ORTHOGONAL
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DOUBLE
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CATEGORIES
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DOUBLE CATEGORIES

"CATEGORIES W. 2 KINDS OF MORPHISMS"

- OBJECTS
- HORIZONTAL ARROWS (FORM A CAT)
- VERTICAL ARROWS (" " ")
- CELLS (FORM CAT IN 2 WAYS)

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha'"] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta'"] & C' \arrow[d] \\
A'' \arrow[r] & B'' \arrow[r] & C''
\end{tikzcd}$$

Ex1: $\underline{A}$ CAT $\qquad \square \underline{A}$ (COMM SQ IN $\underline{A}$)

(VAR $\boxed{\lrcorner}\,\underline{A}$)

$\underline{\underline{A}}$ 2-CAT $\qquad \boxed{\swarrow}\,\underline{\underline{A}}$: CELL
$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}$$

Ex2: SETS, FNS, RELATIONS
$$\begin{tikzcd}
A \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\leq"] & B \arrow[d] \\
A' \arrow[r, dash] & B'
\end{tikzcd}$$

(VAR SETS, FNS, SPANS ; CATS, FUNCT, PROFS)

ORTHOGONAL ADJOINTS

u IS A VERTICAL ADJOINT FOR f IF

THERE ARE CELLS α AND β

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{S.T} \quad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u"] \\
A \arrow[r, equal] & A
\end{tikzcd}
= 1_u
$$

AND

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, equal] \arrow[d, "u"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r, "f"'] & B
\end{tikzcd}
= id_f .
$$

<u>Ex</u>: $\square\underline{A}$, $u = f^{-1}$

$\boxed{\Downarrow}\underline{\underline{A}}$, u LEFT ADJ TO f.

SETS, FNS, REL : EVERY f HAS A VERT ADJ f^{o}

SAME FOR SPANS AND PROFS

$F : \underline{A} \to \underline{B} \;\Rightarrow\; F^* = \underline{B}(F-,-) : \underline{A}^{op} \times \underline{B} \to \underline{\underline{Set}}$

F^* IS VERT ADJ TO F

COMPANIONS

v IS A VERTICAL COMPANION FOR f IF THERE ARE CELLS ε AND η

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{S.T.} \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v"] \\
A \arrow[r, "f"'] \arrow[d, "v"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
= 1_v
$$

AND

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[r, "f"] \arrow[d, "v"] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
= id_f
$$

THIS IS THE (op) DUAL NOTION TO VERT ADJ.

<u>EX</u>: IN $\square\underline{A}$ AND $\boxed{\checkmark}\underline{\underline{A}}$ EVERY f HAS A COMPANION VIZ. f

IN $\mathbb{R}\mathrm{EL}$ EVERY f HAS A COMP: f VIEWED AS A REL

IN $\mathbb{P}\mathrm{ROF}$: $F_* = \underline{B}(-,F-) : \underline{B}^{op} \times \underline{A} \to \underline{\mathrm{SET}}$

<u>SPECIAL CELLS</u>:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

GET A 2-CAT (OR BI-CAT IN CASE OF PSEUDOUBLE CAT)

<u>PROP</u>: (UNIQUENESS) IF u AND u' ARE VERT ADJOINTS OF f, THEN $u \cong u'$ BY SPECIAL ISO.

(COMPOSABILITY) IF $A \xrightarrow{f} B \xrightarrow{h} C$ HAVE VERTICAL ADJOINTS $C \xrightarrow{v} B \xrightarrow{u} A$ THEN $A \xrightarrow{fh} C$ HAS VERTICAL ADJ $C \xrightarrow{v \circ u} A$.

- IF f HAS VERT ADJ u AND COMPANION v, THEN $v \dashv u$ IN VERT 2-CAT.

PROOF OF LAST POINT

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

(VERT ADJ)

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

(COMPANION)

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v"] \\
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

(UNIT) (COUNIT)

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[r, equal] \arrow[d, "v"] \arrow[dr, phantom, "1_v"] & A \arrow[d, "v"] \\
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, equal] \arrow[d, "u"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, equal] \arrow[d, "v"'] \arrow[dr, phantom, "1_v"] & A \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
= 1_v
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "1_u"] & B \arrow[r, equal] \arrow[d, "u"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, equal] \\
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[r, equal] \arrow[d, "u"] \arrow[dr, phantom, "1_u"] & B \arrow[d, "u"] \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
= 1_u
$$

THE 3×2 EQUALITIES
(AKA TRIANGLE EQUALITIES)

PROP: (ADJOINT FLIPPING)

LET
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
BE AN ADJUNCTION.

THEN THERE ARE BIJECTIONS OF CELLS

$$\begin{tikzcd} Y \arrow[r] \arrow[ddd] & C \arrow[d] \\ & B \arrow[d, "u"] \arrow[l, phantom, "\varphi"] \\ & A \\ X \arrow[ur] & \end{tikzcd} \longleftrightarrow \begin{tikzcd} Y \arrow[rr] \arrow[d] \arrow[drr, phantom, "\overline{\varphi}"] & & C \arrow[d] \\ X \arrow[r] & A \arrow[r, "f"'] & B \end{tikzcd}$$

NAT W.R.T. COMP ON LEFT & TOP, I.E.
$\overline{\xi\varphi} = \xi\overline{\varphi}$ AND $\overline{\gamma\cdot\varphi} = \gamma\cdot\overline{\varphi}$.

CONVERSELY, GIVEN SUCH A NATURAL BIJECTION, u IS ADJOINT TO f.

DUALLY
$$\begin{tikzcd} B \arrow[r] \arrow[d, "u"'] & X \arrow[ddd] \\ A \arrow[d] \arrow[r, phantom, "\psi"] & \\ Z \arrow[d, phantom] & \\ & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d, "u"'] & X \arrow[dd] \\ A \arrow[d] \arrow[r, phantom, "\psi"] & \\ Z \arrow[r] & Y \end{tikzcd} \longleftrightarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, phantom, "\tilde{\psi}"] & X \arrow[d] \\ Z \arrow[rr] & & Y \end{tikzcd}$$

.

ALSO DUALLY (SLIDING)

v IS A COMPANION OF f IFF THERE IS A NATURAL BIJECTION

$$\begin{tikzcd} X \arrow[r] \arrow[dd] & A \arrow[d, "v"] \\ & B \arrow[d] \arrow[l, phantom, "\theta"] \\ Y \arrow[r] & C \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, phantom, "\bar{\theta}"] & B \arrow[d] \\ Y \arrow[rr] & & C \end{tikzcd}$$

IFF THERE IS A BIJECTION

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[dd] \\ A \arrow[d, "v"'] \arrow[r, phantom, "\rho"] & \\ B \arrow[r] & Y \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} Z \arrow[rr] \arrow[d] & & X \arrow[d] \\ A \arrow[r, "f"'] & B \arrow[r] \arrow[u, phantom, "\tilde{\rho}"] & Y \end{tikzcd}$$

PROP: $\boxed{\swarrow} : \text{2-CAT} \longrightarrow \text{DOUBCAT}$

HAS A RIGHT ADJ WHICH ASSOCIATES TO A DOUBLE CAT ITS 2-CAT OF MATCHED PAIRS.

THE DOUBLE CATEGORY $\mathbb{M}$ON

OBJ: MONOIDAL CATEGORIES $\underline{V}, \underline{W}, \ldots$

HORIZ: MONOIDAL FUNCT $M : \underline{V} \to \underline{W}$

$$\mu(V_1, V_2) : M(V_1) \otimes M(V_2) \longrightarrow M(V_1 \otimes V_2)$$

$$\eta : I \longrightarrow FI$$

+ ASSOC & UNIT (A $\underline{V}$-GRADED MONOID IN $\underline{W}$)

VERT: COMONOIDAL FUNCT $F : \underline{V} \nrightarrow \underline{W}$

$$\delta(V_1, V_2) : F(V_1 \otimes V_2) \longrightarrow F(V_1) \otimes F(V_2)$$

$$\varepsilon : F(I) \longrightarrow I$$

DOUB:

$$\begin{tikzcd} \underline{V} \arrow[r, "M"] \arrow[d, "F"'] \arrow[dr, phantom, "\varphi"] & \underline{W} \arrow[d, "G"] \\ \underline{V}' \arrow[r, "M'"'] & \underline{W}' \end{tikzcd}$$

$$\varphi : GM \longrightarrow M'F$$

S.T. . .

$$
\begin{tikzcd}[column sep=small]
& GMV_1 \otimes GMV_2 \arrow[r, "\varphi"] & M'FV_1 \otimes M'FV_2 \arrow[dr, "\mu"] & \\
G(MV_1 \otimes MV_2) \arrow[ur, "\delta"] \arrow[dr, "G\mu"'] & & & M'(FV_1 \otimes FV_2) \\
& GM(V_1 \otimes V_2) \arrow[r, "\varphi"'] & M'F(V_1 \otimes V_2) \arrow[ur, "M'\delta"'] &
\end{tikzcd}
$$

+ PENTAGON FOR UNITS.

PROP: ▶/MON IS A DOUBLE CAT.

- VERTICAL ADJOINTS ARE MONOIDAL ADJOINTS.
- M HAS A COMPANION IFF IT IS A STRONG MON FUNCTOR.