}'}$ and $\overline{\mathcal{B}\mathcal{B}'}$; therefore $\sigma(R_A(\mathfrak{g}))$ is the conic through $\mathcal{R}$, $\mathcal{B}$, A and tangent to $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$, i.e.

$$\sigma(R_A(\mathfrak{g})) = S_A(\mathfrak{g}).$$

From the preceding considerations we see that the tangent to $R_A(\mathfrak{L})$ at $\mathfrak{U}$, the tangent to $S_A(\mathfrak{L})$ at A, and $\mathfrak{L}$ are concurrent, i.e. the tangent to $S_A(\mathfrak{L})$ at A meets $\mathfrak{L}$ in a point of $\mathfrak{G}AV$, which agrees with theorem 9.3.1.

Consider the tangents to $S_A(\mathfrak{L})$ through the point $H^{-1}A$. σ transforms $S_A(\mathfrak{L})$ into $R_A(\mathfrak{L})$ and transforms a tangent to $S_A(\mathfrak{L})$ into a tangent to $R_A(\mathfrak{L})$; but all lines through $H^{-1}A$ remain fixed and therefore the tangent to $S_A(\mathfrak{L})$ through $H^{-1}A$ is also tangent to $R_A(\mathfrak{L})$. Thus, we have the following theorem.

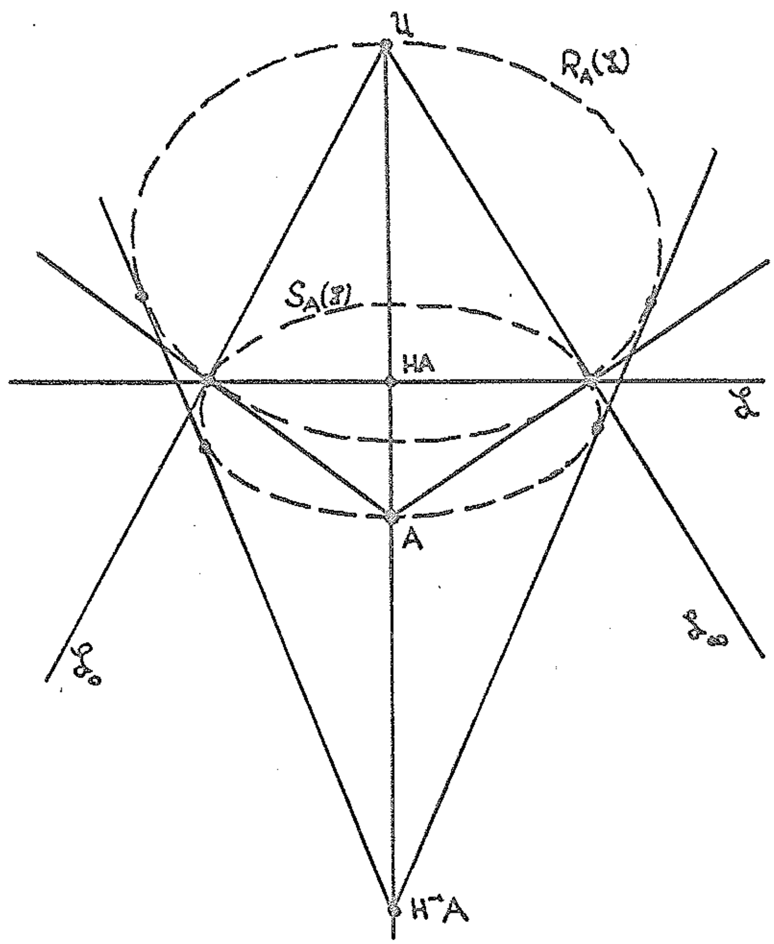

Figure 22.

Theorem: Let $\mathfrak{X}$ be any line not through $\mathfrak{U}$ or A and meeting $\mathfrak{g}A$ in HA. Then the common tangents to $R_A(\mathfrak{X})$ and $S_A(\mathfrak{X})$ meet in $H^{-1}A$. (See figure 22.)

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