Free Double Categories and the Word Problem for Groups

by

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WANT TO DESCRIBE THE FREE DOUBLE CATEGORY GENERATED BY A DOUBLE REFLEXIVE GRAPH

$$\mathbb{D}: \begin{tikzcd} D_{11} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & D_{10} \arrow[l] \arrow[d, shift left=2] \arrow[d, shift right=2] \\ D_{01} \arrow[u] \arrow[r, shift left=2] \arrow[r, shift right=2] & D_{00} \arrow[l] \arrow[u] \end{tikzcd}$$

- DOUBLE GRAPH, CAT GRAPH $\rightarrow$ TOO EASY
- REFLEXIVE GRAPH IN CAT $\rightarrow$ NOT PRACTICAL
- LEFT ADJ EXISTS $\begin{tikzcd} \mathrm{DCat} \arrow[r, shift left=1, "U"] & \mathrm{DGph} \arrow[l, shift left=1, dashed, "F"] \end{tikzcd}$

DESCRIBE EXPLICITLY

- TYPICAL CELL IN $\mathbb{D}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\ C \arrow[r] & B \end{tikzcd}$$

IDENTITIES ON BOUNDARY

$\Rightarrow$ EQUATIONS IN FREE!

$F(\mathbb{D})$: OBJ, HOR ARR, VERT ARROW – EASY

CELLS = EQUIV CLASSES OF WORDS

Ex:

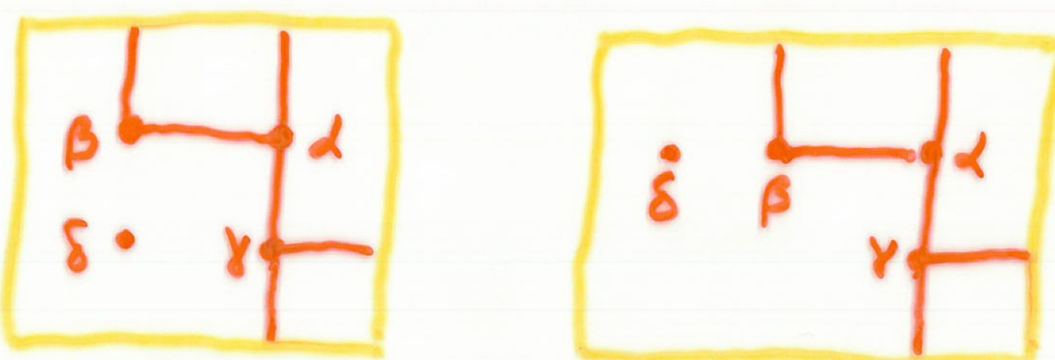
$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal] \\
B \arrow[r, "k"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "h'"] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "k"] \arrow[d, equal] \arrow[dr, phantom, "\gamma"] & C \arrow[d, "w"] \\
B \arrow[r, "k'"'] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

$$((\beta \circ_v \delta) \circ_h \alpha) \circ_v \gamma \quad = \quad (\delta \circ_h \beta) \circ_h (\alpha \circ_v \gamma)$$

$$
\begin{tikzcd}
B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[r] \arrow[d] \arrow[ddr, phantom, "\alpha"] & C \arrow[dd, equal] \\
B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & B \arrow[d, equal] & \\
B \arrow[r, equal] \arrow[d, equal] \arrow[drr, phantom, "\gamma"] & B \arrow[r] & C \arrow[d] \\
B \arrow[rr] & & C'
\end{tikzcd}
\qquad = \qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[dd, equal] \arrow[ddr, phantom, "\delta"] & B \arrow[r] \arrow[dd, equal] \arrow[ddr, phantom, "\beta"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal] \\
 & & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\gamma"] & C \arrow[d] \\
B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r] & C'
\end{tikzcd}
$$

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "1_B"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & C \arrow[d, equal] \\
B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\delta"] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "1_B"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "1_k"] & C \arrow[d, equal] \\
B \arrow[r, equal] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, phantom, "1_B"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\gamma"] & C \arrow[d] \\
B \arrow[rr, equal] & & B \arrow[r] & C'
\end{tikzcd}$$

PLANAR
DUAL'S



"ECKMANN – HILTON"

RECTANGULAR COMPLEXES

- FIRST STUDY THE SHAPE OF CELLS

$K = (R, P, H, V)$

$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$

P = FIN SET OF POINTS — VERTICES

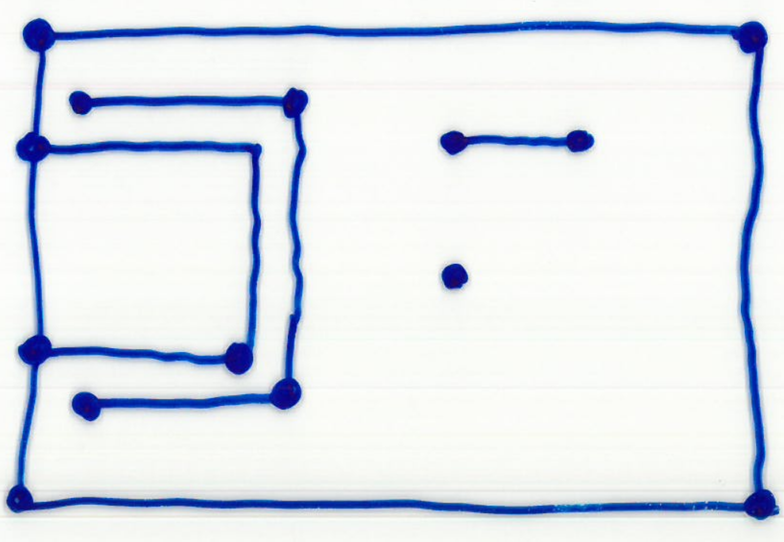
$\left.\begin{array}{l} H = \text{FIN SET OF HOR SEGMENTS} \\ V = \text{" \quad " \quad " VERT \quad "} \end{array}\right\}$ EDGES

1. END PTS OF EDGES ARE VERTICES

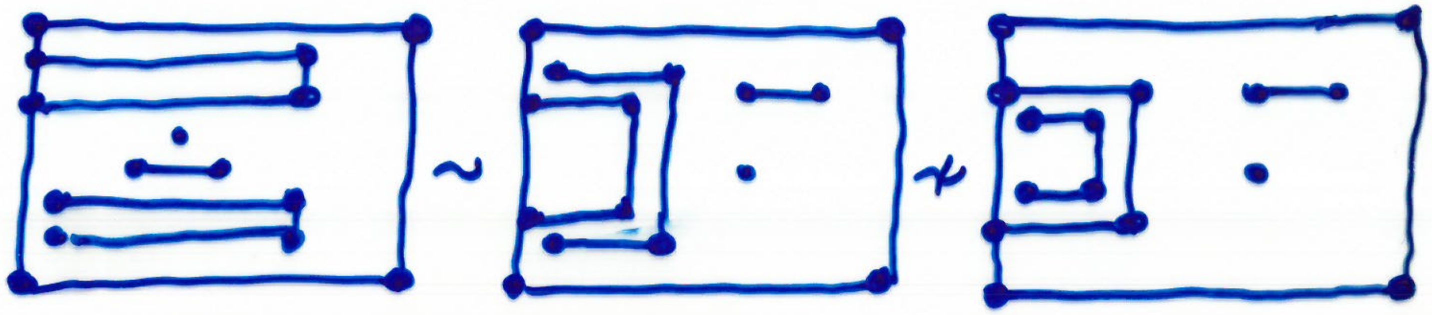
2. EVERY PT OF ∂R IS ON AN EDGE

3. EDGES INTERSECT ONLY IN END POINTS

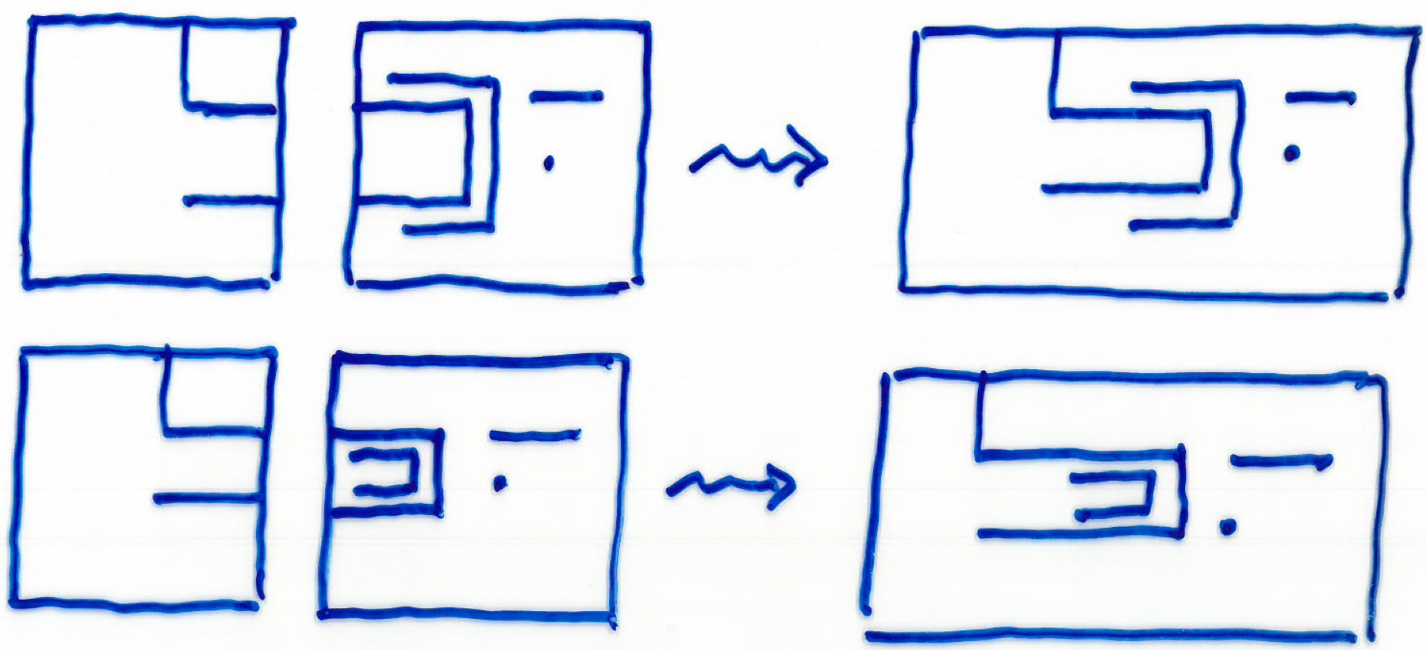
4. ANY VERTEX ON ∂R END PT OF INTERIOR EDGE



TWO COMPLEXES ARE HOMOTOPIC IF EACH CAN BE DEFORMED CONTINUOUSLY INTO THE OTHER IN A WAY RESPECTING 1 - 4.

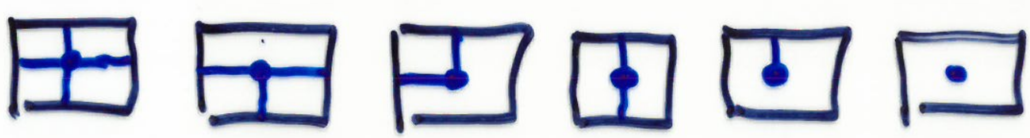


HORIZONTAL MERGING



- COMMON BOUNDARY ERASED
- RESPECTS HOMOTOPY

THM 1: HOMOTOPY CLASSES OF RECT COMPLEXES ARE THE CELLS OF A DOUB CAT FREE ON THE DOUB REFL GPH WITH 1 VERTEX, 1 NON IDENTITY HOR (VERT) ARROW AND 16 NON IDENTY CELLS ONE FOR EACH CHOICE OF IDENTITY OR NON IDENTITY FOR THE ARROWS OF THE BOUNDARY. THEY ARE REPRESENTED BY THE COMPLEXES



(1) (4) (4) (2) (4) (1)

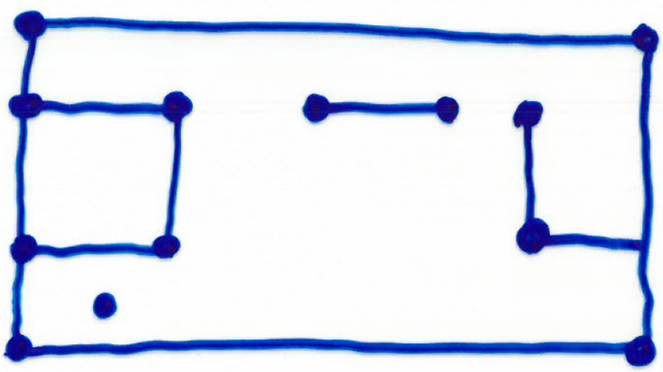
- IDENTITIES ARE

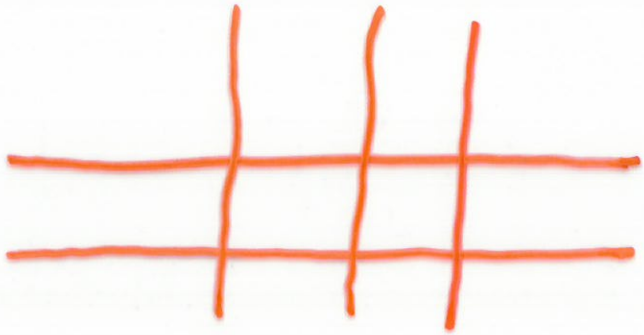
(SO 3 OTHER **BASIC** CELLS

)

$$F : \begin{tikzcd}
{[19]} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & {[2]} \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \\
{[2]} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & {[1]}
\end{tikzcd} \longrightarrow \mathbb{A}$$

WANT TO EXTEND F TO A DOUBLE FUNCTOR $F : \mathbb{RC} \longrightarrow \mathbb{A}$

$K =$ 



ADMISSIBLE SUBDIVISION: AT MOST ONE INTERIOR VERTEX INSIDE EACH SUBRECTANGLE.

$$[K] = \left([K_{11}] \circ_h [K_{12}] \circ_h \cdots \circ_h [K_{1n}]\right) \circ_v \left([K_{21}] \circ_h \cdots\right) \circ_v \cdots$$

EACH SUBRECT IS ONE OF THE 19 BASIC ONES: $[K_{11}] =$  , $[K_{33}] =$ 

MUST HAVE

$$F[K] = \left(F[K_{11}] \circ_h F[K_{12}] \circ_h \cdots\right) \circ_v \left(F[K_{21}] \circ_h \cdots\right) \cdots$$

- UNIQUE
- INDEPENDENT OF SUBDIVISION
- HOMOTOPY INVARIANT
- DOUBLE FUNCTOR

COLOURED COMPLEXES

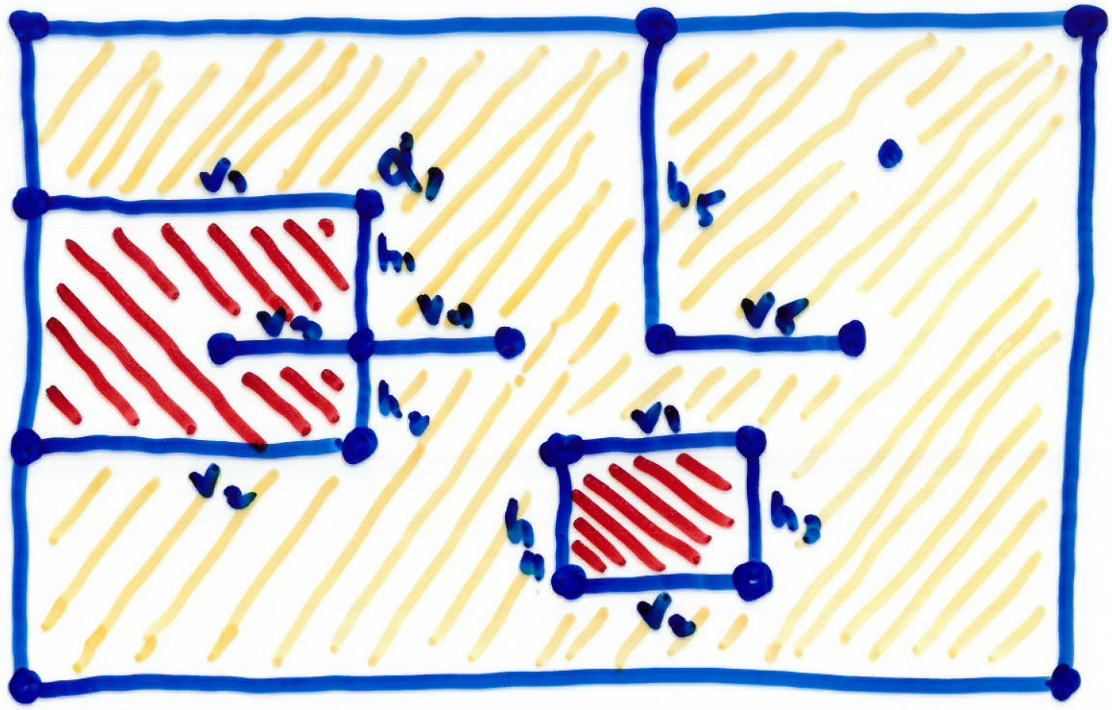
$\mathbb{D}$ A DOUBLE GRAPH (REFLEXIVE)

$K = (R, P, H, V)$ A RECT COMPLEX

A $\mathbb{D}$-COLOURING OF K IS A FN

$$\chi : R \longrightarrow D_{11}$$

- EACH INTERIOR POINT OF R HAS A NBHD LIKE ONE OF THE 19 BASIC CELLS. THAT POINT MUST BE COLOURED BY A CELL OF $\mathbb{D}$ OF SAME BOUNDARY TYPE.
- FURTHER, ALL POINTS OF THE NBHD ARE COLOURED WITH THE APPROPRIATE BOUNDARIES OF THAT CELL
- BOUNDARY POINTS OF R ARE COLOURED LIKE THEIR NBRS.



TWO OBJ: A B

5 VERT ARR:

$$\begin{tikzcd}
A \arrow[loop above, "v_3"] \arrow[d, shift left, "v_2"] \\
B \arrow[u, shift left, "v_1"] \arrow[loop below, "v_4"] \arrow[loop below, "v_5"']
\end{tikzcd}$$

12 CELLS

E.g.

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "v_1"'] \arrow[dr, phantom, "\alpha_1"] & B \arrow[d, equal] \\
A \arrow[r, "h_1"'] & B
\end{tikzcd}$$

α_1

5 HOR ARR:

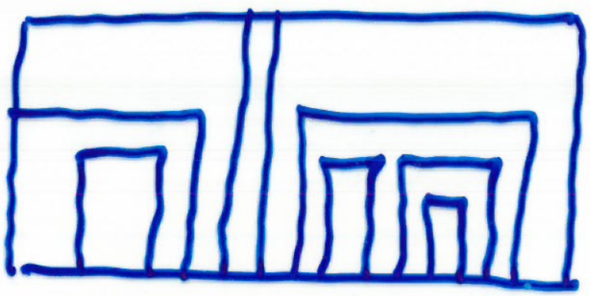
$$\begin{tikzcd}
A \arrow[r, shift left=3, "h_1"] \arrow[r, shift left=1] \arrow[r, shift right=1, "h_3"'] & B \arrow[l, shift left=4, "h_4"] \arrow[loop right, "h_5"]
\end{tikzcd}$$

EXAMPLES

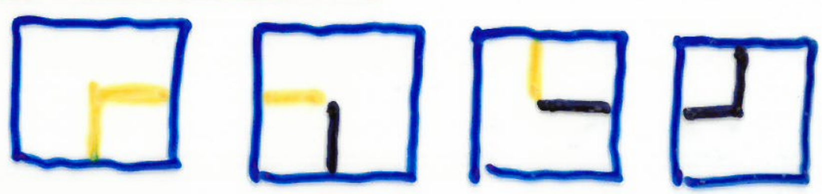
- THE CATALAN DOUBLE CATEGORY

FREE ON 

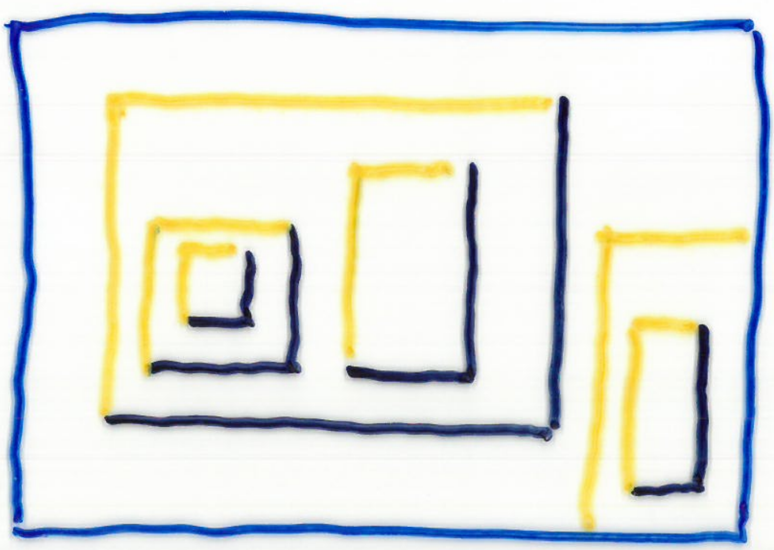
GENERAL CELL



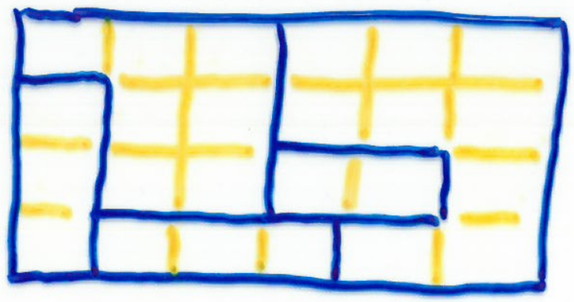
- STACKED RECTANGLES

FREE ON 

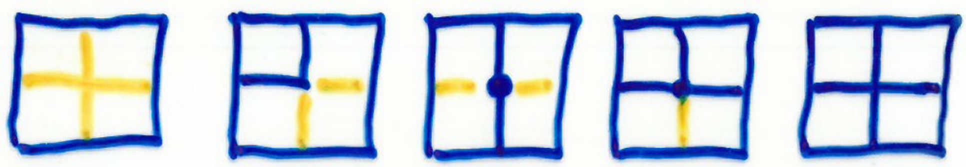
GENERAL CELL



POLYOMINO TILINGS OF RECTANGLES



- 1 OBJ
- 2 HOR → →
- 2 VERT → →
- 12 DOUB



(1) (4) (2) (4) (1)

HOW COMPLEX ARE FREE DOUB CATS?

- ONE MEASURE: DOES THERE EXIST A CELL WITH A GIVEN BOUNDARY?

$$
\begin{tikzcd}
{} \arrow[r, "h_1"] \arrow[d, "v_1"'] & {} \arrow[r, "h_2"] & {} \arrow[r, "h_3"] & {} \arrow[dd, "w"] \\
{} \arrow[d, "v_2"'] & {} \arrow[r, phantom, "?"] & {} & \\
{} \arrow[r, "k_1"'] & {} \arrow[r, "k_2"'] & {} \arrow[r, "k_3"'] & {}
\end{tikzcd}
$$

- FOR FREE CATEGORY ON A REFLEXIVE GRAPH WITH n VERTICIES

  $\exists\, A \to B$ in FREE $\Leftrightarrow$ $\exists$ PATH OF LENGTH n IN GRAPH. IF A IS INCIDENCE MATRIX OF GRAPH, ANS CAN BE READ OFF A^n.

- FOR FREE DOUBLE CATEGORY DIFFERENT STORY: THE WORD PROBLEM FOR GROUPS CAN BE MODELED IN ONE $\Rightarrow$ UNDECIDABLE

WORD PROBLEM FOR GROUPS IS SPECIAL CASE OF WORD PROBLEM FOR CATS.

- $\mathbb{G}$ FINITE GRAPH
- $\mathbb{R}$ FINITE SET OF RELATIONS

$$r:\ \begin{tikzcd} & A_0 \arrow[r, "f_1"] & A_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_m"] & A_m \arrow[dr, equal] & \\ A \arrow[ur, equal] \arrow[dr, equal] & & & & & B \\ & B_0 \arrow[r, "g_1"'] & B_1 \arrow[r, "g_2"'] & \cdots \arrow[r, "g_m"'] & B_m \arrow[ur, equal] & \end{tikzcd}$$

$\langle \mathbb{G} | \mathbb{R} \rangle$ = FREE CAT ON $\mathbb{G}$ / CONG GEN BY $\mathbb{R}$

WHEN ARE 2 WORDS EQUIVALENT?

- CAN ASSUME $\mathbb{R}$ SYMMETRIC.
- CONSTRUCT A DOUBLE CATEGORY WHICH TURNS RELATIONS INTO CELLS.

FREE

THE VERTICIES AND HORIZONTAL ARROWS OF $\mathbb{D}$ ARE THOSE OF $\mathbb{G}$.

• FOR EACH $r \in \mathbb{R}$ DISTINCT VERT ARROWS

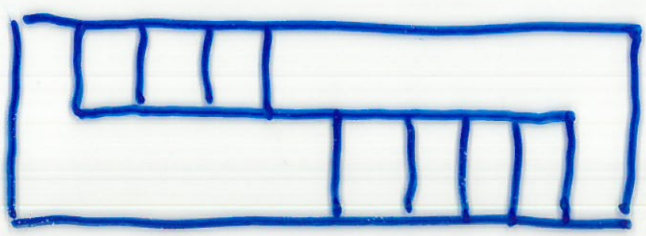
$$\begin{array}{ccccc} A_i & & B & \quad 0 \le i \le m & \quad 0 \le j \le m \\ \big\downarrow x_{r,i} & & \big\downarrow y_{r,j} & \quad x_{r0} = id_A & \quad y_{rm} = id_B \\ A & & B_j & & x_{rm} = y_{r0} \end{array}$$

AND CELLS

$$\begin{tikzcd}
A_{i-1} \arrow[r, "f_i"] \arrow[d, "x_{r,i-1}"'] \arrow[dr, phantom, "\alpha_{ri}"] & A_i \arrow[d, "x_{ri}"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "y_{rj-1}"'] \arrow[dr, phantom, "\beta_{rj}"] & B \arrow[d, "y_{rj}"] \\
B_{j-1} \arrow[r, "g_j"'] & B_j
\end{tikzcd}$$

• ONLY HOR COMPOSITS OF α's & β's IS WITH NEXT IN LINE

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\alpha_{r0}"] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{r1}"] & \cdots \arrow[r] & A_m = B \arrow[r, equal] \arrow[d] \arrow[l, phantom, "\alpha_{rm}"'] \arrow[dr, phantom, "\beta_{r1}"] & B \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\beta_{r2}"] & \cdots \arrow[r, equal] & B \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r, equal] & \cdots \arrow[r, equal] & A = B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_m
\end{tikzcd}$$

γ_r: 

THM: GIVEN TWO PARALLEL WORDS $\alpha_1 \alpha_2 \cdots \alpha_k$ & $d_1 d_2 \cdots d_\ell$, THEN THERE EXISTS A CELL δ WITH BOUNDARY

$$\begin{array}{ccccccccc} C & \xrightarrow{\alpha_1} & C_1 & \xrightarrow{\alpha_2} & C_2 \to & \cdots & \to C_{k-1} & \xrightarrow{\alpha_k} & D \\ \| & & & & & & & & \| \\ C & \xrightarrow[d_1]{} & D_1 & \xrightarrow[d_2]{} & D_2 \to & \cdots & \to D_{\ell-1} & \xrightarrow[d_\ell]{} & D \end{array}$$

IN FREE($\mathbb{D}$) IFF

$\alpha_1 \alpha_2 \cdots \alpha_k = d_1 d_2 \cdots d_\ell$ in $\langle \mathbb{G} | \mathbb{R} \rangle$.

<u>PROOF</u>: ($\Leftarrow$) EASY

($\Rightarrow$) GEOMETRIC + INDUCTION ON # α's & β's.