

**ORTHOGONAL
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CATEGORIES
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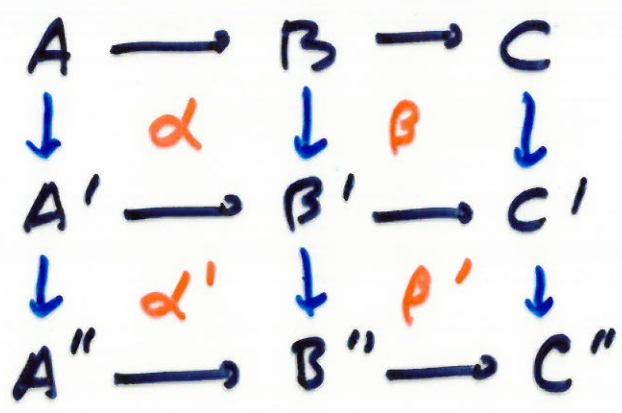
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DOUBLE CATEGORIES

"CATEGORIES W. 2 KINDS OF MORPHISMS"

- OBJECTS
- HORIZONTAL ARROWS (FORM A CAT)
- VERTICAL ARROWS (" " ")
- CELLS (FORM CAT IN 2 WAYS)

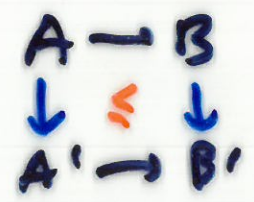


Ex1: A CAT \square A (COMM SQ IN A)
 (VAR \square A)

A 2-CAT \square A : CELL

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \downarrow & \alpha & \downarrow \\
 A' & \longrightarrow & B'
 \end{array}$$

Ex2: SETS, FNS, RELATIONS



(VAR SETS, FNS, SPANS ; CATS, FUNCT, PROFS)

ORTHOGONAL ADJOINTS

μ IS A VERTICAL ADJOINT FOR f IF THERE ARE CELLS α AND β

$$\begin{array}{ccc}
 A \xrightarrow{f} B & B = B & B = B \\
 \parallel \alpha \downarrow \mu & \mu \downarrow \beta \parallel & \mu \downarrow \beta \parallel \\
 A = A & A \xrightarrow{f} B & A \xrightarrow{f} B = 1_{\mu} \\
 & & \parallel \alpha \downarrow \mu \\
 & & A = A
 \end{array}$$

AND

$$\begin{array}{ccc}
 A \xrightarrow{f} B = B & & \\
 \parallel \alpha \downarrow \mu & \beta \parallel & = id_f \\
 A = A \xrightarrow{f} B & &
 \end{array}$$

Ex: $\square A$, $\mu = f^{-1}$

$\square A$, μ LEFT ADJ TO f .

SETS, FNS, REL : EVERY f HAS A VERT ADJ f^o

SAME FOR SPANS AND PROFS

$F: \underline{A} \rightarrow \underline{B} \Rightarrow F^* = \underline{B}(F-, -): \underline{A}^{\text{op}} \times \underline{B} \rightarrow \underline{\text{SET}}$

F^* IS VERT ADJ TO F

COMPANIONS

v IS A VERTICAL COMPANION FOR f IF THERE ARE CELLS ϵ AND η

$$\begin{array}{ccc}
 A \xrightarrow{f} B & A = A & A = A \\
 \downarrow \epsilon \parallel & \parallel \eta \downarrow v & \parallel \eta \downarrow v \\
 B = B & A \xrightarrow{f} B & A \xrightarrow{f} B = 1_v \\
 & & \downarrow \epsilon \parallel \\
 & & B = B
 \end{array}
 \text{ S.T. }$$

AND

$$\begin{array}{ccc}
 A = A \xrightarrow{f} B & & \\
 \parallel \eta \downarrow v \epsilon \parallel & = & \text{id}_f \\
 A \xrightarrow{f} B \longleftarrow B & &
 \end{array}$$

THIS IS THE (OP) DUAL NOTION TO VERT ADJ.

EX: IN $\square A$ AND $\square A$ EVERY f HAS A COMPANION VIZ. f

IN REL EVERY f HAS A COMP: f VIEWED AS A REL

IN PROF: $F_* = \underline{B}(-, f -): \underline{B}^{op} \times \underline{A} \rightarrow \underline{SET}$

SPECIAL CELLS :

$$\begin{array}{ccc} A & = & A \\ \mu \downarrow & \alpha & \downarrow \nu \\ B & = & B \end{array}$$

GET A 2-CAT (OR BI-CAT IN CASE OF PSEUDOUBLE CAT)

PROP: (UNIQUENESS) IF μ AND μ' ARE VERT ADJOINTS OF f , THEN $\mu \cong \mu'$ BY SPECIAL ISO.

(COMPOSABILITY) IF $A \xrightarrow{f} B \xrightarrow{h} C$ HAVE VERTICAL ADJOINTS $C \xrightarrow{v} B \xrightarrow{u} A$ THEN $A \xrightarrow{fh} C$ HAS VERTICAL ADJ $C \xrightarrow{v \cdot u} A$.

- IF f HAS VERT ADJ μ AND COMPANION ν , THEN $\nu \dashv \mu$ IN VERT 2-CAT.

PROOF OF LAST POINT

$$\begin{array}{ccc}
 A \xrightarrow{f} B & B = B \\
 \parallel \alpha \downarrow u & u \downarrow \beta \parallel \\
 A = A & A \xrightarrow{f} B
 \end{array}$$

(VERT ADJ)

$$\begin{array}{ccc}
 A = A & A \xrightarrow{f} B \\
 \parallel \eta \downarrow v & v \downarrow \epsilon \parallel \\
 A \xrightarrow{f} B & B = B
 \end{array}$$

(COMPANION)

$$\begin{array}{ccc}
 A = A \\
 \parallel \eta \downarrow v \\
 A \xrightarrow{f} B \\
 \parallel \alpha \downarrow u \\
 A = A
 \end{array}$$

(UNIT)

$$\begin{array}{ccc}
 B = B \\
 u \downarrow \beta \parallel \\
 A \xrightarrow{f} B \\
 v \downarrow \epsilon \parallel \\
 B = B
 \end{array}$$

(COUNIT)

$$\begin{array}{ccc}
 A = A = A \\
 \parallel \eta \downarrow v & 1_v \downarrow v \\
 A \rightarrow B = B \\
 \parallel \alpha \downarrow u & \beta \parallel = 1_v \\
 A = A \rightarrow B \\
 v \downarrow 1_v & \downarrow v \epsilon \parallel \\
 B = B = B
 \end{array}$$

$$\begin{array}{ccc}
 B = B = B \\
 u \downarrow 1_u & \downarrow u \beta \parallel \\
 A = A \rightarrow B \\
 \parallel \eta \downarrow v & \epsilon \parallel = 1_u \\
 A \rightarrow B = B \\
 \parallel \alpha \downarrow u & 1_u \downarrow u \\
 A = A = A
 \end{array}$$

THE 3x2 EQUALITIES
(AKA TRIANGLE EQUALITIES)

PROP: (ADJOINT FLIPPING)

LET $A \xrightarrow{f} B$ $B = B$ BE AN
 $\parallel \alpha \downarrow u \quad u \downarrow \beta \parallel$ ADJUNCTION.
 $A = A \quad A \xrightarrow{f} B$

THEN THERE ARE BIJECTIONS OF CELLS

$$\begin{array}{ccc}
 Y \longrightarrow C & & Y \longrightarrow C \\
 \downarrow \varphi & \leftrightarrow & \downarrow \bar{\varphi} \\
 X \longrightarrow A & & X \longrightarrow A \xrightarrow{f} B \\
 & & \downarrow L
 \end{array}$$

NAT W.R.T. COMP ON LEFT & TOP, I.E.

$$\overline{\xi \varphi} = \xi \bar{\varphi} \quad \text{AND} \quad \overline{\gamma \cdot \varphi} = \gamma \cdot \bar{\varphi}.$$

CONVERSELY, GIVEN SUCH A NATURAL BIJECTION, u IS ADJOINT TO f .

DUALLY

$$\begin{array}{ccc}
 B \longrightarrow X & & A \xrightarrow{f} B \longrightarrow X \\
 u \downarrow & \leftrightarrow & \downarrow \bar{\psi} \\
 A \xrightarrow{\psi} & & Z \longrightarrow Y \\
 \downarrow & & \downarrow L \\
 Z \longrightarrow Y & &
 \end{array}$$

ALSO DUALLY (SLIDING)

v IS A COMPANION OF f IFF THERE IS A NATURAL BIJECTION

$$\begin{array}{ccc}
 X \rightarrow A & & X \rightarrow A \xrightarrow{f} B \\
 \downarrow \theta & \Downarrow v & \downarrow \theta \\
 Y \rightarrow C & \Leftrightarrow & Y \longrightarrow C
 \end{array}$$

IFF THERE IS A BIJECTION

$$\begin{array}{ccc}
 Z \rightarrow X & & Z \longrightarrow X \\
 \downarrow \rho & \Leftrightarrow & \downarrow \tilde{\rho} \\
 \begin{array}{ccc}
 v \downarrow & & \\
 A & \xrightarrow{\rho} & B \\
 \downarrow & & \downarrow \\
 B & \rightarrow & Y
 \end{array}
 \end{array}$$

PROP: $\boxed{\swarrow} : 2\text{-CAT} \longrightarrow \text{DOUB CAT}$

HAS A RIGHT ADJ WHICH ASSOCIATES TO A DOUBLE CAT ITS 2-CAT OF MATCHED PAIRS.

THE DOUBLE CATEGORY /MON

OBJ: MONOIDAL CATEGORIES $\underline{V}, \underline{W}, \dots$

HORIZ: MONOIDAL FUNCT $M: \underline{V} \rightarrow \underline{W}$

$$\mu(V_1, V_2): M(V_1) \otimes M(V_2) \longrightarrow M(V_1 \otimes V_2)$$

$$\eta: I \longrightarrow FI$$

+ ASSOC & UNIT (A \underline{V} -GRADED MONOID IN \underline{W})

VERT: COMONOIDAL FUNCT $F: \underline{V} \rightarrow \underline{W}$

$$\delta(V_1, V_2): F(V_1 \otimes V_2) \longrightarrow F(V_1) \otimes F(V_2)$$

$$\epsilon: F(I) \longrightarrow I$$

DOUB:

$$\begin{array}{ccc} \underline{V} & \xrightarrow{M} & \underline{W} \\ F \downarrow & \varphi & \downarrow G \\ \underline{V}' & \xrightarrow{M'} & \underline{W}' \end{array}$$

$$\varphi: GM \longrightarrow M'F$$

S.T. ...

$$\begin{array}{ccc}
 & G M V_1 \otimes G M V_2 & \xrightarrow{\varphi} & M' F V_1 \otimes M' F V_2 \\
 \delta \nearrow & & & \searrow \mu \\
 G(M V_1 \otimes M V_2) & & & M'(F V_1 \otimes F V_2) \\
 G \mu \searrow & & & \nearrow M' \delta \\
 & G M (V_1 \otimes V_2) & \xrightarrow{\varphi} & M' F (V_1 \otimes V_2)
 \end{array}$$

+ PENTAGON FOR UNITS.

PROP: $\triangleright / \text{MON}$ IS A DOUBLE CAT.

\triangleright VERTICAL ADJOINTS ARE MONOIDAL ADJOINTS.

\triangleright M HAS A COMPANION IFF IT IS A STRONG MON FUNCTOR.