REPRESENTABLES FOR DOUBLE CATEGORIES

ROBERT PARÉ

- STARTED WHEN STUDYING LIMITS IN DOUBLE CATEGORIES WITH MARCO.
- THERE ARE PROBLEMS !
- ORDINARY LIM – UNDERSTAND IN SET & LIFT USING <u>REPRESENT.</u>
- $\underline{V}$-LIMS DO JUST THAT.
- TRY THIS FOR DOUBLE CATEGORIES. BUT WHAT ARE <u>REPRESENTABLES</u>?

$$H_A : \mathbb{D} \xrightarrow{??} ?$$

WHY STUDY DOUBLE CATEGORIES?

- SUITABLE PLACE TO STUDY CATEGORIES WITH STRUCTURE

- TRUE: MOST ARE 2-CATS OR BI-CATS

- BUT CONSIDER $\underline{\underline{\mathrm{CAT}}}^{2}$

  OBJECTS: FUNCTORS

  MORPHISMS:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \ ?$$

- OTHER EXAMPLE: $\underline{\underline{A}}$ 2-CAT

  $\$\underline{\underline{A}}$ A DOUBLE CAT THAT COMES UP IN THE STUDY OF LAX LIM.

OBS

$f : A \longrightarrow B$

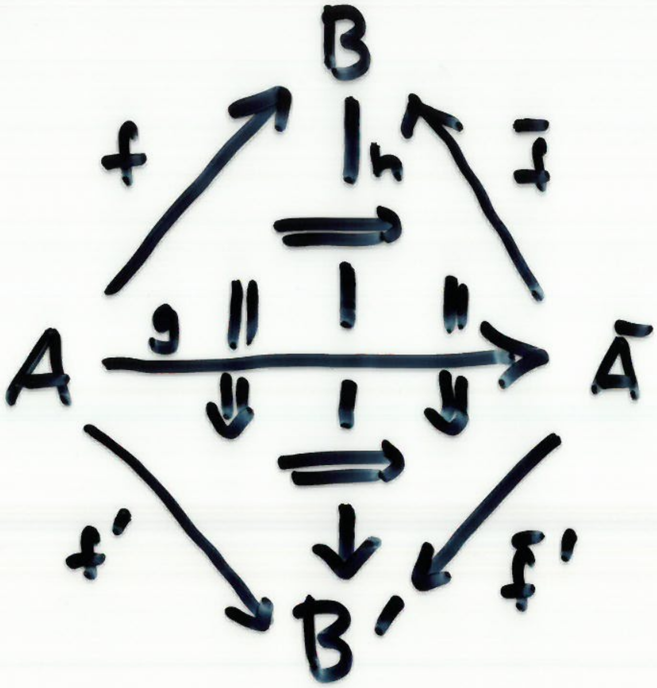
HORIZ

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Rightarrow"] & \bar{A} \arrow[dl, "\bar{f}"] \\ & B & \end{tikzcd}$$

VERT

$$\begin{tikzcd} & B \arrow[d, "h"] \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] \arrow[r, phantom, "\Downarrow"] & B' \end{tikzcd}$$

DOUB



$$\begin{tikzcd} {[f]} \arrow[r] \arrow[d, "|" marking] \arrow[dr, phantom, "\Downarrow"] & {[\bar{f}]} \arrow[d, "|" marking] \\ {[f']} \arrow[r] & {[\bar{f}']} \end{tikzcd}$$

DOUBLE CATEGORIES AS A FOUNDATION

<u>EMBARRASMENT</u>

2-CATS $\quad H_A : \mathbb{A} \longrightarrow \mathbf{Cat}$ $\quad$ 2-FUNCT

BICATS $\quad$ BIFUNCT

DOUB $\quad H_A : \mathbb{D} \longrightarrow \mathbf{Cat}$ $\quad$ DOUB. FUNCT

NO!

HORIZONTAL IS DOMINANT.

$H_A : \mathbb{D} \longrightarrow ?$

$H_A(X) = \{ h : A \to X \mid h \text{ HORIZ} \}$ A SET

$x : X \to Y$ HORIZ

$H_A(x) : H_A(X) \longrightarrow H_A(Y)$ A FUNCTION

$$\begin{tikzcd} X \arrow[d, "v"'] \\ X' \end{tikzcd} \qquad H_A(v) = \left\{ \begin{tikzcd} A \arrow[r] \arrow[dr] \arrow[r, phantom, "\Downarrow", shift right=3] & X \arrow[d, "v"] \\ & X' \end{tikzcd} \right\}$$ A SET

$$\begin{tikzcd} & H_A(X) \\ H_A(v) \arrow[ur] \arrow[dr] & \\ & H_A(X') \end{tikzcd}$$

A SPAN

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow \xi"] & Y \arrow[d, "w"] \\ X' \arrow[r, "x'"'] & Y' \end{tikzcd} \rightsquigarrow \begin{tikzcd} H_A(X) \arrow[rr] & & H_A(Y) \\ H_A(v) \arrow[u] \arrow[d] \arrow[rr, "H_A(\xi)"] & & H_A(w) \arrow[u] \arrow[d] \\ H_A(X') \arrow[rr] & & H_A(Y') \end{tikzcd}$$

MORPH OF SPANS!

$$\begin{tikzcd}
H_A(X) \arrow[r, "H_A(x)"] \arrow[d, "|" marking, "H_A(v)"'] \arrow[dr, phantom, "\overset{H_A(\xi)}{\Longrightarrow}"] & H_A(Y) \arrow[d, "|" marking, "H_A(w)"] \\
H_A(X') \arrow[r, "H_A(x')"'] & H_A(Y')
\end{tikzcd}$$

H_A TAKES ITS VALUES IN THE "DOUBLE CAT" WHOSE

OBJ. SETS

HORIZ. FUNCTIONS

VERT. SPANS

DOUB. MORPH OF SPANS

HORIZ COMPOSITION IS OK

VERT COMPOSITION IS ONLY ASSOCIATIVE UP TO COHERENT ISO !

A **PSEUDOUBLE CATEGORY** IS LIKE A DOUBLE CATEGORY BUT VERTICAL COMPOSITION IS ONLY ASSOCIATIVE & UNITARY UP TO ISOMORPHISMS OF THE FORM

(HORIZ. ISO)

GET A SUB BICATEGORY.

- D. CHAMAILLARD (ESQUISSES - 6)
- G. MOREAU (?)

THE ABOVE IS CALLED

$\mathbb{P}$SET

($\mathbb{P}$ IS SILENT!)

BACK TO REPRESENTABLES

$H_A : \mathbb{D} \longrightarrow \mathbb{P}\mathrm{Set}$

- PRESERVES BOTH HORIZ. COMPS.

- VERT

$$
\begin{tikzcd}
X \arrow[d, "v"'] \\
X' \arrow[d, "v'"'] \\
X''
\end{tikzcd}
\qquad
\begin{tikzcd}
H_A(X) \arrow[d, "H_A(v)"'] \arrow[dd, bend left=60, "H_A(v \cdot v')"] \\
H_A(X') \arrow[d, "H_A(v')"'] \arrow[r, phantom, "\overset{\mu}{\Longrightarrow}"] & {} \\
H_A(X'')
\end{tikzcd}
$$

NOT ISO.

$$
\begin{tikzcd}
 & X \arrow[d, "v"] \\
A \arrow[ur] \arrow[r] \arrow[dr] \arrow[ur, phantom, "\Downarrow", shift right=3] \arrow[dr, phantom, "\Downarrow", shift left=3] & X' \arrow[d, "v'"] \\
 & X''
\end{tikzcd}
\overset{\mu}{\longmapsto}
\begin{tikzcd}
 & X \arrow[dd, "v \cdot v'"] \\
A \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow"] & {} \\
 & X''
\end{tikzcd}
$$

ALSO

$$
\begin{tikzcd}
X \arrow[d, "id_X"'] \\
X
\end{tikzcd}
\qquad
\begin{tikzcd}
H_A(X) \arrow[d, "id_{H_A X}"'] \arrow[d, bend left=60, "H_A(id_X)"] \\
H_A(X)
\end{tikzcd}
\quad \overset{\eta}{\Longrightarrow}
$$

$H_A : \mathbb{D} \longrightarrow \mathbb{P}\mathrm{Set}$

- PRESERVES VERT COMP OF DOUBLE CELL ONCE μ IS FACTORED IN.
- μ, η SATISFY ASSOC AND UNIT LAWS LIKE TRIPLES.

H_A IS A MORPHISM OF PSEUDOUBLE CATEGORIES.

TRANSFORMATIONS

GIVEN $f : B \longrightarrow A$ IN $\mathbb{D}$

WHAT DO WE GET $H_f : H_A \longrightarrow H_B$?

$H_f(X) : H_A(X) \longrightarrow H_B(X)$ A FUNCTION.

NATURAL FOR HORIZ. MORPH.

$$
\begin{tikzcd}
X \arrow[d, "v"'] \\
X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H_A(X) \arrow[rr, "H_f(X)"] & & H_B(X) & \\
H_A(v) \arrow[ur] \arrow[dr] \arrow[rrrr, "H_f(v)"] & & & & H_B(v) \arrow[ul] \arrow[dl] \\
 & H_A(X') \arrow[rr, "H_f(X')"'] & & H_B(X') &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & X \arrow[dd] \\
A \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow"] & \\
 & X'
\end{tikzcd}
\longmapsto
\begin{tikzcd}
 & & X \arrow[dd] \\
B \arrow[r] & A \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow"] & \\
 & & X'
\end{tikzcd}
$$

i.e.

$$
\begin{tikzcd}
H_A(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow H_f(v)"] & H_B(X) \arrow[d] \\
H_A(X') \arrow[r] & H_B(X')
\end{tikzcd}
\quad \text{IN } \mathbb{P}\mathrm{SET}
$$

"DEF": A <u>HORIZONTAL TRANS</u> OF MORPHISMS OF PSEUDOUBLE CATS $t : F \to G : \mathbb{D} \to \mathbb{P}$ IS AS ABOVE WITH OBVIOUS COHERENCE.

<u>PROP</u>: A HORIZONTAL TRANSF $t : H_A \longrightarrow G : \mathbb{D} \longrightarrow \mathbb{P}SET$ IS "THE SAME AS" $x \in G(A)$.

<u>COR</u>: $t : H_A \longrightarrow H_B$ COMES FROM A UNIQUE $f : B \longrightarrow A$ AS ABOVE.

VERTICAL TRANSFORMATIONS

$$
\begin{tikzcd} A \arrow[d, "l"', "|" marking] \\ A' \end{tikzcd}
\qquad\qquad
\begin{tikzcd} H_A \arrow[d, "H_l"', "|" marking] \\ H_{A'} \end{tikzcd}
\qquad ?
$$

$$
\begin{tikzcd}[row sep=small] & H_A(X) \\ H_l(X) \arrow[ur] \arrow[dr] & \\ & H_{A'}(X) \end{tikzcd}
\quad = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "l"', "|" marking] \arrow[dr, phantom, "\Downarrow"] & X \\ A' \arrow[ur] & \end{tikzcd} \right\}
$$

$$
\begin{tikzcd}
X \arrow[r, "x"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& H_A(X) \arrow[rr, "H_A(x)"] & & H_A(Y) \\
H_\ell(X) \arrow[ur, "|" marking] \arrow[dr] \arrow[rr, "H_\ell(x)"] & & H_\ell(Y) \arrow[ur, "|" marking] \arrow[dr] & \\
& H_{A'}(X) \arrow[rr, "H_{A'}(x)"'] & & H_{A'}(Y)
\end{tikzcd}
$$

c.f.

$$
\begin{tikzcd}
H_A(X) \arrow[r] \arrow[d, "|" marking] \arrow[dr, phantom, "\overset{H_\ell(x)}{\Rightarrow}"] & H_A(Y) \arrow[d, "|" marking] \\
H_{A'}(X) \arrow[r] & H_{A'}(Y)
\end{tikzcd}
$$

THESE COMPOSE HORIZONTALLY AND PRESERVE HORIZ. IDENTITIES.

VERTICAL COMPATIBILITY

$$
\begin{tikzcd}
H_A(X) \arrow[r, "|" marking, "H_A(v)"] \arrow[d, "|" marking, "H_\ell(X)"'] & H_A(X') \arrow[d, "|" marking, "H_\ell(X')"] \\
H_{A'}(X) \arrow[r, "|" marking, "H_{A'}(v)"'] & H_{A'}(X')
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_A(X) & H_\ell(X) \otimes H_{A'}(v) \arrow[l] \arrow[d] \\
H_A(v) \otimes H_\ell(X') \arrow[u] \arrow[r] & H_{A'}(X')
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\ell"'] \arrow[dr] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & X \arrow[d, "v"] \\
A' \arrow[r] \arrow[r, phantom, shift left=3ex, "\Uparrow"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "\lambda"'] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & X \arrow[d, "v"] \\
A' \arrow[r] \arrow[ur] \arrow[r, phantom, shift left=3ex, "\Downarrow"] & X'
\end{tikzcd}
$$

ALSO HAVE

$$
\begin{tikzcd}
H_A(X) \arrow[rr, "H_\ell(v)"] & & H_{A'}(X') \\
& H_\ell(v) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\ell"'] \arrow[dr, phantom, "\Searrow"] & X \arrow[d, "v"] \\
A' \arrow[r] & X'
\end{tikzcd}
$$

DEF: A VERTICAL TRANSFORMATION

$u : F \mathrel{\mapstochar\rightarrow} G : \mathbb{D} \longrightarrow \mathbb{P}$ CONSISTS OF

$$\forall \quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\xi"] & Y \arrow[d, "w"] \\ X' \arrow[r, "x'"'] & Y' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d, "u(v)"'] \arrow[dr, phantom, "\Downarrow u(\xi)"] & F(Y) \arrow[d, "u(w)"] \\ G(X') \arrow[r] & G(Y') \end{tikzcd}$$

$$\forall \quad \begin{tikzcd} X \arrow[d, "v"'] \\ X' \arrow[d, "v'"'] \\ X'' \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(v)"] \arrow[d, "u(v)"'] \arrow[dr, "u(v\cdot v')" description] & F(X') \arrow[d, "u(v')"] \\ G(X') \arrow[r, "G(v')"'] & G(X'') \end{tikzcd}$$

$$F(v)\cdot u(v') \xrightarrow{\lambda_{v,v'}} u(v\cdot v')$$

$$u(v)\cdot G(v') \xrightarrow[\rho_{v,v'}]{} u(v\cdot v')$$

SUBJECT TO:

- $u(\xi)$ COMPOSE HORIZ.
- λ, ρ ASSOC. UNIT. COMMUTE (BIMOD)
- A COMPAT FOR VERT COMP.

IS A VERT TRANSF $u : H_A \nrightarrow H_{A'}$ THE SAME AS A VERTICAL MORPH $\ell : A \nrightarrow A'$?

NO! EVEN IF WE TAKE $\mathbb{D} = \mathbb{1}$, THERE IS ONE VERT TRANSF $u : H_* \to H_*$ FOR EACH SET.

TO GET AN IDEA OF THE SITUATION LET US CONSIDER THE CASE WHERE $\mathbb{D}$ IS A VERTICAL CAT

$$\mathbb{D} = \underline{A}^{tr} = \left[\underline{A} \Rrightarrow \underline{A} \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} \underline{A} \right]$$

IN THIS SITUATION, WHAT DO MORPHISMS $\mathbb{D} \longrightarrow \mathbb{P}\mathrm{SET}$ LOOK LIKE? HORIZ. TRANSFS? VERT. TRANSFS?

<u>PROP</u>: IF $\mathbb{D} = \underline{A}^{tr}$, THEN A MORPHISM $F : \mathbb{D} \longrightarrow \mathbb{P}\mathrm{SET}$ IS THE SAME AS A CATEGORY OVER $\underline{A}$

$$\begin{array}{c} \underline{B} \\ \downarrow \\ \underline{A}. \end{array}$$

IF $G : \mathbb{D} \to \mathbb{P}\mathrm{SET}$ IS ANOTHER CORRESPONDING TO $\underline{C} \longrightarrow \underline{A}$, THEN A HORIZ. TRANSF $t : F \to G$ CORRESPONDS TO A COMMUTATIVE TRIANGLE

$$\begin{array}{ccc} \underline{B} & \longrightarrow & \underline{C} \\ & \searrow \quad \swarrow & \\ & \underline{A} & \end{array}$$

A VERTICAL TRANSF $\mu : F \to G$ CORRESPONDS TO

"A PROFUNCTOR OVER $\underline{A}$"

$$\begin{tikzcd} \underline{B} \arrow[rr, "\Phi", "|" marking] \arrow[dr, "P"'] & {} \arrow[d, phantom, "\psi \Rightarrow"] & \underline{C} \arrow[dl, "Q"] \\ & \underline{A} & \end{tikzcd}$$

"PROOF" GIVEN $P : \underline{B} \to \underline{A}$

DEFINE $F : \mathbb{D} \to \mathbb{P}\mathrm{Set}$ BY

- $F(A) = \{ B \mid PB = A \}$

- $\begin{tikzcd} A \arrow[d, "a"', "|" marking] \\ A' \end{tikzcd}$ $F(a) = \{ b \mid Pb = a \}$

$$\begin{tikzcd} & F(A) \\ F(a) \arrow[ur, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"'] & \\ & F(A') \end{tikzcd}$$

EVERYTHING JUST FALLS INTO PLACE.

THE REPRESENTABLE H_A

CORRESPONDS TO $\ulcorner A \urcorner : \mathbb{1} \longrightarrow \underline{A}$

SO A HORIZONTAL TRANSF $t : H_A \longrightarrow F$

IS

$$\begin{tikzcd}
\mathbb{1} \arrow[rr] \arrow[dr, "\ulcorner A \urcorner"'] & \arrow[d, phantom, "="] & \underline{B} \arrow[dl, "p"] \\
& \underline{A} &
\end{tikzcd}$$

IS AN OBJECT OF $\underline{B}$ OVER A.

A VERTICAL TRANSF $\mu : H_A \longrightarrow\!\!\!\!\!\!+\;\; F$

IS

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "\Phi", "\psi"', "|" marking] \arrow[dr, "\ulcorner A \urcorner"'] & \arrow[d, phantom, "\Rightarrow"] & \underline{B} \arrow[dl, "p"] \\
& \underline{A} &
\end{tikzcd}$$

WHICH IS A PAIR

$\Phi : \underline{B}^{op} \longrightarrow \underline{SET}$

+ AN ELEMENT

$x \in LAN_p \Phi (A)$.