ABSOLUTENESS PROPERTIES IN CATEGORY THEORY

by

Robert Paré

A THESIS SUBMITTED TO THE FACULTY
OF GRADUATE STUDIES AND RESEARCH IN
PARTIAL FULFILMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

Department of Mathematics
McGill University
Montreal

June 1969

INTRODUCTION

The aim of this thesis is to investigate those properties of diagrams in a category which are preserved by a given class of functors. We call these absoluteness properties.

In the past, people were interested in classes of functors which preserve a given class of properties, e.g. monopreserving functors, continuous functors, exact functors, etc. But there is a galois correspondence between classes of functors and classes of properties of diagrams, which on the one hand associates to a given class of properties the class of all functors preserving these properties, and on the other hand associates to a given class of functors the class of all properties preserved by these functors. We propose to initiate the study of this second association and to show its relevance in category theory.

Since we will be studying those properties of diagrams which are invariant under certain classes of transformations, we could have titled this thesis "Geometry in the Category of Categories".

Chapter I deals with <u>total absoluteness</u>, the study of properties preserved by all functors. We obtain characterizations in terms of equations which tell us when diagrams possess some of the usual properties in category theory absolutely.

In Chapter II we apply some of the concepts of Chapter I to the theory of <u>triples and their algebras</u>. It becomes apparent here that absoluteness properties are fundamental.

Chapter III is devoted to the study of <u>additive absoluteness</u>.

We give necessary and sufficient conditions for certain properties of diagrams to be preserved by all additive functors. The conditions take the form of equations involving composition, addition, subtraction, identity maps, and 0. Certain examples of additive absoluteness are known in homological algebra, but we shall not go into this here.

Chapter IV studies **Cat**-absoluteness; properties of functors which are preserved by all hyperfunctors of **Cat** (the category of small categories) into itself. The characterizations are in the form of equations involving functors, natural transformations, the four kinds of composition, and both kinds of identities. We shall see that these notions are closely associated with adjointness and tripleableness.

This study is by no means complete. It is apparent that one should investigate *relative absoluteness*, where the word relative is taken in the sense of relative category theory of Eilenberg-Kelly [5]. Then the results of Chapter IV would probably fit under *hyper-absoluteness*. The relationship of absoluteness with homological algebra should also be studied.

The reader is assumed to be familiar with elementary category theory as can be found in the first few chapters of Mitchell [19] or Freyd [7]. The definitions and results which are not easily accessible in the literature are given in the text when they are needed.

I would like to thank Michael Barr for many stimulating conversations. I would especially like to thank my adviser, Professor Lambek, for his guidance and encouragement.

CONTENTS

CHAPTER I

TOTAL ABSOLUTENESS

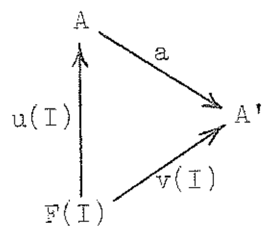
A property of diagrams is called totally absolute if it is preserved by all functors. For example "a is an isomorphism" is a totally absolute property. Some other trivial totally absolute properties are "commutivity of a diagram", "a is an identity map", "there exists a map from A to A' ", etc.

In this chapter we consider some of the usual properties of diagrams in a category and establish necessary and sufficient conditions for these properties to be totally absolute. We obtain equations which imply that a diagram has a given property, and these equations are in the most general form possible.

For the rest of chapter I, when we talk of absoluteness, we mean total absoluteness.

We will not state dual results unless it is not obvious what they should be.

Following Lambek's example [9], we call limits and colimits "infimum" and "supremum" respectively. In fact we refer to them by their usual abbreviations "inf" and "sup". Thus if $F: \underline{I} \longrightarrow \underline{A}$ is a functor, to say that $\sup F = (A, u)$ means that u is a natural transformation from F to the constant functor at A such that if $v(I):F(I) \longrightarrow A'$ is natural in I, then there exists a unique map $a:A \longrightarrow A'$ such that the following diagram commutes for every $I \in |\underline{I}|$ = objects of $\underline{I}$.



§1. ABSOLUTE EPIMORPHISMS

(1.1) PROPOSITION. Let $\underline{A}$ be a category and let $a: A \longrightarrow A'$ be a map in $\underline{A}$. Then a is an absolute epi if and only if there exists a map $a': A' \longrightarrow A$ such that

$$A' \xrightarrow{a'} A \xrightarrow{a} A' = aa' = A'.$$

Proof. The sufficiency of the condition is obvious.

Assume that a is an absolute epi. Apply the hom functor $[A', -]: \underline{A} \longrightarrow \underline{S}$ to a, where $\underline{S}$ = category of sets.

$$[A', a]: [A', A] \longrightarrow [A', A']$$

is therefore epi in $\underline{S}$. But in $\underline{S}$ epi means onto. Therefore there exists $a' \in [A', A]$ such that $[A', a](a') = A'$. Thus $aa' = A'$. ∎

(1.2) DEFINITION. Let $a_i: A_i \longrightarrow A'$ be a family of morphisms of a category $\underline{A}$. $\{a_i\}$ is called a joint epi if $xa_i = ya_i$ for all i, implies that $x = y$.

If $\sup F = (A, u)$ then $\{u(I) \mid I \in |\underline{I}|\}$ is a joint epi. In our characterization of absolute sups we will need the following result.

(1.3) PROPOSITION. Let $a_i : A_i \longrightarrow A'$ be a family of morphisms of a category $\underline{A}$. Then $\{a_i\}$ is an absolute joint epi if and only if one of the a_i is an absolute epi.

Proof. The sufficiency is obvious.

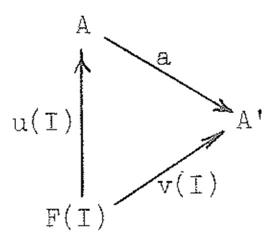
Assume that $\{a_i\}$ is an absolute joint epi. Apply the functor $[A', -]$. Then

$$[A', a_i]: [A', A_i] \longrightarrow [A', A']$$

is a joint epi in $\underline{S}$. But a family of maps in $\underline{S}$ is a joint epi if and only if the union of the images is equal to the codomain. Thus for some index j, there exists $a' \in [A', A_j]$ such that $[A', a_j](a') = A'$. That is to say $a_j a' = A'$.

§2. ABSOLUTE WEAK SUPREMA

(2.1) DEFINITION. Let $F: \underline{I} \longrightarrow \underline{A}$ be a functor. We say that (A, u) is a weak sup of F if $u(I): F(I) \longrightarrow A$ is natural in I and if for any $v(I): F(I) \longrightarrow A'$ which is natural in I, there exists a map $a: A \longrightarrow A'$ such that



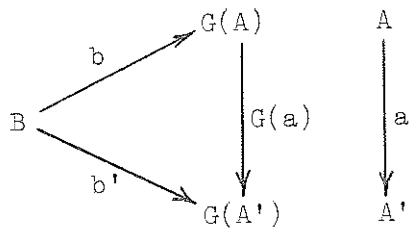
commutes for all I.

Thus we have dropped the uniqueness from the definition of sup.

(2.2) DEFINITION. Two objects A and A' are said to be <u>connected</u> in a category $\underline{A}$ if there exist finitely many objects $A_o, A_1, \ldots, A_n$ such that $A = A_o$, $A' = A_n$, and $[A_{i-1}, A_i] \cup [A_i, A_{i-1}] \neq \emptyset$ for every $i = 1, 2, \ldots, n$.

Obviously connectedness is an absolute property.

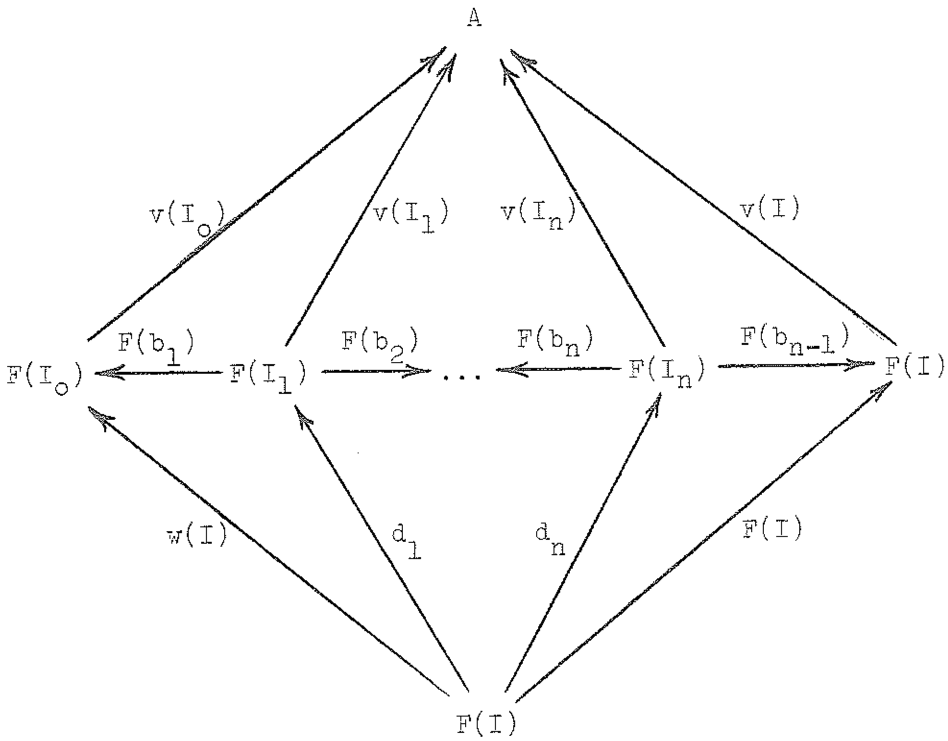
We will find it convenient to use the following special case of Lawvere's [13] comma category. Let $G\colon \underline{A} \to \underline{B}$ be a functor and let $B \in |\underline{B}|$, then (B, G) is the category whose objects are maps $b\colon B \to G(A)$ and whose maps are commutative triangles



(2.3) FUNDAMENTAL LEMMA. Let $F\colon \underline{I} \to \underline{A}$ be a functor, let $w(I)\colon F(I) \to F(I_o)$ be natural in I (I_o is fixed), and assume that for every $I \in |\underline{I}|$, $w(I)$ and $F(I)$ are connected in the comma category $(F(I), F)$. Then if $v(I)\colon F(I) \to A$ is natural in I

$$v(I_o)w(I) = v(I).$$

<u>Proof</u>. That $w(I)$ and $F(I)$ are connected in $(F(I), F)$ means that there exist a finite number of objects of $\underline{I}$, $I_1, I_2, \ldots, I_n$, as many maps in $\underline{A}$, $d_i\colon F(I) \to F(I_i)$, and maps in $\underline{I}$, $I_o \xleftarrow{b_1} I_1 \xrightarrow{b_2} I_2 \ldots I_n \xrightarrow{b_{n+1}} I$ such that the bottom half of the following diagram commutes



But the top half commutes by naturality of v. Proceeding from left to right on the diagram we see that $v(I_o)w(I) = v(I)$ for every $I \in |\underline{I}|$. ∎

This lemma tells us that a natural transformation from F into a constant functor is entirely determined by its value at I_o. Furthermore we have a constructive (and as we shall see, absolute) way of getting it.

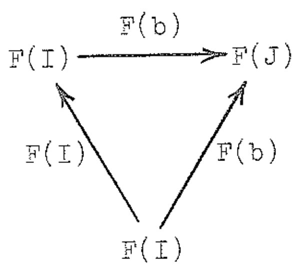
(2.4) THEOREM. Let $\underline{I}$ be a small category and let $F: \underline{I} \longrightarrow \underline{A}$ be a functor. (A, u) is an absolute weak sup of F if and only if $u(I): F(I) \longrightarrow A$ is natural in I, and there exist $I_o \in |\underline{I}|$ and $d_o: A \longrightarrow F(I_o)$ such that for every $I \in |\underline{I}|$, $d_o u(I)$ and $F(I)$ are connected in the comma category $(F(I), F)$.

Proof. Assume that (A, u) is an absolute weak sup of F. By definition $u(I)\colon F(I) \to A$ is natural in I.

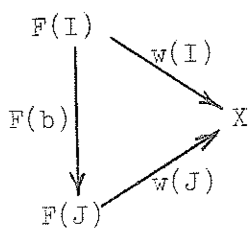
Now we shall construct a category $\underline{A}'$ containing $\underline{A}$ as a full subcategory. Let $|\underline{A}'| = |\underline{A}| + \{X\}$ where X is an arbitrary symbol. Let $[X, B]_{\underline{A}'} = \emptyset$, $[X, X]_{\underline{A}'} = \{X\}$, and $[B, X]_{\underline{A}'} = (B, F)$ the comma category. Composition is the obvious one. This gives us our category $\underline{A}'$.

We now define a congruence relation on $\underline{A}'$. Two maps of $[B, X]_{\underline{A}'}$ are congruent if and only if they are connected in (B, F). Make this relation reflexive by adding the condition that any map of $\underline{A}'$ is congruent to itself. This defines a congruence relation on $\underline{A}'$ and we can form the quotient category $\underline{A}''$ (see Mitchell [19], p.4). The maps of $\underline{A}''$ are congruence classes of maps of $\underline{A}'$ and the objects of $\underline{A}''$ and $\underline{A}'$ are the same. $\underline{A}$ is fully embedded in $\underline{A}''$.

Let $w(I)\colon F(I) \to X$ be the map in $\underline{A}''$ determined by $F(I) \in (F(I), F)$. If $b\colon I \to J$ in $\underline{I}$, then



commutes and so $F(b)$ and $F(I)$ are connected in $(F(I), F)$. This shows that



commutes for every map b of $\underline{I}$. Thus w is natural.

Now (A, u) is an absolute weak sup of F in $\underline{A}$ and thus (A, u) is a weak sup of F in $\underline{A}''$. Therefore there exists a map $d_o: A \longrightarrow X$ such that $d_o u(I) = w(I)$ for all I. This means that $d_o: A \longrightarrow F(I_o)$ for some $I_o \in |\underline{I}|$ and that $d_o u(I)$ is congruent to $F(I)$ for every $I \in |\underline{I}|$. Therefore $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$ for every $I \in |\underline{I}|$.

Conversely, if $G: \underline{A} \longrightarrow \underline{B}$ is any functor, we have a canonically induced functor $\overline{G}: (F(I), F) \longrightarrow (GF(I), GF)$. Connectedness being an absolute property, $\overline{G}(d_o u(I))$ and $\overline{G}(F(I))$ are connected in $(GF(I), GF)$. But $\overline{G}(d_o u(I)) = G(d_o)Gu(I)$ and $\overline{G}(F(I)) = GF(I)$.

Now assume that $v(I): GF(I) \longrightarrow B$ is natural in I. Consider $GA \xrightarrow{Gd_o} GF(I_o) \xrightarrow{v(I_o)} B$. (2.3) now says that

$$v(I_o)G(d_o)Gu(I) = v(I).$$

Therefore (GA, Gu) is a weak sup of GF, and this completes the proof. ∎

It is convenient to state these results in terms of connectedness in comma categories, but it is easier to see the absoluteness by stating these conditions in the form of equations. Even though it seems more complicated in the general case, in special cases it is sometimes simpler and more meaningful. Thus we restate (2.4).

Let $\underline{I}$ be a small category and let $F: \underline{I} \longrightarrow \underline{A}$ be a functor. (A, u) is an absolute weak sup of F if and only if there exist

$I_o \in |\underline{I}|$ and $d_o\colon A \longrightarrow F(I_o)$ such that for all $I \in |\underline{I}|$ there exist

$I_o \xleftarrow{b_1} I_1 \xrightarrow{b_2} I_2 \xleftarrow{b_3} \cdots \xrightarrow{b_n} I$ and $d_i\colon F(I) \longrightarrow F(I_i)$ such that

$$d_o u(I) = F(b_1)d_1$$

$$F(b_2)d_1 = d_2 = F(b_3)d_3$$

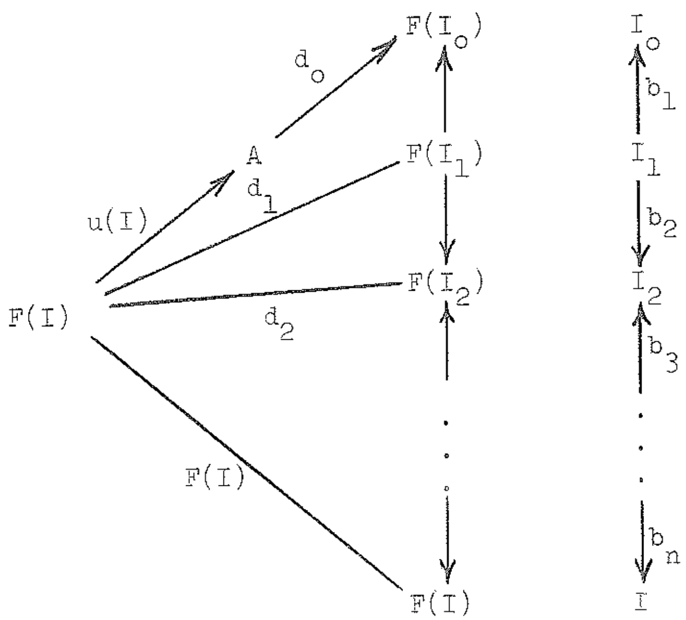
$$F(b_4)d_3 = d_4 = F(b_5)d_5$$

$$\vdots \qquad \vdots$$

$$F(b_{n-2})d_{n-3} = d_{n-2} = F(b_{n-1})d_{n-1}$$

$$F(b_n)d_{n-1} = F(I)$$

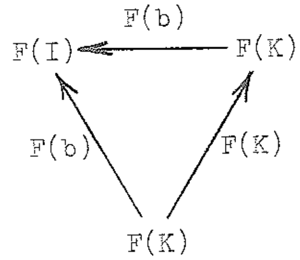
i.e. such that



commutes.

(2.5) REMARK. If $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$ and if $b\colon K \longrightarrow I$ then $d_o u(K)$ and $F(K)$ are connected in $(F(K), F)$. Indeed, b induces a functor

$(F(b), F)$: $(F(I), F) \longrightarrow (F(K), F)$ which sends $d_o u(I)$ to $d_o u(I)F(b)$ and $F(I)$ to $F(I)F(b) = F(b)$. By naturality $d_o u(I)F(b) = d_o u(K)$. Therefore $d_o u(K)$ and $F(b)$ are connected in $(F(K), F)$. But



commutes, showing that $F(b)$ and $F(K)$ are connected in $(F(K), F)$. This proves our statement.

(2.5) is actually very useful when working with special cases.

(2.6) REMARK. We see immediately from theorem (2.4) that if F has an absolute weak sup, then $(F(I_o), d_o u(I))$ is one.

The next result is some sort of a converse to this.

(2.7) THEOREM. Let F: $\underline{I} \longrightarrow \underline{A}$ be a functor. If some weak sup of F is absolute then they all are.

Proof. This follows immediately from the fact that if (B, v) is a weak sup of some functor G, then (B', v') is a weak sup of G if and only if there exist b: $B \longrightarrow B'$ and b': $B' \longrightarrow B$ such that $bv = v'$ and $b'v' = v$. ∎

(2.8) DEFINITION. In this case we say that <u>F has absolute weak sups</u>.

If F has absolute weak sups and if it has a strong sup, then this sup will be absolute as a weak sup but it does not follow from

(2.7) that it will be an absolute sup.

(2.9) THEOREM. Let $F: \underline{I} \to \underline{A}$ have absolute weak sups. Let $G: \underline{I} \to \underline{A}$ be a functor such that there exist natural transformations $G \xrightarrow{t} F \xrightarrow{s} G$. Assume that for every $I \in |\underline{I}|$, $s(I)t(I)$ and $G(I)$ are connected in $(G(I), G)$. Then G has absolute weak sups also.

Proof. Let (A, u) be an absolute weak sup of F. Then (2.4) says that there exist $I_o \in |\underline{I}|$ and $d_o: A \to F(I_o)$ such that for every $I \in |\underline{I}|$, $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$.

For every I, $t(I): G(I) \to F(I)$ and $s: F \to G$ induce a functor

$$(t(I), s): (F(I), F) \to (G(I), G).$$

Since $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$, $(t(I), s)(d_o u(I))$ and $(t(I), s)(F(I))$ are connected in $(G(I), G)$. But by definition of $(t(I), s)$, $(t(I), s)(d_o u(I)) = s(I_o)d_o u(I)t(I)$ and $(t(I), s)(F(I)) = s(I)F(I)t(I) = s(I)t(I)$.

Since $s(I)t(I)$ and $G(I)$ are connected in $(G(I), G)$ by hypothesis, we see that $s(I_o)d_o u(I)t(I)$ and $G(I)$ are connected in $(G(I), G)$. If we take $u'(I) = u(I)t(I): G(I) \to A$ and $d'_o = s(I_o)d_o: A \to G(I_o)$ we conclude that (A, u') is an absolute weak sup of G. ∎

(2.10) COROLLARY. Let $F: \underline{I} \to \underline{A}$ be a functor with absolute weak sups and let $G: \underline{I} \to \underline{A}$ be another functor. If there exist natural transformations such that

$$G \xrightarrow{t} F \xrightarrow{s} G = G$$

then G has absolute weak sups also. ∎

§3. ABSOLUTE SUPREMA

Obviously (A, u) is an absolute sup if and only if (A, u) is an absolute weak sup and $\{u(I) \mid I \in |\underline{I}|\}$ is an absolute joint epi. Thus (1.3) and (2.4) give a characterization of absolute sups. The following results make this more precise.

(3.1) THEOREM. If $F: \underline{I} \to \underline{A}$ is a functor then (A, u) is an absolute sup of F if and only if (A, u) is an absolute weak sup of F and $\{u(I) \mid I \in |\underline{I}|\}$ is a joint epi.

<u>Proof</u>. The necessity of the condition is obvious.

Now assume that (A, u) is an absolute weak sup and $\{u(I) \mid I \in |\underline{I}|\}$ is a joint epi. Since (A, u) is an absolute weak sup there exist $I_o \in |\underline{I}|$ and $d_o: A \to F(I_o)$ such that for every I, $d_o u(I)$ and F(I) are connected in (F(I), F). Then by (2.3)

$$u(I_o)d_o u(I) = u(I)$$

for all I. Since $\{u(I)\}$ is a joint epi we get $u(I_o)d_o = A$.

Consequently, $\{u(I)\}$ is an absolute joint epi, thus proving our theorem. ∎

Theorem (3.1) shows that the map which makes $\{u(I) \mid I \in |\underline{I}|\}$ an absolute joint epi can be chosen to be the d_o of theorem (2.4). We can now state theorem (3.1) in its final form.

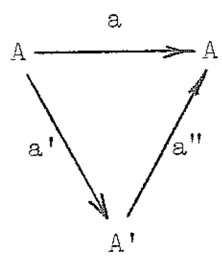
(3.2) THEOREM. Let $\underline{I}$ be a small category and let $F\colon \underline{I} \to \underline{A}$ be a functor. (A, u) is an absolute sup of F if and only if $u(I)\colon F(I) \to A$ is natural in I, and there exist $I_o \in |\underline{I}|$ and $d_o\colon A \to F(I_o)$ such that

$$u(I_o)d_o = A$$

and $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$ for every $I \in |\underline{I}|$. ∎

(3.3) COROLLARY. If $F\colon \underline{I} \to \underline{A}$ has absolute weak sups and has a (strong) sup then it is an absolute (strong) sup. ∎

A map $a\colon A \to A$ is said to be an <u>idempotent</u> if $aa = a$. It is said to be a <u>split idempotent</u> if it factors



with $a'a'' = A'$.

(3.4) THEOREM. Let $F\colon \underline{I} \to \underline{A}$ have absolute weak sups. If (A, u) is one of the absolute weak sups of F, then using the same notation as in (2.4), $d_o u(I_o)$ is an idempotent and the following are equivalent:

(i) $d_o u(I_o)$ is a split idempotent,

(ii) F has an absolute sup,

(iii) F has a sup.

Proof. By (2.4), $d_o u(I)$ and $F(I)$ are connected in $(F(I), F)$ for all I. By lemma (2.3), $(d_o u(I_o))(d_o u(I)) = d_o u(I)$ thus $(d_o u(I_o))(d_o u(I_o)) = d_o u(I_o)$. Thus $d_o u(I_o)$ is an idempotent.

(i) $\Rightarrow$ (ii). Let $d_o u(I_o) = d'_o a'$ such that

$A' \xrightarrow{d'_o} F(I_o) \xrightarrow{a'} A' = A'$. Define $u'(I) = a' d_o u(I)\colon F(I) \longrightarrow A'$.

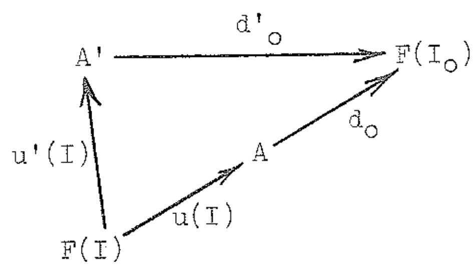
Then $u'(I_o) d'_o = a' d_o u(I_o) d'_o = a' d'_o a' d'_o = A'$.

$$d'_o u'(I) = d'_o a' d_o u(I) = d_o u(I_o) d_o u(I)$$
$$= d_o u(I)$$

Thus $d'_o u'(I)$ is connected to $F(I)$ in $(F(I), F)$ for every I. By theorem (3.2), (A', u') is an absolute sup of F.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i). Let sup $F = (A', u')$. $d_o u(I)\colon F(I) \longrightarrow F(I_o)$ is natural in I, thus there exists a unique $d'_o\colon A' \longrightarrow F(I_o)$ such that



commutes. By lemma (2.3), $u'(I_o) d_o u(I) = u'(I)$, consequently $u'(I_o) d'_o u'(I) = u'(I_o) d_o u(I) = u'(I)$. Since $\{u'(I)\}$ is a joint epi, we get $u'(I_o) d'_o = A'$. ∎

§4. ABSOLUTE COEQUALIZERS

Because absolute coequalizers are important in the theory of triples and their algebras (see [20], and chapter II) we shall investigate these coequalizers in greater detail. This also gives us a chance to see how the conditions break down in special cases.

(4.1) THEOREM. $A_1 \underset{a_1}{\overset{a_0}{\rightrightarrows}} A_0 \xrightarrow{a} A$ is an absolute weak coequalizer in $\underline{A}$ if and only if there exist $b\colon A \to A_0$ and a finite number of maps $b_i\colon A_0 \to A_1$ such that

$$aa_0 = aa_1$$

$$ba = a_{v(1)}b_1$$

$$a_{v(2)}b_1 = a_{v(3)}b_2$$

$$a_{v(4)}b_2 = a_{v(5)}b_3$$

$$\vdots$$

$$a_{v(2n)}b_n = A_0$$

where $v(i) = 0$ or 1, $n \geq 0$.

$A_1 \underset{a_1}{\overset{a_0}{\rightrightarrows}} A_0 \xrightarrow{a} A$ is an absolute coequalizer in $\underline{A}$ if and only if there exist $b\colon A \to A_0$ and a finite number of maps $b_i\colon A_0 \to A_1$ satisfying the same equations as above as well as

$$ab = A.$$

Proof. Remark (2.5) says that we only have to worry about A_0. The first equation expresses naturality of u. It is an easy exercise to verify that the conditions of theorem (2.4) give the above equations.

The second part follows from (3.2). ∎

(4.2) EXAMPLE. $A_1 \underset{a}{\overset{a}{\rightrightarrows}} A_o \xrightarrow{A_o} A_o$ is an absolute coequalizer.

(4.3) EXAMPLE. If $a: A \longrightarrow A'$ is an absolute epi then there exists $b: A' \longrightarrow A$ such that $ab = A'$. It is easy to see that $A \underset{A}{\overset{ba}{\rightrightarrows}} A \xrightarrow{a} A'$ is an absolute coequalizer.

If $a: A \longrightarrow A$ is an idempotent, then $A \underset{A}{\overset{a}{\rightrightarrows}} A \xrightarrow{a} A$ is an absolute weak coequalizer. Thus $A \underset{A}{\overset{a}{\rightrightarrows}} A$ has absolute weak coequalizers and a splits if and only if $A \underset{A}{\overset{a}{\rightrightarrows}} A$ has a coequalizer.

(4.4) EXAMPLE. Beck defines [2] $A_1 \underset{a_1}{\overset{a_o}{\rightrightarrows}} A_o \xrightarrow{a} A$ to be a contractible coequalizer if there exist $b: A \longrightarrow A_o$ and $b_o: A_o \longrightarrow A_1$ such that

$$\begin{aligned} aa_o &= aa_1 \\ ab &= A \\ ba &= a_o b_o \\ a_1 b_o &= A_o. \end{aligned}$$

Contractible coequalizers are absolute.

The natural question arises as to whether there are absolute coequalizers which are not contractible. The answer is yes. Consider $A \underset{a}{\overset{a}{\rightrightarrows}} A' \xrightarrow{A'} A'$ which is absolute but is contractible if and only if

a is an absolute epi.

(4.5) DEFINITION. We shall say that an absolute coequalizer is <u>n-contractible</u> if n is the smallest integer for which (4.1) is valid.

$$A \underset{a}{\overset{a}{\rightrightarrows}} A' \xrightarrow{A'} A' \quad \text{is 0-contractible.}$$

If $A_1 \underset{a_1}{\overset{a_o}{\rightrightarrows}} A_o \xrightarrow{a} A$ is a contractible coequalizer and if $a_o \neq a_1$, then it is 1-contractible.

We shall show that there exist n-contractible coequalizers in $\underline{S}$ for each positive integer n. Before we show this, we shall establish a graph-theoretical characterization of absolute coequalizers in $\underline{S}$.

Given a pair of set maps $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ we define a graph by letting the vertices be the elements of Y and the arrows be the elements of X such that $f_1(x) \xrightarrow{x} f_2(x)$. The coequalizer of f_1 and f_2 is the set of components of this graph with the natural projection

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{f} Z$$

A <u>path</u> from y to y' is a finite number of vertices $y_o, y_1, y_2, \ldots, y_n$ such that $y = y_o$, $y' = y_n$, and such that for each i there is an arrow $y_i \to y_{i+1}$ or $y_{i+1} \to y_i$. The <u>length</u> is said to be n, and the <u>type</u> is defined to be a sequence of +1 or -1, one for each arrow in the path; +1 if the arrow is $y_i \to y_{i+1}$ and -1 if the arrow is $y_{i+1} \to y_i$.

(4.6) THEOREM. If $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ is a pair of set maps, then the coequalizer of f_1 and f_2 is absolute if and only if there exists a path type Δ such that each component of the graph induced by (f_1, f_2) on Y has a vertex, called the centre, such that every vertex of the graph has a path of type Δ to the centre of its component.

Proof. Assume that $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{f} Z$ is an absolute coequalizer, then by (4.1) there exist $g: Z \to Y$ and $g_i: Y \to X$ such that

$$ff_1 = ff_2$$
$$fg = Z$$
$$gf = f_{v(1)}g_1$$
$$f_{v(2)}g_1 = f_{v(3)}g_2$$
$$f_{v(4)}g_2 = f_{v(5)}g_3$$
$$\vdots$$
$$f_{v(2n)}g_n = Y.$$

We can assume that $v(2i - 1) \neq v(2i)$ for if not we can shorten the chain.

If z is a component, define its centre to be $g(z)$. For any $y \in Y$, the component containing y is $f(y)$ and its centre $gf(y)$. The above equations give

$$gf(y) = f_{v(1)}g_1(y)$$
$$f_{v(2)}g_1(y) = f_{v(3)}g_2(y)$$

$$f_{v(4)}g_2(y) = f_{v(5)}g_3(y)$$

$$\vdots$$

$$f_{v(2n)}g_n(y) = y$$

Thus taking the vertices to be $f_{v(2i)}g_i(y)$ we get a path of length n from y to $gf(y)$. The path type is determined by v as follows:

$$\Delta = (v(2n-2i+1) - v(2n-2i+2) \mid i = 1, 2, \ldots, n).$$

Conversely, assume that there exists a path type for which each component has a centre. Define $g\colon Z \to Y$ by putting $g(z)$ = centre of z. Given $y \in Y$, the centre of the component of y is $gf(y)$. By hypothesis we have a path

$$y \xleftrightarrow{x_1} y_1 \xleftrightarrow{x_2} y_2 \xleftrightarrow{x_3} \cdots \xleftrightarrow{x_n} y_n = gf(y)$$

($\longleftrightarrow$ stands for " $\longrightarrow$ or $\longleftarrow$ "). Define $g_i(y) = x_i$ for $i = 1, 2, \ldots, n$. Using the path type to choose the appropriate subscripts for the f's, we see that we have an absolute coequalizer. ∎

Notice that the length of the path is the same as the n in the characterization, thus a coequalizer is n-contractible in $\underline{S}$ if and only if the minimum path length is n.

Let $X = Y = \{0, 1, 2, \ldots, n\}$ and $Z = \{0\}$. $f_0 = X$, $f_1(i) = \min(i+1, n)$, f the only possible map. The graph induced on Y is

$$\overset{0}{\cdot} \longrightarrow \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \quad \cdots \quad \overset{n-1}{\cdot} \longrightarrow \overset{n}{\circlearrowleft}$$

We take n as the centre and the length of the path is n. Thus

$X \underset{f_1}{\overset{f_o}{\rightrightarrows}} Y \overset{f}{\longrightarrow} Z$ is an n-contractible coequalizer. This example was suggested by Michael Barr to replace a more complicated one.

We now give an example in $\underline{S}$ where both f_1 and f_2 are onto (absolute epi) but where the coequalizer is not absolute. Let $X = Y = \{0, 1\}$, $Z = \{0\}$. $f_1 = X$, $f_2(i) = 1 - i$. The induced graph is

$$0 \rightleftarrows 1$$

Clearly there can be no centre.

We also give the following example to show that there exist n-contractible coequalizers in the category $\underline{Ab}$ of abelian groups.

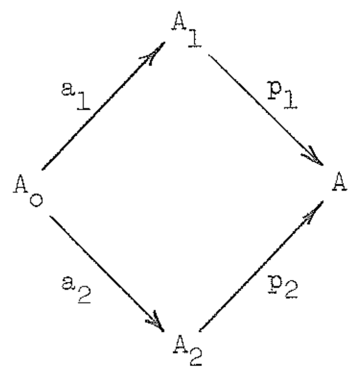
Let C_2 be the cyclic group of order two, and consider the following diagram:

$$C_2{}^n \underset{f}{\overset{C_2{}^n}{\rightrightarrows}} C_2{}^n \overset{0}{\longrightarrow} \{0\}$$

where $f: (a_1, a_2, a_3, \ldots, a_n) \mapsto (a_2, a_3, \ldots, a_n, 0)$. This is an n-contractible coequalizer in $\underline{Ab}$. The details are left to the reader.

§5. ABSOLUTE PUSHOUTS

In view of remark (2.5), the conditions of theorem (2.4) applied to pushouts specialize to:



is an absolute weak pushout if and only if there exist $d_o: A \longrightarrow A_i$ (say A_1, for simplicity) and a finite number of maps d_i, d'_i such that

$$p_1 a_1 = p_2 a_2$$

$$\begin{array}{ll} d_o p_1 = a_1 d_1 & d_o p_2 = a_1 d'_1 \\ a_2 d_1 = a_2 d_2 & a_2 d'_1 = a_2 d'_2 \\ a_1 d_2 = a_1 d_3 & a_1 d'_2 = a_1 d'_3 \\ \vdots & \vdots \\ a_1 d_n = A_1 & a_2 d'_m = A_2 \end{array}$$

This diagram is an absolute pushout if d_o satisfies the further relation

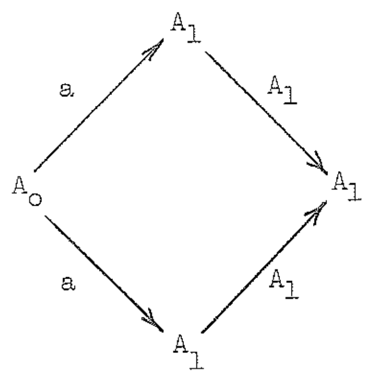
$$p_1 d_o = A.$$

The first column of equations may reduce to $d_o p_1 = A_1$, but the second column never reduces this much. We conclude from $a_2 d'_m = A_2$

that a_2 is an absolute epi. We also see that either a_1 is an absolute epi or p_1 is an absolute mono.

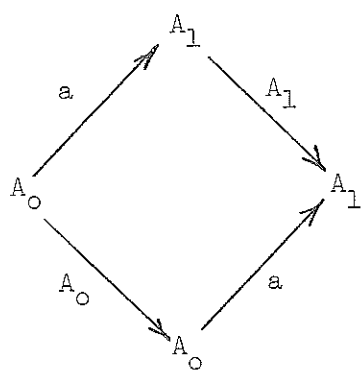
In the case of absolute pushouts we see that a_2 is an absolute epi and either a_1 is an absolute epi or p_1 is an isomorphism.

(5.1) EXAMPLE.



is an absolute pushout if and only if a is an absolute epi.

(5.2) EXAMPLE.



is an absolute pushout for all a.

The cokernel pair $A_o \xrightarrow{a} A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ is just the pushout where $a_1 = a_2 = a$. As we saw above, if $A_o \xrightarrow{a} A_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ is an absolute cokernel pair, then a must be an absolute epi. We conclude that

$p_1 = p_2$ = isomorphism. Thus the only absolute kernel pairs are the trivial ones.

We finish this section with the following characterization of the absolute pullbacks in $\underline{S}$.

(5.3) PROPOSITION. A pullback in $\underline{S}$ is absolute if and only if it is of type (5.2) or a non-empty intersection.

The proof is straight forward and will be left out.

§6. MISCELLANEOUS RESULTS

In $\underline{S}$, the fact that all epimorphisms are absolute is equivalent to the axiom of choice. It is a trivial fact that all monomorphisms with non-empty domains are absolute.

Obviously there are no absolute initial objects (= sup of the empty functor).

If $\underline{I}$ is disconnected, in the sense that $\underline{I} = \underline{I}_1 + \underline{I}_2$ where $\underline{I}_1 \neq \emptyset \neq \underline{I}_2$, then no functor $\underline{I} \longrightarrow \underline{A}$ can have an absolute sup. Indeed, I_o in (2.4) must be connected to every object of $\underline{I}$.

It follows from this that the only absolute coproducts are the 1-fold coproducts, i.e. when $\underline{I}$ has one object.

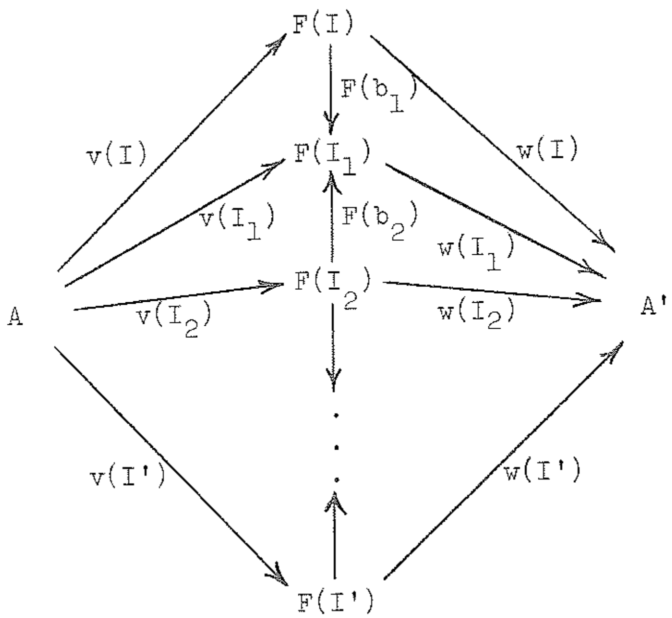
Now we give an alternate proof of a result of Lambek [9].

(6.1) PROPOSITION. Let $F\colon \underline{I} \longrightarrow \underline{A}$ be a functor and let $\underline{I}$ be a connected small category. Assume that there exists $u(I)\colon F(I) \longrightarrow A$, natural in I such that $u(I)$ is an isomorphism for every I, then $(A, u) = \sup F$ and this sup is absolute.

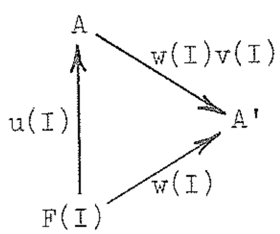
Proof. Let $v(I)\colon A \longrightarrow F(I)$ be the inverse of $u(I)$. $v(I)$ is natural in I. Let $w(I)\colon F(I) \longrightarrow A'$ be natural in I. Let I and I' be any two objects of $\underline{I}$. There exist

$I \xrightarrow{b_1} I_1 \xleftarrow{b_2} I_2 \xrightarrow{b_3} \cdots \; I_n \xleftarrow{b_{n+1}} I'$ since $\underline{I}$ is connected.

The following diagram commutes by naturality of v and w:



Thus $w(I)v(I) = w(I')v(I')$ for any $I, I' \in |\underline{I}|$. But $w(I)v(I)u(I) = w(I)$, thus $w(I)v(I)$ is the unique map making the following diagram commute:



Therefore $(A, u) = \sup F$.

But that u is an isomorphism and that $\underline{I}$ is connected are absolute properties, thus (A, u) is an absolute sup of F. ∎

(6.2) COROLLARY. If $\underline{A}$ is a group considered as a one object category, all sups of functors from a connected category are absolute. ∎

Lambek proved that if $\underline{A}$ is a group with more than one element then a functor from a disconnected category has no sup.

§7. REMARKS

In all of these absoluteness theorems it would have been enough to require that the given property be preserved by full embeddings. Thus these are really results of embedding absoluteness.

For the characterization of absolute epis and absolute sups it would have been enough to demand that these properties be preserved by representable functors. For this one must know what epis and sups are in $\underline{S}$. This process, however, cannot be used to characterize the dual concepts of mono and inf which are always preserved by representable functors. In fact this is how mono and inf are defined. See [20] for an example of this method applied to coequalizers.

CHAPTER II

TRIPLES AND ABSOLUTENESS

In this chapter we examine the relationship between triples and absoluteness properties.

The reader is assumed to be familiar with the material of §1, which is included for completeness. For more details the reader is referred to Manes' thesis [18].

§2 studies the occurrence of absoluteness properties in connection with triples and their algebras.

In §3 we reformulate Beck's tripleableness theorem, replacing his original contractible coequalizers by the more natural absolute coequalizers. Although absolute coequalizers constitute a more general class of coequalizers, in practice the conditions are just as easy to verify. The idea is not to use the characterization in terms of equations, given in chapter I, but to get away from these equations entirely, using only absoluteness (preservation by all functors). This way the proofs are conceptual rather than computational, and thus clearer.

In the VTT, absolute coequalizers do give us a slight advantage, making the conditions weaker. New conditions are given making it easier to prove VTT in some cases.

§4 is devoted to some known examples, just to show how these conditions can be used.

§1. TRIPLES

(1.1) DEFINITION. Let $\underline{A}$ be a category. A *triple* on $\underline{A}$ is a three-triple (T, η, μ) where $T: \underline{A} \to \underline{A}$ is a functor and $\eta: \underline{A} \to T$ and $\mu: T^2 \to T$ are natural transformations such that

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ {\scriptstyle \eta T}\downarrow & \searrow^{T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

commute. η and μ are called the *unit* and the *multiplication* of the triple. The above diagrams express that μ is *associative* and that η is *unitary*.

Let $\underline{B} \underset{F}{\overset{U}{\rightleftarrows}} \underline{A}$ be an adjoint pair of functors, i.e. we have natural transformations $\varepsilon: FU \to \underline{B}$ and $\eta: \underline{A} \to UF$ such that

$$U \xrightarrow{\eta U} UFU \xrightarrow{U\varepsilon} U = U$$
$$F \xrightarrow{F\eta} FUF \xrightarrow{\varepsilon F} F = F.$$

This adjoint pair induces (see Huber [8]) a triple $(UF, \eta, U\varepsilon F)$.

Eilenberg-Moore [6] showed that every triple was of this form by constructing the category of algebras over a triple, together with a canonical underlying - free adjoint pair.

Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\underline{A}$.

(1.2) DEFINITION. A *$\mathbb{T}$-algebra* is a pair (A, a) where

$a: TA \to A$ is an $\underline{A}$ morphism such that

$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta A} & TA \\ & {\scriptstyle A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array}$$

commute. a is called the structure of (A, a) and the above diagrams assert that a is associative and unitary.

(1.3) DEFINITION. A $\mathbb{T}$-homomorphism $f: (A, a) \to (A', a')$ is an $\underline{A}$-morphism $f: A \to A'$ such that

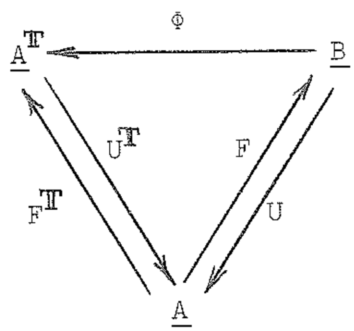
$$\begin{array}{ccc} TA & \xrightarrow{a} & A \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ TA' & \xrightarrow[a']{} & A' \end{array}$$

commutes, i.e. f preserves structure.

We denote the category of $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms by $\underline{A}^{\mathbb{T}}$.

We have the canonical underlying functor $U^{\mathbb{T}}: \underline{A}^{\mathbb{T}} \to \underline{A}$ defined by $U^{\mathbb{T}}(A, a) = A$ and $U^{\mathbb{T}}(f) = f$. This functor has a left adjoint $F^{\mathbb{T}}: \underline{A} \to \underline{A}^{\mathbb{T}}$ which is defined by $F^{\mathbb{T}}(A) = (TA, \mu A)$ and $F^{\mathbb{T}}(f) = Tf$. The adjunctions are $\varepsilon^{\mathbb{T}}: F^{\mathbb{T}}U^{\mathbb{T}} \to \underline{A}^{\mathbb{T}}$ where $\varepsilon^{\mathbb{T}}(A, a) = a$ and $\eta^{\mathbb{T}}: \underline{A} \to U^{\mathbb{T}}F^{\mathbb{T}}$ where $\eta^{\mathbb{T}} = \eta$. The adjoint pair $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ induces the triple $\mathbb{T} = (T, \eta, \mu)$; the one with which we started.

On the other hand, if we start with an adjoint pair $\underline{B} \underset{F}{\overset{U}{\rightleftarrows}} \underline{A}$, $F \xrightarrow{\varepsilon, \eta} U$, form the triple $\mathbb{T} = (UF, \eta, U\varepsilon F)$ and then form $\underline{A}^{\mathbb{T}} \underset{F^{\mathbb{T}}}{\overset{U^{\mathbb{T}}}{\rightleftarrows}} \underline{A}$, we do not in general recover the original category $\underline{B}$ and functors F and U. We have, however, a canonical "comparison" functor Φ



Φ is defined by $\Phi(B) = (UB, U\varepsilon B)$ and $\Phi(g) = Ug$. Φ is the unique functor satisfying the following relations:

$$U^{\mathbb{T}}\Phi = U$$

$$U^{\mathbb{T}}\varepsilon^{\mathbb{T}}\Phi = U\varepsilon$$

$$F^{\mathbb{T}} = \Phi F.$$

(1.4) DEFINITION. We say that a functor $U: \underline{B} \longrightarrow \underline{A}$ is <u>tripleable</u> if U has a left adjoint and if the above defined comparison functor Φ is an equivalence of categories.

§2. TRIPLES AND ABSOLUTENESS

Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\underline{A}$ and let (A, a) be a

$\mathbb{T}$-algebra. By associativity of a,

$$\begin{array}{ccc} T^{n+2}A & \xrightarrow{\;T^{n+1}a\;} & T^{n+1}A \\ {\scriptstyle T^{n}\mu A}\big\downarrow & & \big\downarrow{\scriptstyle T^{n}a} \\ T^{n+1}A & \xrightarrow[\;T^{n}a\;]{} & T^{n}A \end{array} \qquad (2.1)$$

commutes for all $n \geqslant 0$ (T^{0} is defined to be $\underline{A}$).

The μ-associative law shows that the following diagram commutes for all $n \geqslant 1$:

$$\begin{array}{ccc} T^{n+2}A & \xrightarrow{\;T^{n}\mu A\;} & T^{n+1}A \\ {\scriptstyle T^{n-1}\mu TA}\big\downarrow & & \big\downarrow{\scriptstyle T^{n-1}\mu A} \\ T^{n+1}A & \xrightarrow[\;T^{n-1}\mu A\;]{} & T^{n}A \end{array} \qquad (2.2)$$

By naturality of μ we get that

$$\begin{array}{ccc} T^{n+2}A & \xrightarrow{\;T^{n}\mu A\;} & T^{n+1}A \\ {\scriptstyle T^{n-i}\mu T^{i}A}\big\downarrow & & \big\downarrow{\scriptstyle T^{n-i}\mu T^{i-1}A} \\ T^{n+1}A & \xrightarrow[\;T^{n-1}\mu A\;]{} & T^{n}A \end{array} \qquad (2.3)$$

commutes for $n \geqslant i \geqslant 2$.

(2.4) LEMMA. All maps $T^{n}A \longrightarrow A$ built up from $T^{i}a$ and $T^{j}\mu T^{k}A$ are equal.

<u>Proof</u>. For $n = 1$, there is only $TA \xrightarrow{a} A$.

For $n = 2$, the only possibilities are
$T^2A \xrightarrow{Ta} TA \xrightarrow{a} A$ and $T^2A \xrightarrow{\mu A} TA \xrightarrow{a} A$ which are equal by (2.1).

Assume that we have proved the result for $n + 1$ where $n \geqslant 1$. We shall prove it for $n + 2$. Call the unique map $T^iA \longrightarrow A$, a_i for $i \leqslant n + 1$ $(a_o = A)$. Consider

$$T^{n+2}A \xrightarrow{x} T^{n+1}A \xrightarrow{a_{n+1}} A$$

There are four possibilities for x:

(i) $x = T^{n+1}a$. Then since $a_{n+1} = a_n.T^na$, $a_{n+1}.x = a_n.T^na.T^{n+1}a$ and by (2.1) this is $a_n.T^na.T^n\mu A = a_{n+1}.T^n\mu A$.

(ii) $x = T^n\mu A$. Then $a_{n+1}.x = a_{n+1}.T^n\mu A$.

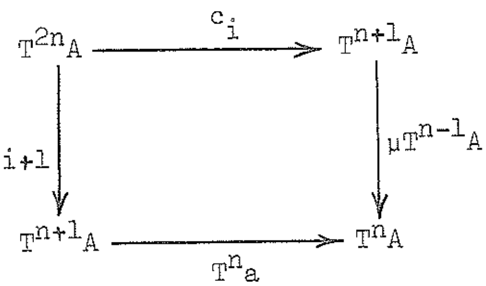
(iii) $x = T^{n-1}\mu TA$. Then since $a_{n+1} = a_n.T^{n-1}\mu A$, $a_{n+1}.x = a_n.T^{n-1}\mu A.T^{n-1}\mu TA$ which, by (2.2) is equal to $a_n.T^{n-1}\mu A.T^n\mu A = a_{n+1}.T^n\mu A$.

(iv) $x = T^{n-i}\mu T^iA$ for $i \geqslant 2$. Then $a_{n+1}.x = a_n.T^{n-1}\mu A.T^{n-i}\mu T^iA$ which by (2.3) is equal to $a_n.T^{n-i}\mu T^{i-1}.T^n\mu A = a_{n+1}.T^n\mu A$.

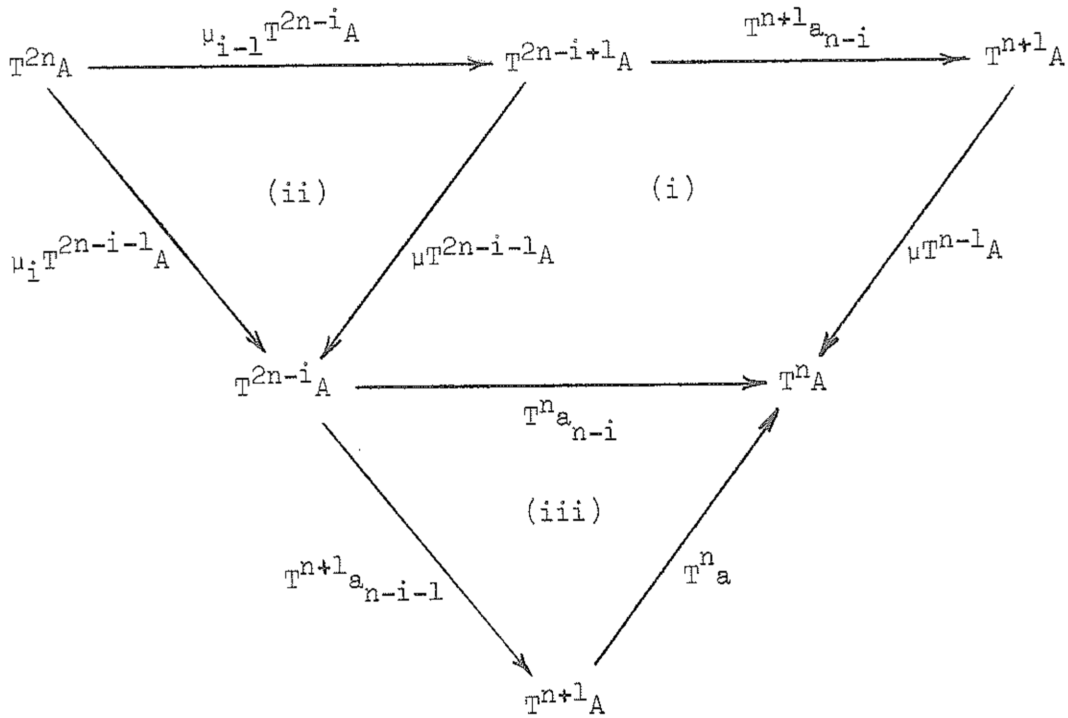
This proves the lemma. ∎

Define $\mu_i = T^{i+1} \xrightarrow{\mu T^{i-1}} T^i \xrightarrow{\mu T^{i-2}} T^{i-1} \quad \ldots \xrightarrow{\mu} T$ and $\mu_o = T\colon T \longrightarrow T$.

(2.5) LEMMA. Let $c_i = T^{n+1}a_{n-i}.\mu_{i-1}T^{2n-i}A$ for a fixed n and $1 \leqslant i \leqslant n$. Then the following diagram commutes:



<u>Proof</u>. Consider the diagram:



(i) commutes by naturality of μ;

(ii) commutes by definition of μ_i;

(iii) commutes by lemma (2.4). Commutativity of the outer diagram is what we had to prove. ∎

Define $\eta_i = \underline{A} \xrightarrow{\eta} T \xrightarrow{T\eta} T^2 \longrightarrow \cdots \xrightarrow{T^{i-1}\eta} T^i$. We are now in a position to prove the main theorem of this section.

(2.6) THEOREM. If $\mathbb{T} = (T, \eta, \mu)$ is a triple on $\underset{\sim}{A}$ and (A, a) is a $\mathbb{T}$-algebra, then

$$T^{n+1}A \underset{\mu T^{n-1}A}{\overset{T^n a}{\rightrightarrows}} T^n A \xrightarrow{a_n} A$$

is an absolute coequalizer in $\underline{A}$ for any $n \geqslant 1$.

Proof. Lemma (2.4) shows that a_n coequalizes $T^n a$ and $\mu T^{n-1}A$. Let $d_o = \eta_n(A)$, $d_o \colon A \longrightarrow T^n A$. Using the fact that $a_n = a.Ta.T^2a \ldots T^{n-1}a$ we show that

$$\begin{array}{ccc} A & \xrightarrow{\eta_n A} & T^n A \\ & \searrow^{A} & \downarrow{\scriptstyle a_n} \\ & & A \end{array}$$

commutes. Now

$$\begin{array}{ccc} T^n A & \xrightarrow{a_n} & A \\ {\scriptstyle \eta_n T^n A}\downarrow & & \downarrow{\scriptstyle \eta_n A} \\ T^{2n} A & \xrightarrow[T^n a_n]{} & T^n A \end{array}$$

commutes by naturality of η_n. Thus

$$\begin{aligned} d_o a_n &= T^n a_n . \eta_n T^n A \\ &= T^n a . T^{n+1} a_{n-1} . \eta_n T^n A \\ &= T^n a . c_1 . \eta_n T^n A \end{aligned}$$

$$\left.\begin{aligned}
\mu T^{n-1}A.c_1.\eta_n T^n A &= T^n a.c_2.\eta_n T^n A \\
\mu T^{n-1}A.c_2.\eta_n T^n A &= T^n a.c_3.\eta_n T^n A \\
&\vdots \\
\mu T^{n-1}A.c_{n-1}.\eta_n T^n A &= T^n a.c_n.\eta_n T^n A
\end{aligned}\right\} \quad \text{(by (2.5))}$$

$$\begin{aligned}
\mu T^{n-1}A.c_n.\eta_n T^n A &= \mu T^{n-1}A.T^{n+1}a_o.\mu_{n-1}T^n A.\eta_n T^n A \\
&= \mu T^{n-1}A.\mu_{n-1}T^n A.\eta_n T^n A \\
&= \mu_n T^{n-1}A.\eta_n T^n A \\
&= (\mu_n.\eta_n T)(T^{n-1}A) \\
&= T^n A
\end{aligned}$$

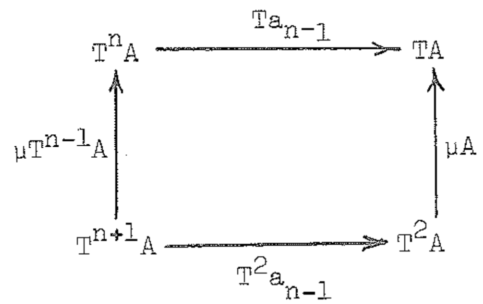
This completes the proof of the theorem. ∎

Of course this is not an elegant proof, but no further reference will be made to the proof nor to the definitions connected with it. The necessary information that the equations contain has been stored in the fact that we have absolute coequalizers (the next theorem testifies to this). From now on we forget these equations (including the ones used in defining contractible coequalizers) and work solely with absoluteness. We see how Beck's theorem can be proved using only absoluteness and in the last section we see how it can be applied.

Note that generally, $T^{n+1}A \underset{\mu T^{n-1}A}{\overset{T^n a}{\rightrightarrows}} T^n A \xrightarrow{a_n} A$ is an n-contractible, but in some cases it may reduce to i-contractible for some $i < n$. In any case it is a good candidate for an n-contractible coequalizer.

(2.7) THEOREM. Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\underline{A}$ and let $a\colon TA \longrightarrow A$ be an $\underline{A}$-morphism. Let $a_i = a.Ta.T^2a \ldots T^{i-1}a$. If for some $n \geqslant 1$, $T^{n+1}A \underset{\mu T^{n-1}A}{\overset{T^n a}{\rightrightarrows}} T^n A \xrightarrow{a_n} A$ is an absolute coequalizer, then (A, a) is a $\mathbb{T}$-algebra.

Proof. We have $a_n.T^n a = a_n.\mu T^{n-1}A$ but $a_n = a.Ta_{n-1}$ and



commutes by naturality of μ, therefore

$$a_n.T^n a = a_n.\mu T^{n-1}A = a.Ta_{n-1}.\mu T^{n-1}A$$
$$= a.\mu A.T^2 a_{n-1}$$

But

$$a_n.T^n a = a_{n+1} = a.Ta.T^2 a_{n-1}$$

thus

$$a.\mu A.T^2 a_{n-1} = a.Ta.T^2 a_{n-1}$$

so

$$a.\mu A.T^2 a_{n-1}.T^{n+1}a = a.Ta.T^2 a_{n-1}.T^{n+1}a$$
$$a.\mu A.T^2 a_n = a.Ta.T^2 a_n.$$

But a_n is an absolute epi, therefore

$$a.\mu A = a.Ta \qquad \text{(The } a\text{-associative law).}$$

Now $a.\eta A.a = a.Ta.\eta TA = a.\mu A.\eta TA = a.$

Thus $a.\eta A.a_n = a_n$ and since a_n is epi

$$a.\eta A = A \qquad \text{(The } a\text{-unitary law).}$$

This completes the proof. ∎

If A and B are categories we let B^{A} denote the category of functors $A \twoheadrightarrow B$ with natural transformations as morphisms. If $G\colon B \longrightarrow C$ is a functor we have an induced functor $G^{A}\colon B^{A} \longrightarrow C^{A}$ defined by composition. If $t\colon G \twoheadrightarrow H$ is a natural transformation then we have an induced natural transformation $t^{A}\colon G^{A} \twoheadrightarrow H^{A}$ also defined by composition. $(\ )^{A}$ sends categories to categories, functors to functors, natural transformations to natural transformations, and respects all kinds of composition and identities.

Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on A. We have an induced triple $\mathbb{T}^{A} = (T^{A}, \eta^{A}, \mu^{A})$ on A^{A}. Since (T, μ) is a $\mathbb{T}^{A}$-algebra we get the following theorem.

(2.8) THEOREM. If $\mathbb{T} = (T, \eta, \mu)$ is a triple on A, then

$$T^{n+2} \underset{\mu T^{n}}{\overset{T^{n}\mu}{\rightrightarrows}} T^{n+1} \xrightarrow{\mu_n} T$$

is an absolute coequalizer in A^{A} for all $n \geqslant 0$. ∎

From (2.8) we see that

$$T^{n+2+i+j} \underset{T^{i}\mu T^{n+j}}{\overset{T^{n+i}\mu T^{j}}{\rightrightarrows}} T^{n+1+i+j} \xrightarrow{T^{i}\mu_n T^{j}} T^{i+j+1}$$

is an absolute coequalizer in A^{A}. This takes care of all possibilities for pairs of maps of the type $T^{p}\mu T^{q}$ between two consecutive powers of T.

If we apply the substitution functor $\text{sub}_A: \underline{A}^{\underline{A}} \longrightarrow \underline{A}$ we see that

$$T^{n+2+i+j}(A) \underset{T^i \mu T^{n+j}(A)}{\overset{T^{n+i}\mu T^j(A)}{\rightrightarrows}} T^{n+1+i+j}(A) \xrightarrow{T^i \mu_n T^j(A)} T^{i+j+1}(A)$$

is an absolute coequalizer in $\underline{A}$. Theorem (2.6) tells us that

$$T^{n+1+i}(A) \underset{T^i \mu T^{n-1}(A)}{\overset{T^{n+i}a}{\rightrightarrows}} T^{n+i}(A) \xrightarrow{T^i a_n} T^i(A)$$

is an absolute coequalizer in $\underline{A}$, where (A, a) is a $\mathbb{T}$-algebra. This takes care of all possibilities of pairs of maps $T^{p+1}(A) \longrightarrow T^p(A)$ where (A, a) is a $\mathbb{T}$-algebra.

§3. TRIPLEABLENESS

Let $U: \underline{B} \longrightarrow \underline{A}$ be a functor.

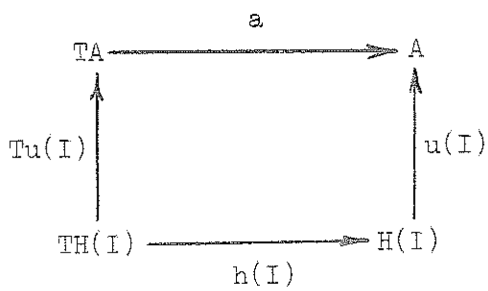
(3.1) DEFINITION. We say that $\underline{B}$ has U-absolute sups if every functor $G: \underline{I} \longrightarrow \underline{B}$ such that UG has an absolute sup in $\underline{A}$, has a sup (not necessarily absolute) in $\underline{B}$.

(3.2) DEFINITION. We say that U preserves U-absolute sups if whenever a functor $G: \underline{I} \longrightarrow \underline{B}$ has a sup in $\underline{B}$ and UG has an absolute sup in $\underline{A}$, then $U.\sup G \cong \sup UG$, i.e. if $\sup G = (B, v)$ and $\sup UG = (A, u)$ then there should exist an isomorphism $a: UB \longrightarrow A$ such that $a.Uv = u$.

(3.3) DEFINITION. We say that U reflects U-absolute sups if for every $v(I): G(I) \longrightarrow B$, natural in I, such that $(UB, Uv) \cong \sup UG$ which is absolute, then $(B, v) \cong \sup G$ (not necessarily absolute).

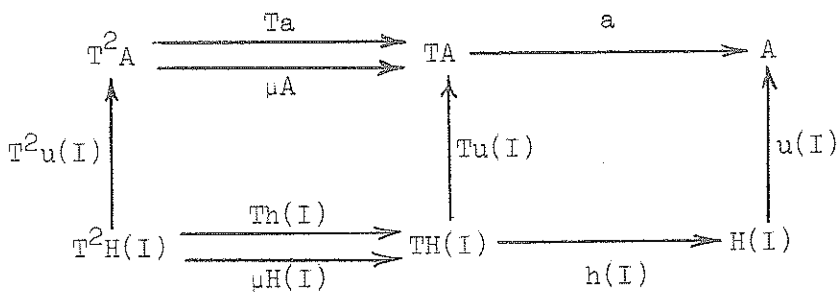
(3.4) THEOREM. Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\underline{A}$. Then $\underline{A}^{\mathbb{T}}$ has $U^{\mathbb{T}}$-absolute sups, $U^{\mathbb{T}}$ preserves $U^{\mathbb{T}}$-absolute sups, and $U^{\mathbb{T}}$ ~~creates~~ reflects $U^{\mathbb{T}}$-absolute sups.

Proof. Let $G\colon \underline{I} \to \underline{A}^{\mathbb{T}}$ be a functor. Let $U^{\mathbb{T}}G = H$ and $U^{\mathbb{T}}\varepsilon^{\mathbb{T}}G = h$, then $G(I) = (H(I), h(I))$. Assume that H has an absolute sup in $\underline{A}$, $(A, u) = \sup H$. Applying T we see that $\sup TH = (TA, Tu)$. Since $TH(I) \xrightarrow{h(I)} H(I) \xrightarrow{u(I)} A$ is natural in I, there exists a unique $a\colon TA \to A$ such that



commutes.

We shall show that (A, a) is a $\mathbb{T}$-algebra. Consider the following diagram:

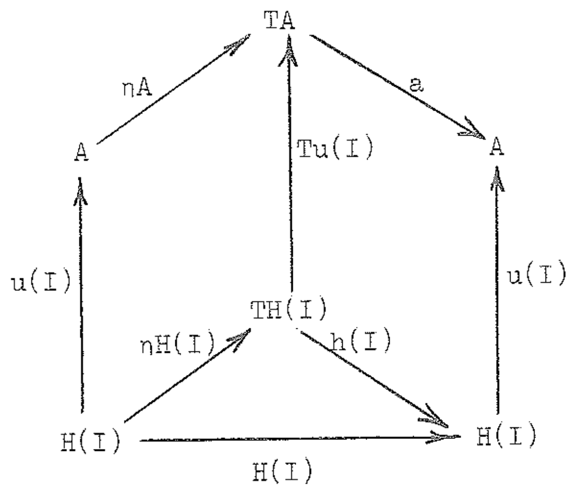


The square on the right commutes by definition of a, the upper square on the left for the same reason, and the lower square on the left commutes by naturality of μ. Therefore

$$a.Ta.T^2u(I) = a.\mu A.T^2u(I)$$

for every $I \in |\underline{I}|$. Since (A, u) is an absolute sup, $\{u(I) | I \in |\underline{I}|\}$ is an absolute joint epi, therefore $a.Ta = a.\mu A$.

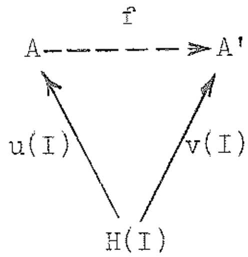
Next consider the following diagram:



The right square commutes by definition of a, the left by naturality of η, the bottom triangle by the $h(I)$ - unitary axiom. Therefore we have $a.\eta A.u(I) = u(I)$ for every $I \in |\underline{I}|$. Since $\{u(I)\}$ is a joint epi, we see that $a.\eta A = A$. This finishes the proof that (A, a) is a $\mathbb{T}$-algebra.

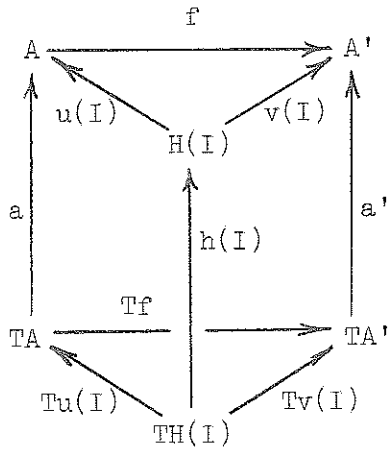
That $u(I): (H(I), h(I)) \longrightarrow (A, a)$ is a $\mathbb{T}$-homomorphism follows from the definition of a. That $u(I)$ is natural in I follows from the fact that it is natural in $\underline{A}$.

Now let $v(I): (H(I), h(I)) \longrightarrow (A', a')$ be natural in I. Then applying $U^{\mathbb{T}}$, $v(I): H(I) \longrightarrow A'$ is natural in I. Since $(A, u) = \sup H$, there exists a unique $\underline{A}$-morphism $f: A \longrightarrow A'$ such that



commutes.

To see that f is a $\mathbb{T}$-homomorphism consider the following diagram:



$$f.a.Tu(I) = f.u(I).h(I) = v(I)h(I) = a'.Tv(I)$$
$$= a'.Tf.Tu(I)$$

But since $\{u(I)\}$ is an absolute joint epi, $f.a = a'.Tf$. Thus f is a $\mathbb{T}$-homomorphism and we conclude that $((A, a), u) = \sup G$. Thus $\underline{A}^{\mathbb{T}}$ has $U^{\mathbb{T}}$-absolute sups.

It is obvious that $U^{\mathbb{T}}$ preserves these sups.

Since the a was uniquely determined by the requirement that $u(I)$ be a homomorphism it is also evident that $U^{\mathbb{T}}$ reflects these sups. ▮

We now state Beck's tripleableness theorem.

(3.5) THEOREM. A functor $U: \underline{B} \to \underline{A}$ is tripleable if and only if U has a left adjoint, $\underline{B}$ has U-absolute coequalizers, and U preserves and reflects U-absolute coequalizers.

Proof. If U is tripleable, we can assume that $\underline{B} = \underline{A}^{\mathbb{T}}$ and that $U = U^{\mathbb{T}}$. Then U has a left adjoint and the rest follows from (3.4).

Now assume that U has a left adjoint F with adjunctions $\varepsilon: FU \to \underline{B}$ and $\eta: \underline{A} \to UF$, and that $\underline{B}$ has U-absolute coequalizers and U preserves and reflects them. We obtain $\overline{\Phi}$, the inverse of Φ as the following coequalizer

$$FT^{n+1}A \underset{\varepsilon FT^nA}{\overset{FT^na}{\rightrightarrows}} FT^nA \to \overline{\Phi}(A, a)$$

where n is a fixed integer $(n \geqslant 0)$ and (A, a) is any $\mathbb{T}$-algebra. This coequalizer exists since, applying U to the pair of maps, we get

$$T^{n+2}A \underset{\mu T^nA}{\overset{T^{n+1}a}{\rightrightarrows}} T^{n+1}A \to A$$

which is an absolute coequalizer by (2.6).

The details which show that Φ and $\overline{\Phi}$ are inverse equivalences are standard (see Beck [1], [2] or Manes [18]) and are left out. ∎

In Beck's original statement of the theorem, contractible coequalizers (see (I, 4.4)) were used instead of absolute coequalizers.

Consider the following situation $\underline{C} \xrightarrow{V} \underline{B} \xrightarrow{U} \underline{A}$ where U and

V are tripleable. It does not follow that UV is tripleable (e.g. Torsion free abelian groups $\longrightarrow$ Ab $\longrightarrow$ S).

Beck has conditions (VTT) which, when imposed on U, ensure that UV is tripleable. We now give these conditions replacing contractible coequalizers by absolute ones.

(3.6) VTT. Let U: $\underline{B} \longrightarrow \underline{A}$ be tripleable and assume that the U-absolute coequalizers that $\underline{B}$ has and U preserves and reflects are themselves absolute. Then for any tripleable V: $\underline{C} \longrightarrow \underline{B}$, UV is tripleable. ∎

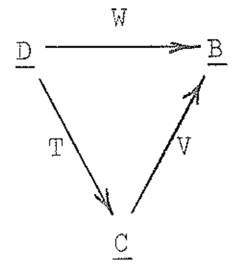
This theorem is not equivalent to Beck's VTT, for it is quite conceivable that coequalizers of U-contractible pairs be absolute but not contractible.

We now prove a sort of VTT which seems to be easier to use in practice.

(3.7) THEOREM. (RVTT = relatively vulgar tripleableness theorem.) Let $\underline{C} \xrightarrow{V} \underline{B} \xrightarrow{U} \underline{A}$ be such that U has a left adjoint, $\underline{B}$ has split idempotents, there exists a functor G: $\underline{A} \longrightarrow \underline{B}$ and there exist natural transformations s and t such that

$$V \xrightarrow{s} GUV \xrightarrow{t} V = V$$

then for any tripleable functor W: $\underline{D} \longrightarrow \underline{B}$ factoring through V,

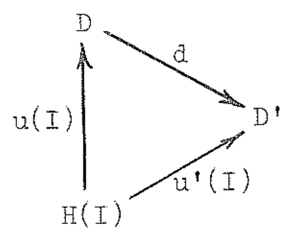


UW is tripleable.

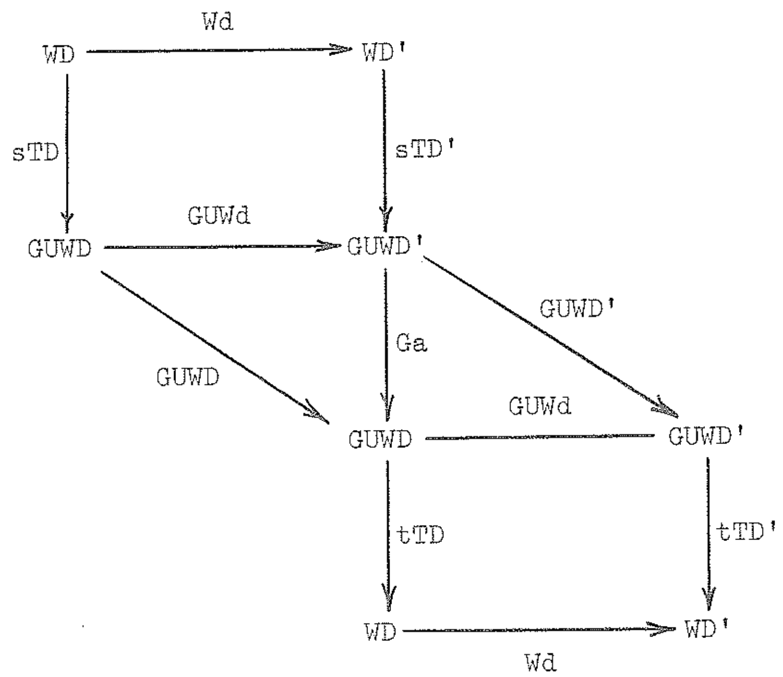
Proof. Since $V \xrightarrow{s} GUV \xrightarrow{t} V = V$, $VT \xrightarrow{sT} GUVT \xrightarrow{tT} VT = VT$, thus $W \xrightarrow{sT} GUW \xrightarrow{tT} W = W$.

Let $H: \underline{I} \longrightarrow \underline{D}$ be such that UWH has an absolute sup in $\underline{A}$. Then GUWH has an absolute sup in $\underline{B}$. Since $WH \longrightarrow GUWH \longrightarrow WH = WH$ by (I, 2.10) WH has absolute weak sups. (I, 3.14) and split idempotents imply that WH has an absolute sup in $\underline{B}$. Since W is tripleable, H has a sup in $\underline{D}$. W preserves this sup (since it is W-absolute) and then UW obviously preserves it.

To see that UW reflects UW-absolute sups let sup H = (D, u) be UW-absolute. Assume that $u'(I): H(I) \longrightarrow D'$ is natural in I, then there exists a unique map $d: D \longrightarrow D'$ such that



commutes. Assume that UWd is an isomorphism in $\underline{A}$ and let its inverse be $a: UWD' \longrightarrow UWD$.



commutes, the left half showing that

$$(tTD.Ga.sTD').Wd = tTD.sTD = WD$$

and the right half showing that

$$Wd.(tTD.Ga.sTD') = tTD'.sTD' = WD'.$$

Therefore Wd is an isomorphism. This shows that U reflected this sup. Since W is tripleable (thus it reflects W-absolute sups) and since this sup which U reflected is absolute in B, W also reflects it, thus showing that UW reflects UW-absolute sups. Thus UW is tripleable. ∎

(3.8) COROLLARY. Let $C \xrightarrow{V} B \xrightarrow{U} A$ be such that U has a left adjoint, B has split idempotents, V is tripleable, there exists a functor $G: A \longrightarrow B$ and there exist natural transformations s and t such that

$$V \xrightarrow{s} GUV \xrightarrow{t} V = V.$$

Then UV is tripleable.

(3.9) COROLLARY. Let $U\colon \underline{B} \longrightarrow \underline{A}$ have a left adjoint and assume that $\underline{B}$ has split idempotents. Assume, furthermore, that there exists a functor $G\colon \underline{A} \longrightarrow \underline{B}$ and there exist natural transformations s and t such that

$$\underline{B} \xrightarrow{s} GU \xrightarrow{t} \underline{B} = \underline{B}.$$

Then U is VTT.

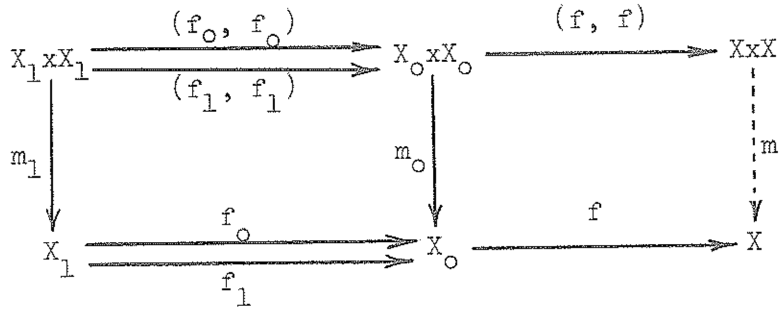
§4. EXAMPLES

The category of groups is tripleable over the category of sets. Let $U\colon \underline{Gr} \longrightarrow \underline{S}$ be the usual underlying functor. It is well known that U has a left adjoint. Let $X_1 \underset{f_1}{\overset{f_o}{\rightrightarrows}} X_o$ be two groups and two group homomorphisms and assume that

$$X_1 \underset{f_1}{\overset{f_o}{\rightrightarrows}} X_o \xrightarrow{f} X$$

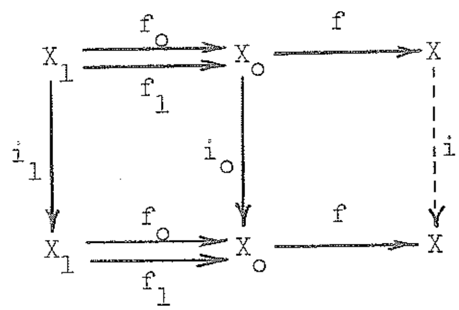
is an absolute coequalizer in sets. We must define a group structure on X in such a way that f is a homomorphism and the coequalizer of f_o and f_1 in $\underline{Gr}$.

Consider the diagram:



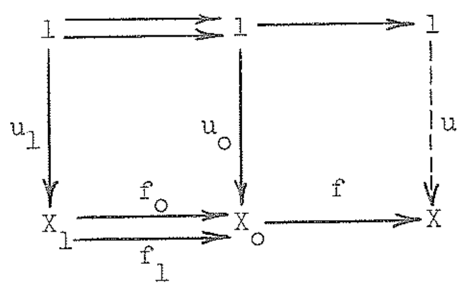
where m_o and m_1 are the multiplications of X_o and X_1 respectively. The upper and lower squares on the left commute because f_o and f_1 are homomorphisms. Thus $f.m_o.(f_o, f_o) = f.m_o.(f_1, f_1)$. But the upper row is a coequalizer diagram because it is the result of applying the squaring functor to an absolute coequalizer diagram. Therefore there exists a <u>unique</u> $m: XxX \longrightarrow X$ making the square on the right commute.

Next we use the same reasoning on the following diagram



where i_o and i_1 are the inverses of X_o and X_1, to get a unique $i: X \longrightarrow X$ such that the right hand square commutes.

The following diagram gives us the unit of the group:



One easily verifies that this defines a group structure on X. This structure was entirely determined by the requirement that f be a homomorphism. Furthermore $X_1 \overset{f_o}{\underset{f_1}{\rightrightarrows}} X_o \xrightarrow{f} X$ is a coequalizer in $\underline{Gr}$. The details are very similar to those of theorem (3.4).

This sketches the proof that $U: \underline{Gr} \to \underline{S}$ is tripleable. This example is typical of varietal categories (see Linton [15]). Indeed varietal categories are known to be tripleable over $\underline{S}$.

Now let $\underline{B}$ be a full reflective subcategory of $\underline{A}$ and $U: \underline{B} \to \underline{A}$ the inclusion. Then if F is the left adjoint of U we have $FU \cong \underline{B}$. If we assume that $\underline{B}$ has split idempotents, then (3.9) implies that U is VTT.

A refinement of (3.7) and thus of (3.8) and (3.9), specifying which idempotents must split, gives the result that the inclusion of any full reflective subcategory is VTT.

Let $\underline{X}$ be a discrete category and let $\underline{A}$ be a pointed category having $\underline{X}$ indexed products. Assume that $\underline{A}$ has split idempotents. We have an adjoint pair $\underline{A}^{\underline{X}} \underset{F}{\overset{U}{\rightleftarrows}} \underline{A}$ where U is the product and F is const, the functor which associates to any object of $\underline{A}$ the constant functor with that value.

We have natural transformations

$$\underline{A}^{\underline{X}} \xrightarrow{s} FU \xrightarrow{t} \underline{A}^{\underline{X}}$$

where $s(A_i): (A_i) \to (\Pi A_i)$ is defined by $s(A_i)_j: A_j \text{——} \Pi A_i$ which is determined by

$$A_j \longrightarrow A_i = \begin{cases} A_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

t is defined by the projection maps. Then $t.s = \underline{A}^{\underline{X}}$ thus U is VTT.

Let $\underline{B} \underset{F}{\overset{U}{\rightleftarrows}} \underline{A}$ be an adjoint pair and assume that $\underline{B}$ has split idempotents. Then one of the adjunction equations says

$$F \xrightarrow{F\eta} FUF \xrightarrow{\varepsilon F} F = F.$$

Thus U is RVTT through F, i.e. if $W\colon \underline{D} \longrightarrow \underline{B}$ is a tripleable functor which factors through F, then UW is tripleable.

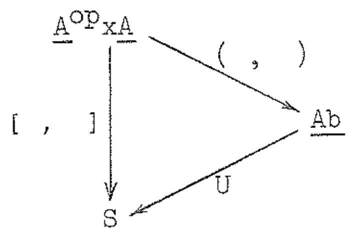
Let $\underline{B} \underset{F}{\overset{U}{\rightleftarrows}} \underline{A}$ be an adjoint pair and assume that $\underline{B}$ has split idempotents and assume also that U has a right adjoint R. Then U is RVTT through R.

CHAPTER III

ADDITIVE ABSOLUTENESS

In this chapter we propose to study additive absoluteness, i.e. properties of diagrams in additive categories which are preserved by all additive functors. We obtain characterizations of some of the usual properties of category theory. These characterizations take the form of equations involving composition, addition, subtraction, identities, and zero.

By an additive category we mean a pointed category $\underline{A}$ such that the hom functor $[\ ,\ ] : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{S}$ factors through the usual underlying functor from the category of abelian groups to sets.



A functor $G: \underline{A} \longrightarrow \underline{A}'$ between two additive categories is called an additive functor if the induced maps

$$(A_o, A_1) \longrightarrow (GA_o, GA_1)$$

are abelian group homomorphisms.

Additive absoluteness has been used often in homological algebra, e.g. split short exact sequences are absolute short exact sequences and for chain complexes, contractible means absolutely acyclic. We leave all questions of homological algebra for a later work.

<u>It is understood that when we speak of absoluteness in this chapter we mean additive absoluteness.</u>

§1. ABSOLUTE EPIMORPHISMS

(1.1) PROPOSITION. Let $\underline{A}$ be an additive category and let $a: A \longrightarrow A'$ be an $\underline{A}$ morphism. Then a is an absolute epi if and only if there exists a map $a': A' \longrightarrow A$ such that $aa' = A'$.

<u>Proof</u>. The sufficiency is obvious.

Now assume that a is an absolute epi. Then $(A', a): (A', A) \longrightarrow (A', A')$ is an epi in $\underline{Ab}$. Thus (A', a) is onto and therefore there exists $a' \in (A', A)$ such that $aa' = A'$. ∎

(1.2) PROPOSITION. Let $\underline{A}$ be an additive category and let $a_i: A_i \longrightarrow A'$ be a family of maps in $\underline{A}$. Then $\{a_i\}$ is an absolute joint epi if and only if there exist maps $a'_i: A' \longrightarrow A_i$, only finitely many non-zero, such that

$$\Sigma a_i a'_i = A'.$$

<u>Proof</u>. Assume that $\Sigma a_i a'_i = A'$ and assume that $xa_i = ya_i$ for all i. Then $xa_i a'_i = ya_i a'_i$ and $\Sigma xa_i a'_i = \Sigma ya_i a'_i$ but $\Sigma xa_i a'_i = x$ and $\Sigma ya_i a'_i = y$ thus $x = y$ and $\{a_i\}$ is a joint epi. However the fact that $\Sigma a_i a'_i = A'$ is preserved by all additive functors, thus $\{a_i\}$ is an absolute joint epi.

Now assume that $\{a_i\}$ is an absolute joint epi. Apply the functor $(A', -)$. $(A', a_i): (A', A_i) \longrightarrow (A', A')$ is a joint epi in $\underline{Ab}$. But this is a joint epi if and only if the induced map $\coprod(A', A_i) \longrightarrow (A', A')$ is epi in $\underline{Ab}$, i.e. onto. Therefore there exists an element of $\coprod(A', A_i)$ which is sent onto A', i.e. there exist $a'_i: A' \longrightarrow A_i$, only finitely many non-zero, such that $\Sigma a_i a'_i = A'$.

§2. ABSOLUTE WEAK SUPREMA

Let $\underline{A}$ be an additive category, $\underline{I}$ a small category (not necessarily additive), and $F: \underline{I} \longrightarrow \underline{A}$ a functor.

(2.1) DEFINITION. An F-matrix is a matrix $(a_{I, J})$ with rows and columns indexed by the objects of $\underline{I}$ such that $a_{I, J}: F(J) \longrightarrow F(I)$ where $a_{I, J}$ is the entry in the I^{th} row and the J^{th} column, and such that only finitely many rows are non-zero.

(2.2) DEFINITION. Let $b: J_o \longrightarrow I_o$ be a map in $\underline{I}$. Define $M(b)$ to be the F-matrix $(a_{I, J})$ where all $a_{I, J}$ are zero except $a_{I_o, J_o} = F(b)$ and $a_{J_o, J_o} = -F(J_o)$. It is understood that if $I_o = J_o$ then $a_{I_o, J_o} = a_{J_o, J_o} = F(b) - F(J_o)$.

(2.3) DEFINITION. Define a D-column to be a column matrix indexed by the objects of $\underline{I}$ such that the I^{th} entry is an $\underline{A}$-morphism $D \longrightarrow F(I)$, all except finitely many being zero.

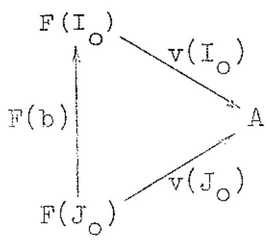
Note that the domains and codomains of the maps in F-matrices and D-columns are so arranged that we can compose F-matrices together and we can compose an F-matrix with a D-column. The finiteness conditions in (2.1) and (2.3) insure that all sums make sense.

(2.4) DEFINITION. Let $E(I)$ denote the $F(I)$-column whose I^{th} entry is $F(I)$ and whose other entries are zero.

The terminology of definitions (2.1), (2.2), (2.3), and (2.4) is not standard.

(2.5) LEMMA. Let $[v(I)]$ be a row matrix indexed by objects of $\underline{I}$ where the I^{th} entry is an $\underline{A}$-morphism $v(I)\colon F(I) \longrightarrow A$ for a fixed A. Then $v(I)$ is natural in I if and only if $[v(I)]M(b) = 0$ for all maps b of $\underline{I}$.

<u>Proof</u>. Let $b\colon J_o \longrightarrow I_o$, then $[v(I)]M(b) = (0, 0, \ldots, 0, v(I_o)F(b) - v(J_o), 0, \ldots)$ which is $[0]$ if and only if $v(I_o)F(b) = v(J_o)$, i.e. if and only if the following diagram commutes



The lemma is now obvious. ∎

We can now state and prove the main theorem of this chapter.

(2.6) THEOREM. Let $\underline{A}$ be an additive category and $\underline{I}$ a small (not necessarily additive) category. If $F\colon \underline{I} \longrightarrow \underline{A}$ is a functor, then (A, u) is an absolute weak sup of F if and only if $u(I)\colon F(I) \longrightarrow A$ is natural in I and there exists an A-column (a_I)

such that for each $J \in |\underline{I}|$ there exist a finite number of $\underline{I}$-morphisms $b_1, b_2, \ldots, b_n$ and as many $F(J)$-columns $B_1, B_2, \ldots, B_n$ such that

$$(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i.$$

Proof. Assume that there exists an A-column (a_I), $a_I\colon A \longrightarrow F(I)$, such that the above conditions are satisfied. Let $v(I)\colon F(I) \longrightarrow A'$ be natural in I. We want a map $f\colon A \longrightarrow A'$ such that $fu(I) = v(I)$ for all I.

Define $f = \sum_I v(I)a_I$ and let $[v(I)]$ be the row matrix, indexed by objects of $\underline{I}$, whose I^{th} entry is $v(I)$.

$$(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i$$

$$[v(I)](a_I)u(J) = [v(I)]E(J) + \sum_{i=1}^{n} [v(I)]M(b_i)B_i$$

Therefore $fu(J) = v(J)$ for all J. This proves that (A, u) is a weak sup of F. But the above conditions, being equations involving composition, addition, subtraction, identities, and zero, are preserved by all additive functors. Thus (A, u) is an absolute weak sup of F.

Assume that (A, u) is an absolute weak sup of F.

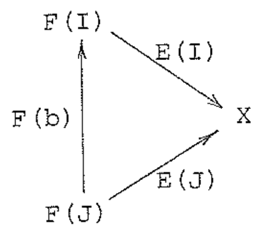
We shall construct an additive category $\underline{A}'$ which contains $\underline{A}$ as a full subcategory. $|\underline{A}'| = |\underline{A}| + \{X\}$ where X is an arbitrary symbol. $(X, X)_{\underline{A}'} = Z$, the ring of integers. For $C \in |\underline{A}|$, $(X, C)_{\underline{A}'} = \{0\}$ and $(C, X)_{\underline{A}'} = \coprod_I (C, F(I))_{\underline{A}'}$ (the coproduct in $\underline{Ab}$). Thus a map $C \longrightarrow X$ can be thought of as a C-column. Composition in $\underline{A}'$ is just ordinary multiplication of a matrix by a scalar. As we have defined it $\underline{A}'$ is an additive category containing $\underline{A}$ as a full subcategory.

Next we define an additive congruence relation, $\equiv$, on the hom sets of $\underline{A}'$. (1) For all $\underline{A}'$-morphisms f, we have $f \equiv f$.
(2) For (a_I), (b_I): $C \longrightarrow X$, $(a_I) \equiv (b_I)$ if and only if there exist a finite number of maps of $\underline{I}$, $b_1, b_2, \ldots, b_n$, and as many C-columns $B_1, B_2, \ldots, B_n$ such that $(a_I - b_I) = \sum_{i=1}^{n} M(b_i)B_i$.

This relation is reflexive, symmetric, transitive, additive, and multiplicative, i.e. it is an additive congruence relation on $\underline{A}'$.

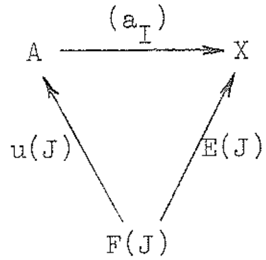
Form the quotient category $\underline{A}'/\equiv$. This category is additive and contains $\underline{A}$ as a full subcategory. We have the following maps $E(I)$: $F(I) \longrightarrow X$ (we represent a congruence class by any one of its elements).

Now $E(I)$ is natural in I, for if b: $J \longrightarrow I$, then $E(I)F(b) - E(J) = M(b)E(J)$ and so $E(J)F(b) \equiv E(I)$ in $\underline{A}'$. Therefore in $\underline{A}'/\equiv$



commutes.

Now since (A, u) is an absolute weak sup in $\underline{A}$ it is also a weak sup in $\underline{A}'/\equiv$. Therefore there exists a map (a_I): $A \longrightarrow X_1$ such that



commutes for all J. Thus $(a_I)u(J) = E(J)$ in $\underline{A}'/\equiv$, i.e. $(a_I)u(J) \equiv E(J)$ in $\underline{A}'$ for every J. Therefore, for every $J \in |\underline{I}|$ there exists a finite number of $\underline{I}$-morphisms $b_1, b_2, \ldots, b_n$, and as many $F(J)$-columns $B_1, B_2, \ldots, B_n$, such that

$$(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i.$$ ∎

(2.7) REMARK. If the condition of theorem (2.6) is satisfied for J and if $b\colon K \longrightarrow J$, then this condition is also satisfied for K.

Indeed, if $(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i$ then $(a_I)u(J)F(b) = E(J)F(b) + \sum_{i=1}^{n} M(b_i)B_iF(b)$. By naturality $u(J)F(b) = u(K)$. Also $B_iF(b)$ is an $F(K)$-column for all i. Finally, note that $E(J)F(b) = E(K) + M(b)E(K)$, thus

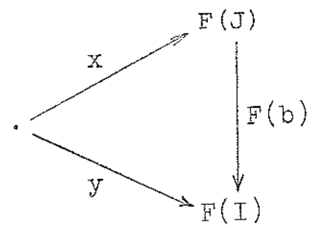
$$(a_I)u(K) = E(K) + M(b)E(K) + \sum_{i=1}^{n} M(b_i)B_iF(b).$$

This proves our claim.

(2.8) REMARK. If one weak sup of F is (additively) absolute, then all weak sups of F are. In this case we say that <u>F has absolute weak sups</u>.

If $F\colon \underline{I} \longrightarrow \underline{A}$ is a functor from a small category $\underline{I}$ to an additive category $\underline{A}$ then it is obvious that if F has a totally absolute weak sup then this weak sup is additively absolute. We now show how the conditions of theorem (I, 2.4) imply the conditions of theorem (2.6).

Let x and y be two maps of $\underline{A}$ such that



commutes. Then $M(b)E(J)x = E(I)y - E(J)x$ thus $E(J)x \equiv E(I)y$ in $\underline{A}'$ (defined in (2.6)). Therefore if $x: F(J) \longrightarrow F(I)$ and $z: F(J) \longrightarrow F(K)$ are connected in $(F(J), F)$ then $E(I)x \equiv E(K)z$.

Since by theorem (I, 2.4) $d_o u(I)$ and $F(J)$ are connected in $(F(J), F)$ for all J, we see that

$$E(I_o)d_o u(J) \equiv E(J)$$

therefore

$$E(I_o)d_o u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i$$

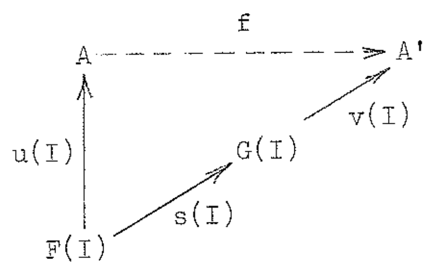
for some b_i and B_i. This shows that (A, u) is an additively absolute weak sup.

(2.9) THEOREM. Let $F: \underline{I} \longrightarrow \underline{A}$ have absolute weak sups. Let $G: \underline{I} \longrightarrow \underline{A}$ be a functor such that there are natural transformations $G \xrightarrow{t} F \xrightarrow{s} G$. Assume that for every I, there exist $\underline{I}$-morphisms $b_1, b_2, \ldots, b_n$, and $G(I)$-columns $B'_1, B'_2, \ldots, B'_n$ such that

$$s(I)t(I)E'(I) = E'(I) + \sum_{i=1}^{n} M'(b_i)B'_i$$

where the primes indicate that the matrices are defined with respect to G. Under these conditions G has absolute weak sups also.

<u>**Proof.**</u> Let (A, u) be a weak sup of F. Now assume that $v(I)\colon G(I) \longrightarrow A'$ is natural in I. Then we have $F(I) \xrightarrow{s(I)} G(I) \xrightarrow{v(I)} A'$ thus there exists $f\colon A \longrightarrow A'$ such that



commutes. Then $fu(I)t(I) = v(I)s(I)t(I)$. But

$$\begin{aligned} v(I)s(I)t(I)E'(I) &= v(I)E'(I) + \sum_{i=1}^{n} v(I)M'(b_i)B'_i \\ &= v(I)E'(I). \end{aligned}$$

Therefore $v(I)s(I)t(I) = v(I)$ for all I, thus $fu(I)t(I) = v(I)$. We see that $(A, u.t)$ is a weak sup of G. Since F has absolute weak sups and the hypotheses are preserved by additive functors, then $(A, u.t)$ is an absolute weak sup of G. ∎

(2.10) COROLLARY. If $F\colon \underline{I} \longrightarrow \underline{A}$ has absolute weak sups and if there exist natural transformations s and t such that

$$G \xrightarrow{t} F \xrightarrow{s} G = G$$

then G has absolute weak sups also. ∎

§3. ABSOLUTE SUPREMA

Let $\underline{I}$ be a small category and $\underline{A}$ an additive category. If $F: \underline{I} \longrightarrow \underline{A}$ is a functor then (A, u) is an absolute sup of F if and only if (A, u) is an absolute weak sup of F (see (2.6)) and $\{u(I) \mid I \in |\underline{I}|\}$ is an absolute joint epi (see (1.2)). The following results tell us even more.

(3.1) THEOREM. Let $F: \underline{I} \longrightarrow \underline{A}$ be a functor from a small category to an additive one. Then (A, u) is an absolute sup of F if and only if (A, u) is an absolute weak sup of F and $\{u(I) \mid I \in |\underline{I}|\}$ is a joint epi.

<u>Proof</u>. The necessity of the condition is obvious.

Now assume that (A, u) is an absolute weak sup and $\{u(I) \mid I \in |\underline{I}|\}$ is a joint epi. By theorem (2.6) there exists an A-column (a_I) such that for each $J \in |\underline{I}|$ there exist a finite number of $\underline{I}$-morphisms $b_1, b_2, \ldots, b_n$, and as many $F(J)$-columns $B_1, B_2, \ldots, B_n$ such that $(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i$. Thus $[u(I)](a_I)u(J) = [u(I)]E(J) + \sum_{i=1}^{n} [u(I)]M(b_i)B_i$. Since $u(I)$ is natural, by lemma (2.5) $[u(I)]M(b_i) = [0]$ thus we have

$$(\sum_I u(I)a_I)u(J) = u(J)$$

But since $\{u(J)\}$ is a joint epi we get

$$\sum_I u(I)a_I = A.$$

This shows that $\{u(I)\}$ is an absolute joint epi and thus proves the theorem. ∎

We now state the characterization of additively absolute sups in its final form.

(3.2) THEOREM. Let $\underline{I}$ be a small category, $\underline{A}$ an additive category, and $F: \underline{I} \longrightarrow \underline{A}$ a functor. (A, u) is an absolute sup of F if and only if there exists an A-column (a_I) such that for every $J \in |\underline{I}|$ there exist $\underline{I}$-morphisms $b_1, b_2, \ldots, b_n$, and $F(J)$-columns $B_1, B_2, \ldots, B_n$, such that

$$(a_I)u(J) = E(J) + \sum_{i=1}^{n} M(b_i)B_i$$

and

$$\sum_I u(I)a_I = A. \quad ▮$$

(3.3) COROLLARY. If $F: \underline{I} \longrightarrow \underline{A}$ has absolute weak sups and has a (strong) sup then this (strong) sup is absolute. ▮

(3.4) THEOREM. Let $F: \underline{I} \longrightarrow \underline{A}$ have absolute weak sups. If (A, u) is one of the absolute weak sups of F, then using the notation of (2.6), $\sum_I u(I)a_I$ is an idempotent and the following are equivalent:

(i) $\sum_I u(I)a_I$ is a split idempotent,

(ii) F has an absolute sup,

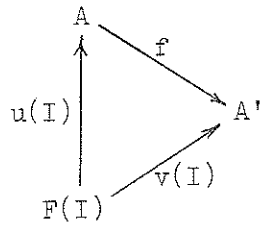
(iii) F has a sup.

Proof. Assume that

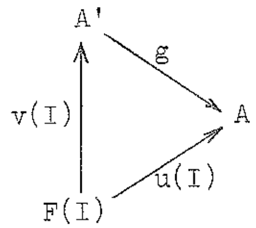
$$(a_I)u(J) = E(J) + \Sigma M(b_i)B_i.$$

Then $[u(I)](a_I)u(J) = [u(I)]E(J) + \Sigma[u(I)]M(b_i)B_i$ and so $[u(I)](a_I)u(J) = u(J)$ i.e. $(\sum_I u(I)a_I)u(J) = u(J)$. Therefore $\sum_I u(I)a_I$ is an idempotent.

Now to see that (iii) implies (i) assume that F has a sup (A', v). By (2.6) we see that $f = \sum_I v(I)a_I$ is such that the following diagram commutes:



Since $(A, v) = \sup F$, there exists a unique $g: A' \longrightarrow A$ such that



commutes and by the usual arguments $fg = A'$. But

$$gf = g\sum_I v(I)a_I = \sum_I gv(I)a_I$$

$$= \sum_I u(I)a_I.$$

This shows that our idempotent is split.

Now to see that (i) implies (iii) assume that $\Sigma u(I)a_I = gf$ such that $fg = A'$. Let $w(I): F(I) \longrightarrow B$ be natural in I. Then there exists $x: A \longrightarrow B$ such that $xu(I) = w(I)$. Consider $xg: A' \longrightarrow B$. Then $xgfu(I) = x. \Sigma u(I)a_I. u(I)$

$$= x.u(I) = w(I).$$

Thus (A', fu) is a weak sup of F.

Assume that $xfu(I) = yfu(I)$ for all I. Then

$xfu(I)a_I = yfu(I)a_I$ and $\sum_I xfu(I)a_I = \sum yfu(I)a_I$ thus

$xf\,\Sigma u(I)a_I = yf\,\Sigma u(I)a_I$ i.e. $xfgf = yfgf$ thus

$$xf = yf$$

$$xfg = yfg$$

$$x = y$$

Therefore $\{fu(I)\}$ is a joint epi and $(A', fu) = \sup F$.

Notice that (ii) is equivalent to (iii) by (3.3) and this finishes the proof. ∎

§4. ABSOLUTE COEQUALIZERS

We now show how these rather complicated conditions simplify in the case of coequalizers.

(4.1) THEOREM. Let $A_1 \xrightarrow{a_o} A_o \xrightarrow{a} A$ be maps of $\underline{A}$. Then a is an absolute weak cokernel of a_o if and only if there exist d_o and d_1, $A \xrightarrow{d_o} A_o \xrightarrow{d_1} A_1$ such that

$$aa_o = 0$$

$$d_o a + a_o d_1 = A_o$$

Moreover, a is an absolute cokernel of a_o if and only if there exist d_o and d_1 satisfying the above conditions as well as $ad_o = A$.

Proof. To say that a is a weak cokernel of a_o means that a is a weak coequalizer of a_o and 0.

Assume that a is an absolute weak cokernel of a_o. Then

$aa_o = 0$. Theorem (2.6) implies that there exist maps $c_1, c_2, \ldots, c_6$ such that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} a = \begin{bmatrix} A_o \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & a_o \\ 0 & -A_1 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -A_1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_6 \end{bmatrix}$$

which implies that

$$c_1 a = A_o + a_o c_4.$$

Taking $d_o = c_1$ and $d_1 = -c_4$ we get the required equation.

Now assume that there exist d_o and d_1 such that $aa_o = 0$ and $d_o a + a_o d_1 = A_o$. Let $x: A_o \longrightarrow X$ be such that $xa_o = 0$. Define $y: A \longrightarrow X$ to be xd_o. Then $xd_o a = x(A_o - a_o d_1) = x - xa_o d_1 = x$. This shows that the conditions imply that we have a weak cokernel, therefore an absolute weak cokernel.

If we have the extra relation $ad_o = A$ then a is an absolute epi, thus we have an absolute cokernel.

Finally if a is an absolute cokernel of a_o then $aa_o = 0$ and $d_o a + a_o d_1 = A_o$. Thus $ad_o a + aa_o d_1 = ad_o a = a$. Since a is epi we have $ad_o = A$. This completes the proof of the theorem. ∎

The dual of (4.1) is the following:

$A \xrightarrow{a} A_o \xrightarrow{a_o} A_1$ is an absolute weak kernel diagram if and only if there exist maps $A_1 \xrightarrow{d_1} A_o \xrightarrow{d_o} A$ such that

$$a_o a = 0$$

$$d_1 a_o + ad_o = A_o$$

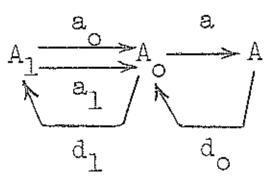
Moreover, a is an absolute kernel of a_o if and only if there exist d_o and d_1 satisfying the above equations as well as

$$d_o a = A.$$

We note that a is an absolute weak kernel of a_o if and only if a_o is an absolute weak cokernel of a.

In view of the fact that $xa_o = xa_1$ if and only if $x(a_o - a_1) = 0$, we see that $A_1 \overset{a_o}{\underset{a_1}{\rightrightarrows}} A_o \xrightarrow{a} A$ is a (weak) coequalizer if and only if a is a (weak) cokernel of $a_o - a_1$, and thus we state the following result.

(4.2) THEOREM. Let $A_1 \overset{a_o}{\underset{a_1}{\rightrightarrows}} A_o \xrightarrow{a} A$ be a diagram in $\underline{A}$. Then this is an absolute weak coequalizer diagram if and only if there exist d_o and d_1



such that $aa_o = aa_1$

$$d_oa + a_od_1 = A_o + a_1d_1.$$

Moreover this is an absolute coequalizer if and only if d_o satisfies the extra condition

$$ad_o = A.$$ ▮

In the case of coequalizers it is easy to see how the conditions for total absoluteness reduce to those for additive absoluteness.

Let $A_1 \overset{a_o}{\underset{a_1}{\rightrightarrows}} A_o \xrightarrow{a} A$ be a totally absolute coequalizer. Then by theorem (I, 4.1) there exist $b\colon A \longrightarrow A_o$ and a finite number of maps $b_i\colon A_o \longrightarrow A_1$ such that

$$\begin{aligned}
aa_o &= aa_1 \\
ab &= A \\
ba &= a_{v(1)}b_1 \\
a_{v(2)}b_1 &= a_{v(3)}b_2 \\
a_{v(4)}b_2 &= a_{v(5)}b_3 \\
&\vdots \\
a_{v(2n)}b_n &= A_o
\end{aligned}$$

where $v(i) = 0$ or 1, $n \geqslant 0$.

The first two equations are the same as in the additive case ($d_o = b$). Now, adding the remaining equations we get

$$ba + (a_{v(2)} - a_{v(1)})b_1 + (a_{v(4)} - a_{v(3)})b_2 + \dots + (a_{v(2n)} - a_{v(2n-1)})b_n = A_o$$

But

$$(a_{v(2i)} - a_{v(2i-1)}) = \begin{cases} a_o - a_1 \\ -(a_o - a_1) \\ 0 \end{cases}$$

Thus $ba + (a_o - a_1)(\Sigma\, \varepsilon_i b_i) = A_o$ where $\varepsilon_i = v(2i-1) - v(2i)$. Therefore $d_1 = \Sigma \varepsilon_i b_i$.

The following theorem is interesting in that it ties up absolute coequalizers with contractible ones. It is also interesting because it is an example where certain simple conditions on a diagram ensure that additive absoluteness implies total absoluteness.

(4.3) THEOREM. Let $A_1 \underset{a_1}{\overset{a_o}{\rightrightarrows}} A_o$ be a reflexive diagram in $\underline{A}$, an additive category. That is to say there is $b\colon A_o \longrightarrow A_1$ such that $a_o b = A_o = a_1 b$. Then $A_1 \underset{a_1}{\overset{a_o}{\rightrightarrows}} A_o \overset{a}{\longrightarrow} A$ is an additively absolute coequalizer if and only if it is a contractible coequalizer.

Proof. Obviously if $A_o \rightrightarrows A_1 \longrightarrow A$ is contractible then it is additively absolute.

Assume now that it is additively absolute. Thus we have d_o and d_1 such that

$$aa_o = aa_1$$

$$ad_o = A$$

$$d_o a + a_o d_1 = A_o + a_1 d_1.$$

Define $d'_1 = b + ba_1 d_1 - d_1\colon A_o \longrightarrow A_1$.

$$a_o d'_1 = a_o b + a_o b a_1 d_1 - a_o d_1$$

$$= A_o + a_1 d_1 - a_o d_1 = d_o a$$

$$a_1 d'_1 = a_1 b + a_1 b a_1 d_1 - a_1 d_1$$

$$= A_o + a_1 d_1 - a_1 d_1 = A_o$$

Thus we have

$$aa_o = aa_1$$

$$ad_o = A$$

$$d_o a = a_o d'_1$$

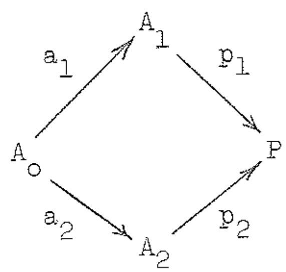
$$a_1 d'_1 = A_o$$

i.e. $A_1 \rightrightarrows A_o \longrightarrow A$ is a contractible coequalizer. ∎

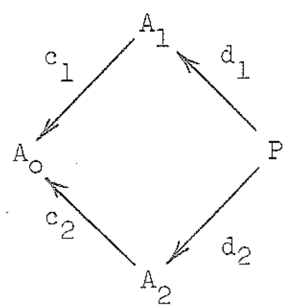
We never used the relation $ad_o = A$ in the proof, thus we could have stated the result for weak coequalizers. In this case a contractible weak coequalizer is the same as the contractible pairs of Manes [18].

§5. ABSOLUTE PUSHOUTS

(5.1) THEOREM. Consider the following diagram in $\underline{A}$



This is an absolute weak pushout diagram if and only if there exist d_1, d_2, c_1, c_2



such that

$$p_1a_1 = p_2a_2$$
$$d_1p_1 = a_1c_1 + A_1$$
$$d_2p_1 = -a_2c_1$$
$$d_1p_2 = -a_1c_2$$
$$d_2p_2 = a_2c_2 + A_2.$$

Furthermore, this diagram is an absolute pushout if and only if we have the extra relation

$$p_1d_1 + p_2d_2 = P.$$

<u>Proof</u>. Assume that we have an absolute weak pushout. Then theorem (2.6) implies that we have maps d_o, d_1, d_2, c_1, c'_1, c_2 c'_2 such that

$$\begin{bmatrix} d_o \\ d_1 \\ d_2 \end{bmatrix} p_1 = \begin{bmatrix} 0 \\ A_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -A_o & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ * \\ * \end{bmatrix} + \begin{bmatrix} -A_o & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c'_1 \\ * \\ * \end{bmatrix}$$

and

$$\begin{bmatrix} d_o \\ d_1 \\ d_2 \end{bmatrix} p_2 = \begin{bmatrix} 0 \\ 0 \\ A_2 \end{bmatrix} + \begin{bmatrix} -A_o & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_2 \\ * \\ * \end{bmatrix} + \begin{bmatrix} -A_o & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c'_2 \\ * \\ * \end{bmatrix}$$

We get

$$d_op_1 = -c_1 - c'_1$$

$$d_1p_1 = A_1 + a_1c_1$$

$$d_2p_1 = a_2c'_1$$

$$d_op_2 = -c_2 - c'_2$$

$$d_1p_2 = a_1c_2$$

$$d_2p_2 = A_2 + a_2c'_2.$$

Solving for c'_1 and c'_2 in the first and the fourth equations and substituting in the third and sixth we get

$$d_1p_1 = A_1 + a_1c_1$$

$$(d_2 + a_2d_o)p_1 = -a_2c_1$$

$$d_1p_2 = a_1c_2$$

$$(d_2 + a_2 d_o)p_2 = A_2 - a_2 c_2$$

Now set $d'_2 = d_2 + a_2 d_o$ and $c''_2 = -c_2$. The equations now look like

$$d_1 p_1 = A_1 + a_1 c_1$$

$$d'_2 p_1 = -a_2 c_1$$

$$d_1 p_2 = -a_1 c''_2$$

$$d'_2 p_2 = A_2 + a_2 c''_2$$

the required result.

Now assume that we have d_1, d_2, c_1, c_2 satisfying the given equations. Let $x_1\colon A_1 \longrightarrow X$ and $x_2\colon A_2 \longrightarrow X$ be such that $x_1 a_1 = x_2 a_2$. Define $P \longrightarrow X$ to be $x_1 d_1 + x_2 d_2$.

$$\begin{aligned}(x_1 d_1 + x_2 d_2)p_1 &= x_1 d_1 p_1 + x_2 d_2 p_1 \\ &= x_1(a_1 c_1 + A_1) + x_2(-a_2 c_1) \\ &= x_1 a_1 c_1 + x_1 - x_2 a_2 c_1 \\ &= x_1\end{aligned}$$

$$\begin{aligned}(x_1 d_1 + x_2 d_2)p_2 &= x_1 d_1 p_2 + x_2 d_2 p_2 \\ &= x_1(-a_1 c_2) + x_2(a_2 c_2 + A_2) \\ &= -x_1 a_1 c_2 + x_2 a_2 c_2 + x_2 \\ &= x_2\end{aligned}$$

Thus the equations force the given diagram to be a weak pushout, therefore an absolute weak pushout.

Next, if we have $p_1 d_1 + p_2 d_2 = P$ then $\{p_1, p_2\}$ is an absolute joint epi, therefore this extra condition ensures that we have an absolute pushout.

Finally, assume again the equations in the statement and suppose that the diagram is a pushout. Then

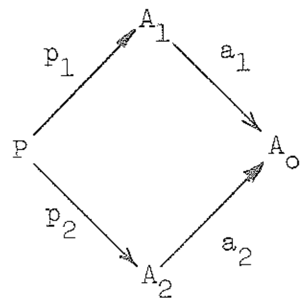
$$p_1d_1p_1 + p_2d_2p_1 = p_1a_1c_1 + p_1 - p_2a_2c_1$$
$$= p_1$$
$$p_1d_1p_2 + p_2d_2p_2 = -p_1a_1c_2 + p_2a_2c_2 + p_2$$
$$= p_2$$

Since $\{p_1, p_2\}$ is a joint epi, we get

$$p_1d_1 + p_2d_2 = P.$$

This completes the proof of our theorem. ∎

We now state the dual result.



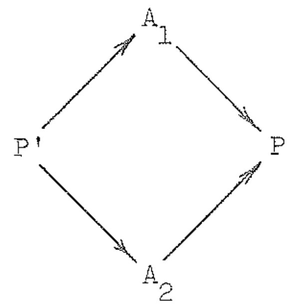
is an absolute weak pullback diagram if and only if there exist d_1, d_2, c_1, c_2 such that

$$a_1p_1 = a_2p_2$$
$$p_1d_1 = c_1a_1 + A_1$$
$$p_1d_2 = -c_1a_2$$
$$p_2d_1 = -c_2a_1$$
$$p_2d_2 = c_2a_2 + A_2.$$

Furthermore, this is an absolute pullback if and only if we have the extra relation

$$d_1p_1 + d_2p_2 = P.$$

We see, by interchanging the c_i with the d_i and by changing the sign, that



is an absolute weak pushout if and only if it is an absolute weak pullback.

From this we see that if

$$A \longrightarrow A_o \rightrightarrows A_1$$

is an absolute weak cokernel pair diagram then it is also an absolute weak equalizer diagram.

§6. ABSOLUTE COPRODUCTS

In the case of total absoluteness, a disconnected diagram never had an absolute sup. For additive absoluteness the ban on disconnected diagrams is lifted. In fact disconnected diagrams are particularly well disposed to having additively absolute sups, as long as there are not too many components.

Trivially the zero object is absolute, thus initial objects are absolute (initial object = sup of the empty diagram = empty coproduct). Thus weak initial objects are also absolute, but this is trivially so, because all objects in an additive category are weak initial.

(6.1) THEOREM. A (weak) coproduct is absolute if and only if there are only finitely many non-zero terms.

Proof. Let $\{A_i\}$ be a family of objects of $\underline{A}$ and let (A, u) be an absolute weak coproduct of the A_i. Since the index category $\underline{I}$ is discrete the only maps are identities but $M(id) =$ zero matrix. Therefore the conditions of theorem (2.6) become: (A, u) is an absolute weak coproduct if and only if there exists a family of maps $a_i: A \longrightarrow A_i$, only finitely many non-zero, such that

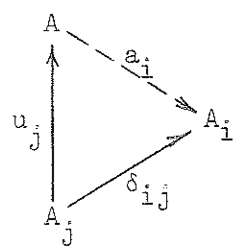
$$(a_i)u_j = E_j$$

or

$$a_i u_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ A_i & \text{if } i = j. \end{cases}$$

Since only finitely many a_i are non-zero, then only finitely many A_i can be non-zero.

Now assume that only finitely many A_i are non-zero and let (A, u) be a weak coproduct of $\{A_i\}$. For a fixed i we have a family of maps $\delta_{i,j}: A_j \longrightarrow A_i$ thus there exists a map $a_i: A \longrightarrow A_i$ such that



Thus $a_i u_j = \delta_{ij}$ and since almost all A_i are zero then almost all

a_i are zero. Then by the above characterization, (A, u) is an absolute weak coproduct of the $\{A_i\}$. ∎

We state the characterization in terms of equations for reference.

(6.2) THEOREM. Let $\{A_i\}$ be a family of objects of $\underline{A}$. Let A be an object of $\underline{A}$ and $u_i\colon A_i \longrightarrow A$ a family of maps. Then (A, u) is an absolute weak coproduct of the A_i if and only if almost all A_i are zero, and there exist maps $a_i\colon A \longrightarrow A_i$ such that

$$a_i u_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ A_i & \text{if } i = j. \end{cases}$$

Moreover (A, u) is an absolute coproduct if and only if we have the extra condition

$$\Sigma u_i a_i = A.$$ ∎

We remark that the notion of absolute (weak) coproduct is self-dual. Thus A is an absolute (weak) coproduct of $\{A_i\}$ if and only if A is an absolute (weak) product of $\{A_i\}$.

(6.3) THEOREM. Let $F\colon \underline{I} \longrightarrow \underline{A}$ be a functor into an additive category. Assume that $\underline{I} = \underline{I}' + \underline{I}''$. Then by restriction we have $F'\colon \underline{I}' \longrightarrow \underline{A}$ and $F''\colon \underline{I}'' \longrightarrow \underline{A}$. If F has a weak sup then so do F' and F''. If F has absolute weak sups so do F' and F''. If F' and F'' have weak sups (A', u') and (A'', u'') respectively and if A' and A'' have a weak coproduct then F has a weak sup. If, furthermore, F' and F'' have absolute weak sups then so does F.

Proof. Let (A, u) be a weak sup of F. Let u' be u

restricted to $\underline{I}'$, then it is easy to see that (A, u') is a weak sup of F'. It follows that if F has absolute weak sups then so does F'. We have the same results for F''.

Assume that F' and F'' have weak sups (A', u') and (A'', u'') respectively. Let A be a weak coproduct with injections $a': A' \longrightarrow A$ and $a'': A'' \longrightarrow A$. Define $u(I): F(I) \longrightarrow A$ by

$$u(I) = \begin{cases} a'u'(I) & \text{if } I \in |\underline{I}'| \\ a''u''(I) & \text{if } I \in |\underline{I}''| \end{cases}$$

u is natural and (A, u) is a weak sup of F.

Since finite weak coproducts are absolute then if F' and F'' have absolute weak sups, so does F. ∎

Most of the interesting additive categories have finite coproducts and then we have the following result.

(6.4) COROLLARY. If $\underline{A}$ has finite coproducts then a disconnected diagram with a finite number of components has absolute weak sups if and only if each component has absolute weak sups. ∎

§7. REMARKS

As we see from the proofs, it is not necessary to demand that the properties be preserved by all functors but only by additive full embeddings.

It is possible to characterize additively absolute sups by demanding only that they be preserved by representable functors. For this one must know what sups are in $\underline{Ab}$. We show how this can be done by working out a typical example.

Let $A_1 \overset{a_0}{\underset{a_1}{\rightrightarrows}} A_0 \xrightarrow{a} A$ be an absolute coequalizer in $\underline{A}$, an additive category. Apply $(A, -)$ to this diagram.

$$(A, A_1) \rightrightarrows (A, A_0) \xrightarrow{(A,\ a)} (A, A)$$

is a coequalizer in $\underline{\text{Ab}}$. Thus (A, a) is epi in $\underline{\text{Ab}}$, but epi is the same as onto in $\underline{\text{Ab}}$. Thus there exists a map $b \in (A, A_0)$ such that $(A, a)(b) = ab = A$.

We now apply the functor $(A_0, -)$.

$$(A_0, A_1) \rightrightarrows (A_0, A_0) \xrightarrow{(A_0,\ a)} (A_0, A)$$

is a coequalizer in $\underline{\text{Ab}}$.

Coequalizers are constructed as follows in $\underline{\text{Ab}}$: Let $G_1 \overset{g_0}{\underset{g_1}{\rightrightarrows}} G_0$ be two abelian group homomorphisms. Let $H = \{g_0(x) - g_1(x) \mid x \in G_1\}$. H is a subgroup and the coequalizer is G_0/H with the canonical surjection.

Since $(A_0, a)(ba) = aba = a = (A_0, a)(A_0)$, thus $ba - A_0 \in \{a_0 x - a_1 x \mid x \in (A_0, A_1)\}$. Therefore we have a map $x \in (A_0, A_1)$ such that

$$ba - A_0 = a_0 x - a_1 x.$$

This is the required characterization.

CHAPTER IV

Cat - ABSOLUTENESS

In this chapter we are concerned with those properties of functors which are preserved by all hyperfunctors from Cat to itself. Here we take Cat to be the hypercategory (see Eilenberg-Kelly [5]) of small categories with functors as morphisms and natural transformations as hypermorphisms. A hyperfunctor from Cat to itself sends categories to categories, functors to functors between the corresponding categories, and natural transformations to natural transformations between the corresponding functors. Furthermore a hyperfunctor is required to preserve both kinds of identities and the four kinds of composition.

One can also take Cat to be Lawvere's hypercategory of all categories.

We shall see that Cat-absolute properties can be expressed in terms of equations involving functors, natural transformations, and identities. The Cat-absolute property derived from some property is, in a sense, the closest one can come to defining the given property in terms of the hyperstructure of Cat.

The fact that a functor is an equivalence of categories is a Cat-absolute property. To say that (T, η, μ) is a triple is a Cat-absolute property (we used this fact in chapter II, where we applied the hyperfunctor $(\ )^{\underline{A}}$ to a given triple).

The example which motivated this study is the following. Clearly, the fact that a functor U has a left adjoint is a Cat-absolute property. Functors with left adjoints preserve monos, thus they preserve monos absolutely. As we shall see, assuming certain mild completeness properties, this is the whole story; i.e. functors which preserve monos absolutely usually have left adjoints.

From now on, when we talk of absoluteness we mean Cat-absoluteness. We also make the convention to underline hyperfunctors.

§1. ABSOLUTE PRESERVATION

(1.1) THEOREM. Let $U\colon \underline{B} \longrightarrow \underline{A}$ be a functor between two small categories. Then the following statements are equivalent:

(i) U preserves weak infs absolutely

(ii) U preserves (strong) infs absolutely

(iii) U preserves monos absolutely

(iv) U preserves those monos which are also epis absolutely (only the property of being mono is to be preserved)

(v) There exist a functor $F\colon \underline{A} \longrightarrow \underline{B}$ and two natural transformations $\varepsilon\colon FU \longrightarrow \underline{B}$ and $\eta\colon \underline{A} \longrightarrow UF$ such that

$$U \xrightarrow{\ \eta U\ } UFU \xrightarrow{\ U\varepsilon\ } U = U.$$

Proof. Plan of proof:

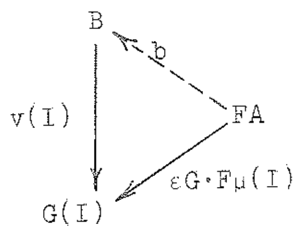
$$\begin{array}{ccccccc} & \nearrow & (i) & \searrow & & & \\ (v) & & & & (iii) \Longrightarrow (iv) \Longrightarrow (v) & & \\ & \searrow & (ii) & \nearrow & & & \end{array}$$

To see that (v) implies (i) assume that $G\colon \underline{I} \longrightarrow \underline{B}$ is a functor with a weak inf (B, v) and let F, ε, η be as in (v).

Assume that $\mu(I): A \longrightarrow UG(I)$ is natural in I. Then we have

$$FA \xrightarrow{F\mu(I)} FUG(I) \xrightarrow{\varepsilon G(I)} G(I)$$

which is also natural in I. Thus there exists a map $b: FA \longrightarrow B$ such that the following diagram commutes:



Applying U we get

$$Uv(I).Ub = U\varepsilon G(I).UF\mu(I)$$

Thus

$$\begin{aligned} Uv(I).Ub.\eta A &= U\varepsilon G(I).UF\mu(I).\eta A \\ &= U\varepsilon G(I).\eta UG(I).\mu(I) \\ &= \mu(I) \end{aligned}$$

This shows that (UB, Uv) is a weak inf of UG and thus (v) implies that U preserves weak infs. But (v) is obviously an absolute property therefore U preserves weak infs absolutely.

That (i) implies (iii) and (ii) implies (iii) follow from the fact that a map $b: B \longrightarrow B'$ is mono if and only if $B \overset{B}{\underset{B}{\rightrightarrows}} B \xrightarrow{b} B'$ is a weak kernel pair, and this is so if and only if $B \overset{B}{\underset{B}{\rightrightarrows}} B \xrightarrow{b} B'$ is a strong kernel pair.

That (iii) implies (iv) is obvious.

We now show that (iv) implies (v). For this we shall construct a hypercategory $\underline{\underline{A}}$ by adding to $\underline{\underline{Cat}}$ an extra object $\underline{X}$ and morphisms and hypermorphisms as described below.

There are three kinds of morphisms $\underline{X} \to \underline{C}$: $(F, 1)$ where $F: \underline{A} \to \underline{C}$ and $(G, 2)$ and $(G, 3)$ where $G: \underline{B} \to \underline{C}$.

There are no morphisms $\underline{C} \to \underline{X}$ and only $\underline{X}: \underline{X} \to \underline{X}$.

The hypermorphisms $(F, i) \to (F', i)$ are (t, i) where $t: F \to F'$ and $i = 1, 2, 3$.

The hypermorphisms $(F, 1) \to (G, 2)$ are (t, u) and (t, v) where $t: FU \to G$ and u and v are arbitrary symbols.

The hypermorphisms $(G, 2) \to (H, 3)$ are of the form (s, w) where $s: G \to H$ and w is an arbitrary symbol.

The hypermorphisms $(F, 1) \to (H, 3)$ are of the form (r, w') where $r: FU \to H$ and w' is an arbitrary symbol.

There are no hypermorphisms $(F, i) \to (F', j)$ for $i > j$.

We define composition by the following relations:

$$K(F, i) = (KF, i)$$

$$K(t, *) = (Kt, *)$$

$$t(F, i) = (tF, i)$$

$$(s, i).(t, i) = (s.t, i)$$

$$(s, 2).(t, u) = (s.t, u)$$

$$(s, 2).(t, v) = (s.t, v)$$

$$(r, 3).(s, w) = (r.s, w)$$

$$(r, 3).(s.w') = (r.s, w')$$

$$(s. w).(r, 2) = (s.r, w)$$

$$(s, w).(t, u) = (s.t, w') = (s, w).(t, v)$$

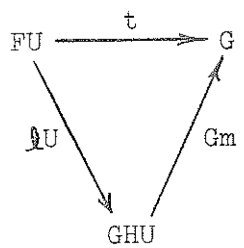
$$(t, u).(s, 1) = (t.sU, u)$$

$$(t, v).(s, 1) = (t.sU, v)$$

$$(s, w').(r, 1) = (s.rU, w')$$

With composition defined this way, A is a hypercategory and the embedding of Cat in A is a hyperfunctor. The details are straightforward and are left to the reader.

Next we define a congruence relation on the hypermorphisms of A. Every hypermorphism is congruent to itself and $(t, u), (t, v): (F, 1) \longrightarrow (G, 2)$ are congruent if and only if there exist $H: A \longrightarrow B$, $\ell: F \longrightarrow GH$, and $m: HU \longrightarrow B$ such that



commutes. This indeed defines a congruence relation on the hypermorphisms, in the sense that it respects all possible kinds of composition in A.

Form the quotient hypercategory B which has the same objects and morphisms as A, and whose hypermorphisms are congruence classes of hypermorphisms of A. Cat is still embedded in B and this embedding is a hyperfunctor.

Define the hyperfunctor F to be the composition

$$\text{Cat} \hookrightarrow B \xrightarrow{(X, -)} \text{Cat}.$$

$\underline{F}(U)$: $\underline{F}(\underline{B}) \longrightarrow \underline{F}(\underline{A})$. Therefore $\underline{F}(U)$ preserves those monos which are epis.

The objects of $\underline{F}(\underline{B}) = (\underline{X}, \underline{B})$ are $(F, 1)$ where $F: \underline{A} \longrightarrow \underline{B}$ and $(G, 2)$ and $(G, 3)$ where $G: \underline{B} \longrightarrow \underline{B}$. Consider the map $(\underline{B}, w): (\underline{B}, 2) \longrightarrow (\underline{B}, 3)$.

$(\underline{B}, w)$ is epi in $\underline{F}(\underline{B})$ for if $(s, 3).(\underline{B}, w) = (s', 3).(\underline{B}, w)$ then

$$(s, w) = (s', w)$$

thus

$$s = s'$$

therefore

$$(s, 3) = (s', 3)$$

$(\underline{B}, w)$ is mono in $\underline{F}(\underline{B})$ for if $(\underline{B}, w).(r, 2) = (\underline{B}, w).(r', 2)$ then

$$(r, w) = (r', w)$$

so

$$r = r'$$

therefore

$$(r, 2) = (r', 2).$$

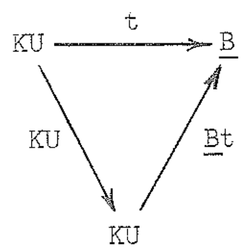
If $(\underline{B}, w).(t, u_1) = (\underline{B}, w).(t', u_2)$ for some $t, t': KU \longrightarrow \underline{B}$ then

$$(t, u_1) = (t', u_2)$$

thus

$$t = t'.$$

Since

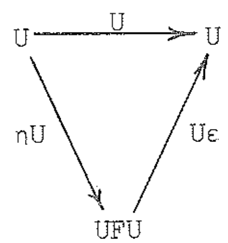


commutes then $(t, u_1) \equiv (t', u_2)$ in $\underline{\underline{A}}$, i.e. $(t, u_1) = (t', u_2)$ in $\underline{\underline{B}}$ where $u_i = u$ or v. This shows that $(\underline{B}, w)$ is mono.

Consequently $\underline{F}(U)(\underline{B}, w) = U(\underline{B}, w) = (U, w)$ is mono in $\underline{F}(\underline{A})$. The objects of $\underline{F}(\underline{A})$ are of the form $(F, 1)$ where $F\colon \underline{A} \longrightarrow \underline{A}$ and $(G, 2)$ and $(G, 3)$ where $G\colon \underline{B} \longrightarrow \underline{A}$. Consider

$$(\underline{A}, 1) \underset{(U,\ v)}{\overset{(U,\ u)}{\rightrightarrows}} (U, 2) \xrightarrow{(U,\ w)} (U, 3).$$

Both compositions are equal to (U, w'). Therefore $(U, u) = (U, v)$ in $\underline{F}(\underline{A})$, i.e. $(U, u) \equiv (U, v)$ in $\underline{\underline{A}}$. By definition of $\equiv$, there exist $F\colon \underline{A} \longrightarrow \underline{B}$, $\eta\colon \underline{A} \longrightarrow UF$, and $\varepsilon\colon FU \longrightarrow \underline{B}$ such that



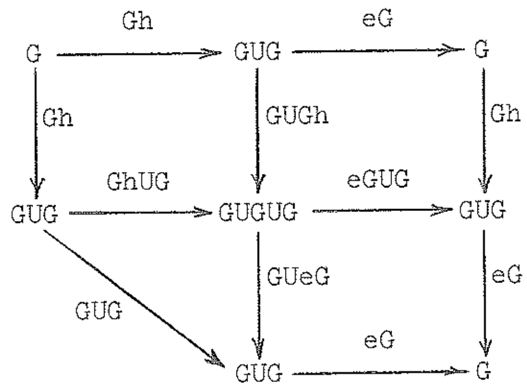
commutes. This completes the proof of the theorem. ∎

We now establish when such a functor has a left adjoint.

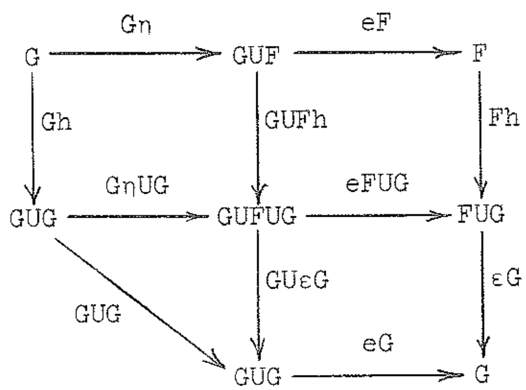
(1.2) THEOREM. Assume that $\underline{B} \underset{G}{\overset{U}{\rightleftarrows}} \underline{A}$ are functors such that there exist $e\colon GU \longrightarrow \underline{B}$ and $h\colon \underline{A} \longrightarrow UG$ such that

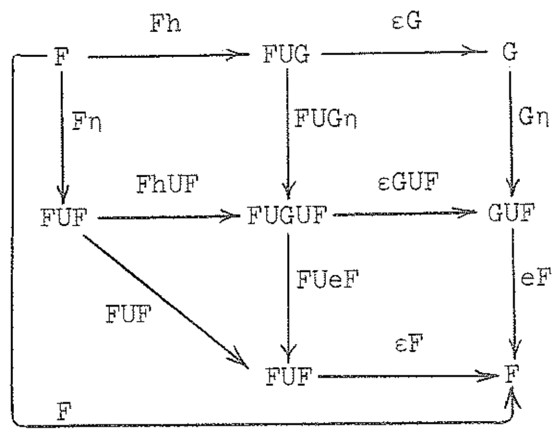
$U \xrightarrow{hU} UGU \xrightarrow{Ue} U = U$. Then $G \xrightarrow{Gh} GUG \xrightarrow{eG} G$ is an idempotent in $\underline{B}^{\underline{A}}$ and U has a left adjoint F, if and only if $eG.\,Gh$ is a split idempotent.

Proof. The commutativity of the following diagram shows that $eG.\,Gh$ is an idempotent.

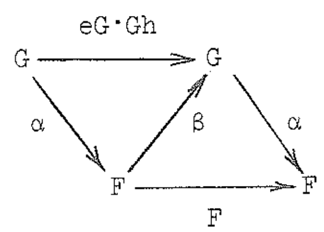


Assume that U has a left adjoint F with adjunctions $\varepsilon\colon FU \to \underline{B}$ and $\eta\colon \underline{A} \to UF$. Then the following diagrams show how $eG.\,Gh$ splits.

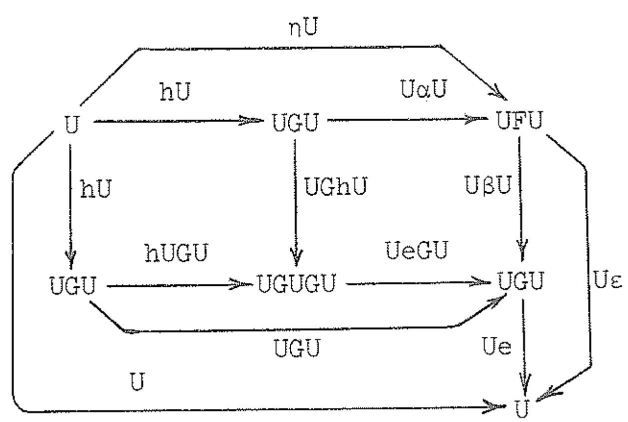


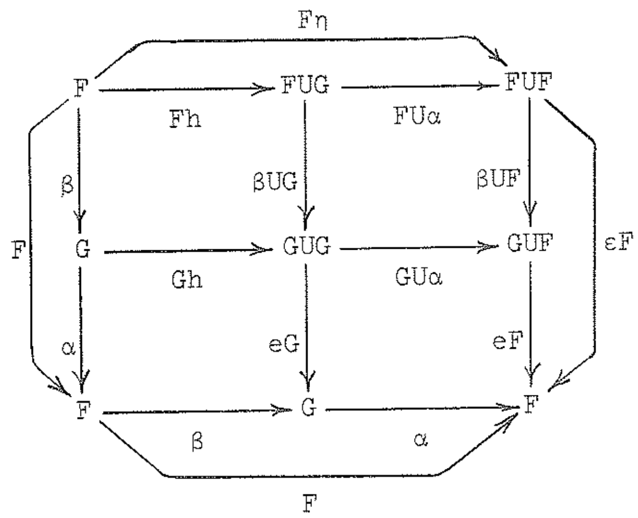


Assume that eG. Gh splits making the following diagram commute.



Define $\varepsilon = FU \xrightarrow{\beta U} GU \xrightarrow{e} \underline{B}$ and $\eta = \underline{A} \xrightarrow{h} UG \xrightarrow{U\alpha} UF$. Now the following diagrams commute showing that F is left adjoint to U with adjunctions η and ε.





∎

(1.3) COROLLARY. Let $\underline{A}$ and $\underline{B}$ be small categories and assume that $\underline{B}$ has split idempotents. Then $\underline{B}^{\underline{A}}$ has split idempotents and $U: \underline{B} \longrightarrow \underline{A}$ preserves monos absolutely if and only if it has a left adjoint. ∎

We now give an example of a functor which preserves monos absolutely but has no adjoint.

Let $\underline{B}$ be the category with one object B and two maps $\{0, 1\}$, composition being ordinary multiplication. Let $\underline{A}$ be the one morphism category $\mathbb{1}$. $U: \underline{B} \longrightarrow \underline{A}$ is the only functor and $G: \underline{A} \longrightarrow \underline{B}$ is also the only functor. Now $UG: \underline{A} \longrightarrow \underline{A}$ must be the identity thus define $\eta: \underline{A} \longrightarrow UG$ to be $\underline{A}: \underline{A} \longrightarrow \underline{A}$. Define $\varepsilon: GU \longrightarrow \underline{B}$ by $\varepsilon(B) = 0$. Then we have

$$U \xrightarrow{\eta U} UGU \xrightarrow{U\varepsilon} U = U.$$

However there can be no adjoint to U, for the only hom set in $\underline{B}$ has 2 elements and the only hom set in $\underline{A}$ has 1 (they cannot be isomorphic).

We obtain the dual statements as follows. Assume, for example, that $U: \underline{B} \to \underline{A}$ preserves epis absolutely. Consider $\underline{D}: \underline{\underline{Cat}} \to \underline{\underline{Cat}}$, $\underline{D}(\underline{C}) = \underline{C}^{op}$. $\underline{D}$ is not a hyperfunctor since it interchanges the order of composition of hypermorphisms, but it has all the desirable properties. If $\underline{G}: \underline{\underline{Cat}} \to \underline{\underline{Cat}}$ is any hyperfunctor then $\underline{DGD}$ is also a hyperfunctor. Thus $\underline{DGD}(U)$ preserves epis. Then we see that $\underline{GD}(U)$ preserves monos, i.e. $\underline{G}(U^{op}): \underline{G}(\underline{B}^{op}) \to \underline{G}(\underline{A}^{op})$ preserves monos for all hyperfunctors G. Therefore $U^{op}: \underline{B}^{op} \to \underline{A}^{op}$ preserves monos absolutely, and by (1.1) there exist $G: \underline{A}^{op} \to \underline{B}^{op}$, $e: GU^{op} \to \underline{B}^{op}$, $h: \underline{A}^{op} \to U^{op}G$ such that

$$U^{op} \xrightarrow{hU^{op}} U^{op}GU^{op} \xrightarrow{U^{op}e} U^{op} = U^{op}$$

Putting $F = G^{op}$, $\varepsilon = h^{op}$, $\eta = e^{op}$ we get $\varepsilon: UF \to \underline{A}$ and $\eta: \underline{B} \to FU$ such that

$$U \xrightarrow{U\eta} UFU \xrightarrow{\varepsilon U} U = U.$$

If $\underline{B}$ has split idempotents then U has a right adjoint.

§2. ABSOLUTE REFLECTION

If P is a property, then to say that U reflects P means that U preserves "not P", thus reflection properties are really preservation properties.

(2.1) THEOREM. Let $U: \underline{B} \to \underline{A}$ be a functor between small categories. Then the following are equivalent:

(i) U reflects isos absolutely,

(ii) U reflects monos absolutely,

(iii) U reflects epis absolutely,

(iv) U reflects weak infs absolutely,

(v) U reflects weak sups absolutely,

(vi) U reflects joint monos absolutely,

(vii) U reflects joint epis absolutely,

(viii) U reflects infs absolutely,

(ix) U reflects sups absolutely,

(x) U reflects totally absolute monos absolutely,

(xi) U reflects totally absolute epis absolutely,

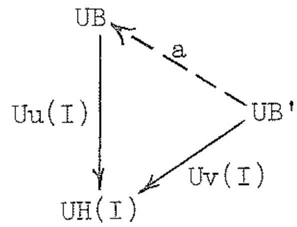
(xii) U reflects totally absolute infs absolutely (where by U reflects totally absolute infs we mean that U not only reflects the inf but also the absoluteness),

(xiii) U reflects totally absolute sups absolutely,

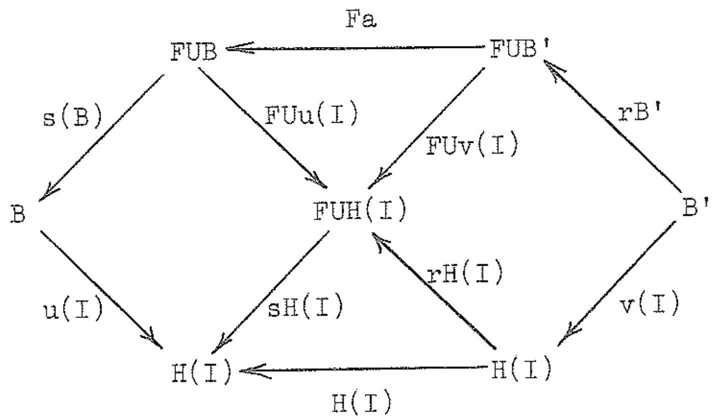
(xiv) there exist a functor $F\colon \underline{A} \longrightarrow \underline{B}$ and two natural transformations $r\colon \underline{B} \longrightarrow FU$ and $s\colon FU \longrightarrow \underline{B}$ such that

$$\underline{B} \xrightarrow{r} FU \xrightarrow{s} \underline{B} = \underline{B}.$$

<u>Proof</u>. (xiv)$\Longrightarrow$(iv). Assume that we have F, r, s as above. Let $u(I)\colon B \longrightarrow H(I)$ be natural in I, where $H\colon \underline{I} \longrightarrow \underline{B}$. Assume that (UB, Uu) is a weak inf of UH in $\underline{A}$. Now let $v(I)\colon B' \longrightarrow H(I)$ be natural in I. Then there exists a map a such that



commutes. Therefore



commutes, showing that (B, u) is a weak sup of H.

By duality, (xiv) $\Longrightarrow$ (v).

The following implications are straightforward:

(iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (ii)
$\searrow$ (xi)

(v) $\Longrightarrow$ (vii) $\Longrightarrow$ (iii)
$\searrow$ (x)

((ii) $\wedge$ (xi)) $\Longrightarrow$ (i)

((iv) $\wedge$ (vi)) $\Longrightarrow$ (viii)

((v) $\wedge$ (vii)) $\Longrightarrow$ (ix)

((I, 2.10) $\wedge$ (xiv) $\wedge$ (ix)) $\Longrightarrow$ (xii). Indeed, let u(I): H(I) $\longrightarrow$ B be natural in I and assume that (UB, Uu) is an absolute sup of UH. Then (ix) $\Longrightarrow$ (B, u) = sup H. (xiv) $\Longrightarrow$ H $\longrightarrow$ FUH $\longrightarrow$ H = H thus (I, 2.10) $\Longrightarrow$ H has absolute weak sups. Therefore (B, u) is an absolute sup of H. (dual of (I, 2.10) $\wedge$ (xiv) $\wedge$ (viii)) $\Longrightarrow$ (xii) This shows that (xiv) implies all the other statements.

To show the converse we have the following straightforward implications:

(iv) $\Longrightarrow$ (ii)

(v) $\Longrightarrow$ (iii)

(vi) $\Longrightarrow$ (ii)

(vii) $\Longrightarrow$ (iii)

(viii) $\Longrightarrow$ (i)

(ix) $\Longrightarrow$ (i)

(xii) $\Longrightarrow$ (i)

(xiii) $\Longrightarrow$ (i)

Now each of (i), (ii), and (x) implies that U reflects isos to monos absolutely (U reflects isos to monos meaning that Um iso $\Longrightarrow$ m mono). Also (iii) is the dual of (ii) and (xi) is the dual of (x). Therefore it will be sufficient to show that (U reflects isos to monos absolutely) $\Longrightarrow$ (xiv).

We shall construct a hypercategory $\underline{\underline{A}}$ containing $\underline{\underline{Cat}}$, by adding a new object $\underline{X}$ and morphisms and hypermorphisms as described below.

The morphisms $\underline{X} \longrightarrow \underline{C}$ are of the form (F, i) where F: $\underline{B} \longrightarrow \underline{C}$ and i = 1, 2, 3. There are no morphisms $\underline{C} \longrightarrow \underline{X}$ and only $\underline{X}$: $\underline{X} \longrightarrow \underline{X}$.

The hypermorphisms $(F, i) \longrightarrow (F', i)$ are of the form (t, i) where $t: F \longrightarrow F'$ for $i = 1, 2, 3$.

The hypermorphisms $(F, 1) \longrightarrow (G, 2)$ are of the form (t, u_1) and (t, u_2) for $t: F \longrightarrow G$ and where u_1 and u_2 are arbitrary symbols.

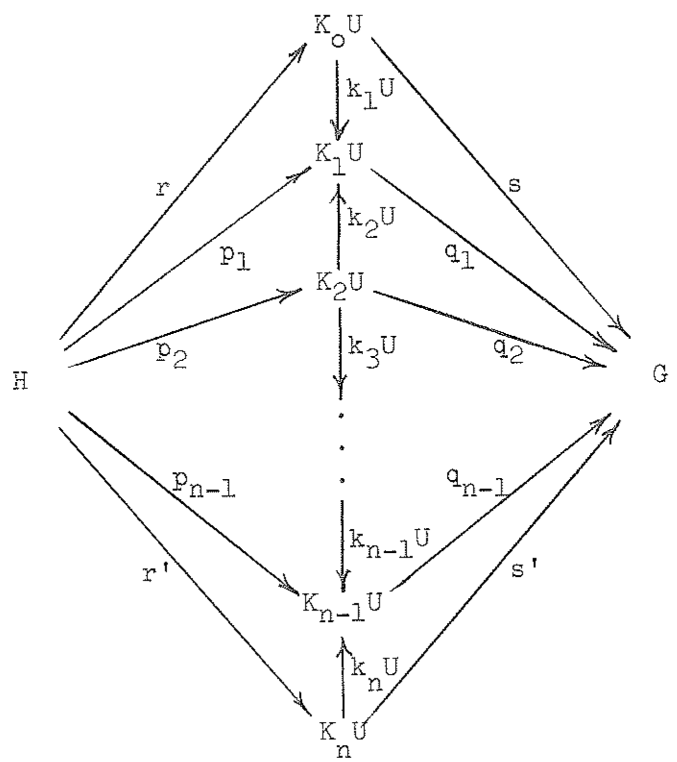
The hypermorphisms $(F, 1) \longrightarrow (H, 3)$ are of the form (t, v) for $t: F \longrightarrow H$ and v an arbitrary symbol.

The hypermorphisms $(G, 2) \longrightarrow (H, 3)$ are of the form (t, m) for $t: G \longrightarrow H$ and m an arbitrary symbol.

The hypermorphisms $(H, 3) \longrightarrow (G, 2)$ are equivalence classes of (s, K, r) where $r: H \longrightarrow KU$ and $s: KU \longrightarrow G$. $(s, K, r) \sim (s', K', r')$ if and only if there exist a finite number of functors $K_o, K_1, K_2, \ldots$ $\ldots, K_n$, natural transformations

$$K_o \xrightarrow{k_1} K_1 \xleftarrow{k_2} K_2 \xrightarrow{k_3} K_3 \quad \ldots\ldots \quad K_{n-1} \xleftarrow{k_n} K_n,$$

and natural transformations p_i and q_i such that $K = K_o$, $K' = K_n$, and



commutes.

There are no hypermorphisms $(G, 3) \twoheadrightarrow (H, 1)$ nor $(G, 2) \twoheadrightarrow (H, 1)$.

Composition of hypermorphisms is defined by the following relations:

$$G(F, i) = (GF, i)$$

$$G(t, *) = (Gt, *)$$

$$G(s, K, r) = (Gs, GK, Gr)$$

$$t(F, i) = (tF, i)$$

$$(t, *).\ (r, *) = (t.r, *)$$

where $*$ on the right is determined by the domain and codomain and in the case of u_i the i's are the same as on the left.

$$(t, 2).(s, K, r) = (t.s, K, r)$$

$$(s, K, r).(t, 3) = (s, K, r.t)$$

$$(t, m).(s, K, r) = (t.s.r, 3)$$

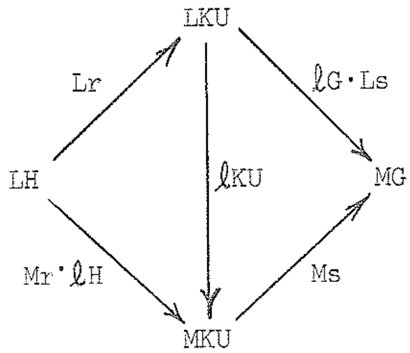
$$(s, K, r).(t, m) = (s.r.t, 2)$$

The verifications that composition is well defined and that this defines a hypercategory structure on $\underline{\underline{A}}$ are left to the reader. We work out the only case which is not straightforward, to show why we had to take equivalence classes as hypermorphisms.

Let $(s, K, r): (H, 3) \twoheadrightarrow (G, 2)$ and $\ell: L \twoheadrightarrow M$, then the following must commute:

$$\begin{array}{ccc} L(H, 3) & \xrightarrow{L(s, K, r)} & L(G, 2) \\ {\scriptstyle \ell(H, 3)} \downarrow & & \downarrow {\scriptstyle \ell(G, 2)} \\ M(H, 3) & \xrightarrow{M(s, K, r)} & M(G, 2) \end{array}$$

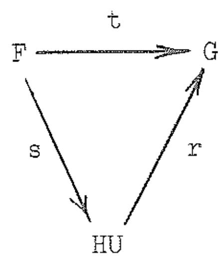
The following diagram commutes by naturality



Therefore $(\ell G.Ls, LK, Lr) = (Ms, MK, Mr.\ell H)$, i.e. $(\ell G, 2).(Ls, LK, Lr) = (Ms, MK, Mr).(\ell H, 3)$. Which is what we had to prove.

Therefore we have a hypercategory $\underline{\underline{A}}$ and $\underline{\underline{Cat}}$ is contained in it.

Next we define a congruence relation on the hypermorphisms of $\underline{\underline{A}}$ which identifies (t, u_1) with (t, u_2) for some $t: F \longrightarrow G$. $(t, u_1) \equiv (t, u_2)$ if and only if there exist H and r and s such that



commutes. We make this relation reflexive by requiring that all hypermorphisms be in relation with themselves. The details that this is a congruence relation are left to the reader.

Form the quotient hypercategory $\underline{\underline{B}}$. $\underline{\underline{Cat}}$ is embedded (hyper-embedded) in $\underline{\underline{B}}$.

Consider the hyperfunctor

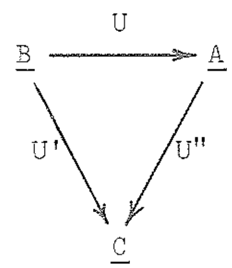
$$\underline{F} = \underline{\underline{Cat}} \hookrightarrow \underline{\underline{B}} \xrightarrow{(\underline{X},\ -)} \underline{\underline{Cat}}.$$

By hypothesis $\underline{F}(U): \underline{F}(\underline{B}) \longrightarrow \underline{F}(\underline{A})$ reflects isos to monos. $\underline{F}(U)(\underline{B}, m) = U(\underline{B}, m) = (U, m): (U, 2) \longrightarrow (U, 3)$. But $(U, m).(U, \underline{A}, U) = (U.U.U, 3) = (U, 3) =$ identity on $(U, 3)$. Also $(U, \underline{A}, U).(U, m) = (U.U.U, 2) = (U, 2) =$ identity on $(U, 2)$. Therefore $\underline{F}(U)(\underline{B}, m)$ is an iso in $\underline{F}(\underline{A})$. Thus $(\underline{B}, m)$ is mono in $\underline{F}(\underline{B})$. Now $(\underline{B}, m).(\underline{B}, u_1) = (\underline{B}, m).(\underline{B}, u_2)$ since both equal $(\underline{B}, v)$. This implies that $(\underline{B}, u_1) = (\underline{B}, u_2)$ in $\underline{F}(\underline{B})$, i.e. $(\underline{B}, u_1) \equiv (\underline{B}, u_2)$ in $\underline{\underline{A}}$. Therefore there exist a functor H and natural transformations s and r such that

$$\underline{B} \xrightarrow{s} HU \xrightarrow{r} \underline{B} = \underline{B}.$$

This completes the proof of the theorem. ∎

Consider the following diagram of small categories and functors:



Then by "absolute nonsense" we see that (U" reflects monos absolutely) $\wedge$ (U' preserves monos absolutely) $\Longrightarrow$ (U preserves monos absolutely). We should be able to see this from the characterizations. Assume that we have $G, F, r, s, \eta, \varepsilon$ such that

$$\underline{A} \xrightarrow{r} GU'' \xrightarrow{s} \underline{A} = \underline{A}$$

$$U' \xrightarrow{\eta U'} U'FU' \xrightarrow{U'\varepsilon} U' = U'.$$

Then for U we get

$$U \xrightarrow{rU} GU''U \xrightarrow{G\eta U''U} GU''UFU''U \xrightarrow{sUFU''U} UFU''U \xrightarrow{U\varepsilon} U = U$$

i.e.

$$U \xrightarrow{(sUFU''.G\eta U''.r)U} UFU''U \xrightarrow{U\varepsilon} U = U.$$

We also see from the diagram that (U'' preserves monos absolutely) $\wedge$ (U' reflects monos absolutely) $\Rightarrow$ (U reflects monos absolutely). We see more from the characterization: if U_1U_2 reflects monos absolutely then so does U_2.

§3. ABSOLUTE FAITHFULNESS

(3.1) THEOREM. Let $U: \underline{B} \to \underline{A}$ be a functor between two small categories. Then U is absolutely faithful if and only if there exist a functor $G: \underline{A} \to \underline{B}$ and natural transformations r and s such that

$$\underline{B} \xrightarrow{r} GU \xrightarrow{s} \underline{B} = \underline{B}.$$

Proof. Assume that U is absolutely faithful. Since (U faithful) $\Rightarrow$ (U reflects monos) then (U absolutely faithful) $\Rightarrow$ (U reflects monos absolutely) and by (2.1) there exist G, r, s such that

$$\underline{B} \xrightarrow{r} GU \xrightarrow{s} \underline{B} = \underline{B}.$$

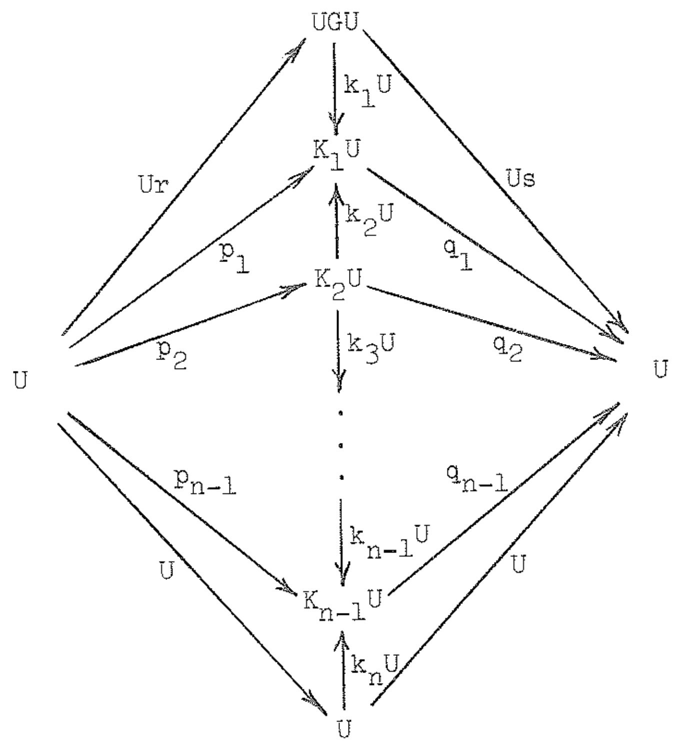
Assume that we have G, r, s as above. Let $b_1, b_2 \colon B \to B'$ be such that $Ub_1 = Ub_2$. Then $sB'.GUb_1 = sB'.GUb_2$ and by naturality $b_1.sB = b_2.sB$ and composing with rB we get $b_1 = b_2$. Thus U is faithful. But the above conditions are absolute thus U is absolutely faithful. ∎

A functor U which is tripleable is faithful thus absolute tripleableness implies absolute faithfulness. Therefore U must have a left adjoint and there must exist G, r, s such that $\underline{B} \xrightarrow{r} GU \xrightarrow{s} \underline{B} = \underline{B}$. We saw (II, 3.9) that if $\underline{B}$ has split idempotents these conditions imply tripleableness, even V.T.T. Of course this is not absolute tripleableness but it is close enough. No doubt one could find an exact characterization of absolute tripleableness but it would probably not be worth the effort, considering how weak a condition idempotent splitting is.

§4. ABSOLUTE FULNESS

(4.1) THEOREM. Let $U \colon \underline{B} \to \underline{A}$ be a functor between small categories. U is absolutely full if and only if there exist functors $G \colon \underline{A} \to \underline{B}$, $K_i \colon \underline{A} \to \underline{A}$ $(i = 1, 2, 3, \ldots, n-1)$, and natural transformations $r \colon \underline{B} \to GU$, $s \colon GU \to \underline{B}$,

$UG \xrightarrow{k_1} K_1 \xleftarrow{k_2} K_2 \xrightarrow{k_3} \cdots \to K_{n-1} \xleftarrow{k_n} \underline{A}$, $p_i \colon U \to K_i U$, and $q_i \colon K_i U \to U$ such that the following diagram commutes:



Proof. Assume that U is absolutely full. Let $\underline{A}$ be the hypercategory described in (2.1). Define a hyperfunctor $\underline{G}$: $\underline{\underline{Cat}} \longrightarrow \underline{\underline{Cat}}$ to be the composition $\underline{\underline{Cat}} \hookrightarrow \underline{A} \xrightarrow{(\underline{X},\ -)} \underline{\underline{Cat}}$. $\underline{G}(U)$: $\underline{G}(\underline{B}) \longrightarrow \underline{G}(\underline{A})$ is full. Now $\underline{G}(U)(\underline{B}, 3) = U(\underline{B}, 3) = (U, 3)$ and $\underline{G}(U)(\underline{B}, 2) = U(\underline{B}, 2) = (U, 2)$. We have $(U, \underline{A}, U)$: $(U, 3) \longrightarrow (U, 2)$ a map of $\underline{G}(\underline{A})$. Therefore there exists a map $(\underline{B}, 3) \longrightarrow (\underline{B}, 2)$ in $\underline{G}(\underline{B})$, say (s, G, r): $(\underline{B}, 3) \longrightarrow (\underline{B}, 2)$, such that $\underline{G}(U)(s, G, r) = (U, \underline{A}, U)$. But $\underline{G}(U)(s, G, r) = U(s, G, r) = (Us, UG, Ur)$. To say that $(Us, UG, Ur) = (U, \underline{A}, U)$ means that there exist functors K_i: $\underline{A} \longrightarrow \underline{A}$ and natural transformations

$UG \xrightarrow{k_1} K_1 \xleftarrow{k_2} K_2 \quad \ldots\ldots \quad K_{n-1} \xleftarrow{k_n} \underline{A}$, $p_i \colon U \longrightarrow K_iU$, and $q_i \colon K_iU \longrightarrow U$ such that the diagram in the statement of the theorem commutes.

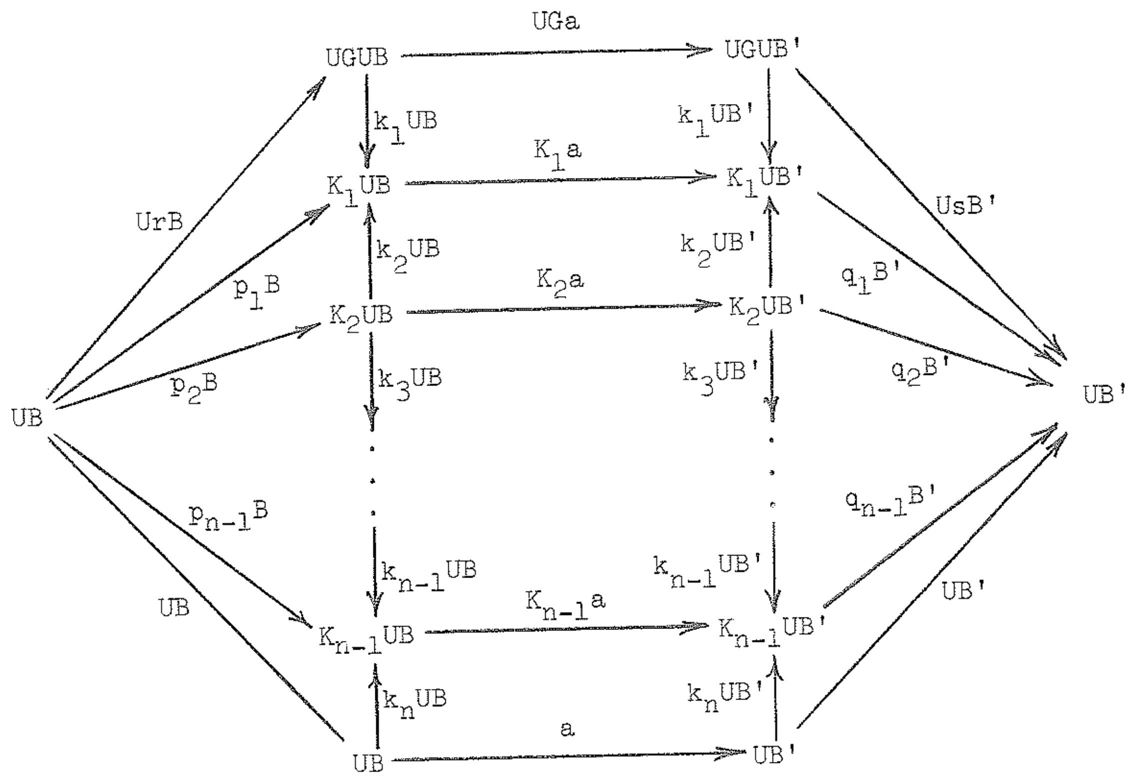
Now assume that we have functors and natural transformations such as in the statement of the theorem. Let B and B' be objects of $\underline{B}$ and $a \colon UB \longrightarrow UB'$ an $\underline{A}$-morphism. We have the following morphism in $\underline{B}$:

$$B \xrightarrow{rB} GUB \xrightarrow{Ga} GUB' \xrightarrow{sB'} B'.$$

Applying U we get

$$UB \xrightarrow{UrB} UGUB \xrightarrow{UGa} UGUB' \xrightarrow{UsB'} UB'.$$

However the following diagram commutes

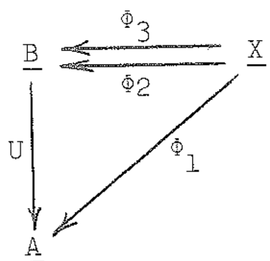


thus proving that $UsB'.UGa.UrB = a$, and U is full. The conditions are obviously absolute therefore U is absolutely full. ∎

§5. REMARKS

We include a few remarks to clarify the constructions of theorems (1.1) and (2.1). Effectively we have pursued the following course.

In (1.1) we constructed a hypercategory $\underline{A}$ by adding to $\underline{\underline{Cat}}$ a new object $\underline{X}$, three new morphisms, Φ_1, Φ_2, Φ_3 as follows



and three new hypermorphisms $\Phi_1 \overset{u}{\underset{v}{\rightrightarrows}} U\Phi_2$ and $\Phi_2 \xrightarrow{m} \Phi_3$ such that $Um.u = Um.v$, and all consequences. Keeping in mind that our hyperfunctor $\underline{F}$ will be essentially $(\underline{X}, -)$ we identify all pairs of hypermorphisms coequalized by m and all consequences of this. Then in $\underline{F}(\underline{B})$, m is mono; since $\underline{F}(U)$ preserves monos, Um is mono in $\underline{F}(\underline{A})$ and thus $u = v$. Interpreting the consequences of this we obtain the characterization.

In (2.1), the idea is the same. We add a new object $\underline{X}$ to $\underline{\underline{Cat}}$, three new morphisms Φ_1, Φ_2, Φ_3: $\underline{X} \longrightarrow \underline{B}$ and four new hypermorphisms

$$\Phi_1 \underset{v}{\overset{u}{\rightrightarrows}} \Phi_2 \xrightarrow{m} \Phi_3$$

and

$$U\Phi_3 \xrightarrow{w} U\Phi_2$$

such that $m.u = m.v$ and $Um.w = U\Phi_3$ and $w.Um = U\Phi_2$ and all consequences. The same idea as in (1.1) gives the characterization.

This construction should have occurred in (4.1) but luckily it was possible to use the construction of (2.1) again.

CONCLUSION

The methods used in this thesis are quite general. Given a certain property, one can find necessary and sufficient conditions for the diagram to possess this property absolutely. The characterization takes the form of equations involving all the peculiarities of the extra structure considered.

These characterizations are found by generating categories with prescribed properties. However, once the characterizations have been found it is fairly easy to find more elegant proofs, eliminating the messiness involved in generating categories.

It is amazing that one has a "mechanical" means of extracting the absoluteness from a given property. We obtain a "best approximation" of a given property by an absolute one.

It is our thesis that absoluteness properties in category theory are fundamental in mathematics and should be studied more extensively. One should be able to recognize these properties in a diagram and make full use of them.

REFERENCES

[1] J. Beck, Triples, algebras and cohomology, Dissertation, Columbia, 1967.

[2] J. Beck, Untitled manuscript, Cornell, 1966.

[3] M. Bunge, Relative Functor Categories and Categories of Algebras, J. Algebra, 11 (1969), 64-101.

[4] P. M. Cohn, Universal Algebra, New York, Harper and Row, 1965.

[5] S. Eilenberg and G. M. Kelly, Closed Categories, Proceedings of the Conference on Categorical Algebra, La Jolla, Springer (1966).

[6] S. Eilenberg and J. C. Moore, Adjoint Functors and Triples, Ill. J. Math. 9 (1965), 381-398.

[7] P. Freyd, Abelian Categories, Harper and Row, New York, 1964.

[8] P. J. Huber, Homotopy Theory in General Categories, Math. Ann., vol. 144 (1961), 361-385.

[9] J. Lambek, Completions of Categories, Springer Lecture Notes in Mathematics, 24 (1966).

[10] J. Lambek, A Fixpoint Theorem for Complete Categories, Math. Zeitschr., 103 (1968), 151-161.

[11] J. Lambek, Deductive Systems and Categories (I), Math. Systems Theory, 2, 287-318.

[12] J. Lambek, Deductive Systems and Categories (II). Standard Constructions and Closed Categories, Category Theory, Homology Theory and their Applications I, Seattle, Springer Lecture Notes (86), 1969.

[13] F. W. Lawvere, Functorial Semantics of Algebraic Theories, Dissertation, Columbia (1963).

[14] F. W. Lawvere, The Category of Categories as a Foundation for Mathematics, Proceedings of the Conference on Categorical Algebra, La Jolla, Springer, 1966.

[15] F. E. J. Linton, Some Aspects of Equational Categories, Proceedings of the Conference on Categorical Algebra, La Jolla, Springer, 1966.

[16] F. E. J. Linton, An Outline of Functorial Semantics, Seminar on Triples and Categorical Homology Theory, Springer Lecture Notes (80), Zurich (1969).

[17] F. E. J. Linton, Coequalizers in Categories of Algebras, Seminar on Triples and Categorical Homology Theory, Springer Lecture Notes (80), Zurich (1969).

[18] E. G. Manes, A Triple Miscellany: Some Aspects of the Theory of Algebras over a Triple, Dissertation, Wesleyan University, (1967).

[19] B. Mitchell, Theory of Categories, Academic Press, New York, 1965.

[20] R. Paré, Absolute Coequalizers, Category Theory, Homology Theory and their Applications I, Springer Lecture Notes (86), Seattle, 1969.