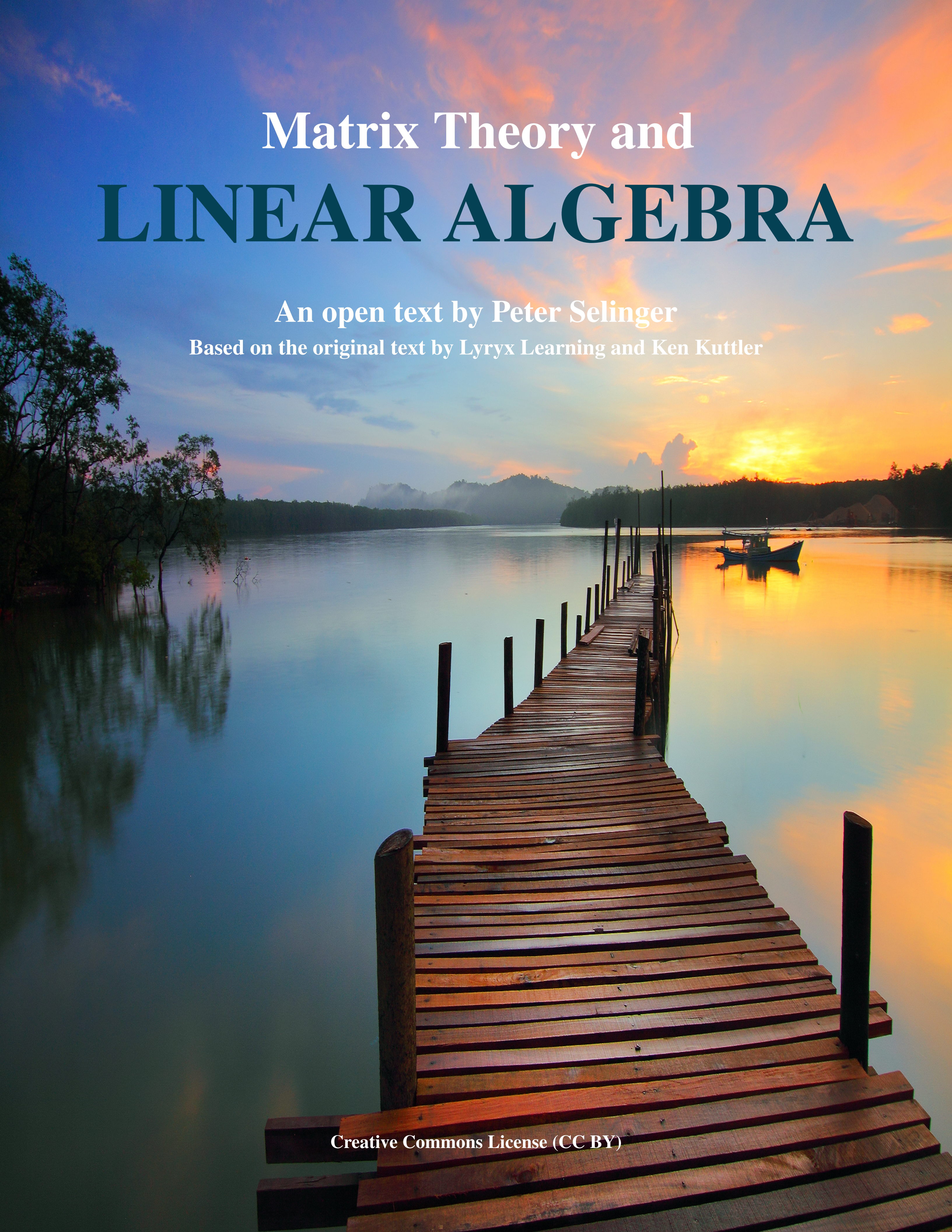
Matrix Theory and
LINEAR ALGEBRA

An open text by Peter Selinger

Based on the original text by Lyryx Learning and Ken Kuttler



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First edition

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Dal 2018 A	• P. Selinger: updated title and front matter. • P. Selinger: Extensive revisions and additions. Rewrote most of Chapters 9–11 from scratch. Added application on perspective rendering, error correcting codes, least squares approximations and curve fitting, an audio demo for Fourier series, sketching of quadratic forms, and principal component analysis.
Dal 2017 B	• P. Selinger, M.B. Langlois: Re-ordered chapters. Extensive revisions to Chapters 1–8. Added new sections on fields, cryptography, geometric interpretation of linear transformations, recurrences, systems of linear differential equations.
2017 A	• Lyryx: Front matter has been updated including cover, copyright, and revision pages. • I. Farah: contributed edits and revisions, particularly the proofs in the Properties of Determinants II: Some Important Proofs section.
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2012 A	• **Original text** by K. Kuttler of Brigham Young University. That version is used under Creative Commons license CC BY (`https://creativecommons.org/licenses/by/3.0/`) made possible by funding from The Saylor Foundation's Open Textbook Challenge. See Elementary Linear Algebra for more information and the original version.

Contents

Preface

Matrix Theory and Linear Algebra is an introduction to linear algebra for students in the first or second year of university. The book contains enough material for a 2-semester course. Major topics of linear algebra are presented in detail, and many applications are given. Although it is not a proof-oriented book, proofs of most important theorems are provided.

Each section begins with a list of desired outcomes which a student should be able to achieve upon completing the chapter. Throughout the text, examples and diagrams are given to reinforce ideas and provide guidance on how to approach various problems. Students are encouraged to work through the suggested exercises provided at the end of each section. Selected solutions to these exercises are given at the end of the text.

Open text

This textbook has a website at `https://www.mathstat.dal.ca/~selinger/linear-algebra/`. There, you can find the most up-to-date version. The website also contains supplementary material, a link to the source code and license, options for purchasing a printed version of this book, and more.

Reporting typos

Like all books, this book likely contains some typos and other errors. However, since it is an open text, typos can easily be fixed and an updated version posted online. It is my intention to fix all typos. If you find a typo (no matter how small), please report it to me at selinger@mathstat.dal.ca. Thanks to the following people who have already reported typos: Yaser Alkayale, Daniele Arcara, Hassaan Asif, Courtney Baumgartner, Kieran Bhaskara, Junwen Deng, Serena Drouillard, Robert Earle, Warren Fisher, Esa Hannila, Melissa Huggan, Xiaoyu Jia, Arman Kerimbek, Peter Lake, Marie-Andrée Langlois, Brenda Le, Sarah Li, Ian MacIntosh, Li Wei Men, Deklan Mengering, Dallas Sawtell, Alain Schaerer, Dinesh Sequeira, Yi Shu, Bruce Smith, Asmita Sodhi, Michael St Denis, Daniele Turchetti, Liu Yuhao, and Ziqi Zhang.

1. Systems of linear equations

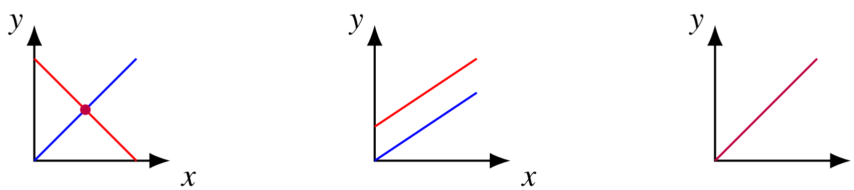
1.1 Geometric view of systems of equations

Outcomes

A. *Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in 3-dimensional space)*

As you may remember, linear equations like $2x+3y=6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y. Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x,y) that satisfies both equations. If the point (x,y) is a point of intersection, we say that (x,y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in the plane, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the plane) involving two variables.



One solution　　No solutions　　Infinitely many solutions

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. There is no solution because the two lines are parallel and they never intersect. The third situation that can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x+y=1$ and $2x+2y=2$ are two equations that yield the same line when graphed. In this case there are infinitely many points that are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations.

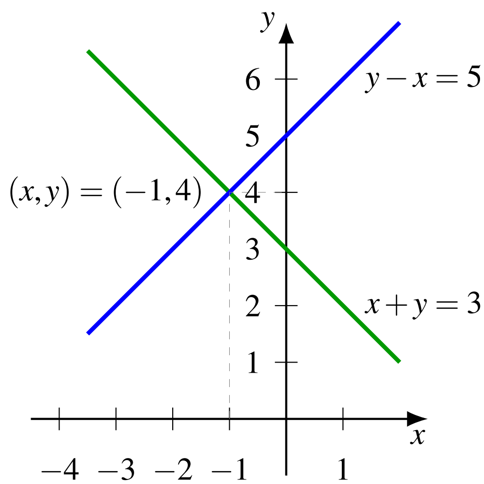
When considering linear systems of equations, there are always three possibilities for the number of solutions: there is exactly one solution, there are infinitely many solutions, or there is no solution. When we speak of *solving* a system of equations, we usually mean finding *all* of its solutions. This can mean finding one solution (if the solution is unique), finding infinitely many solutions, or finding that there is no solution.

Example 1.1: A graphical solution

Use a graph to solve the following system of equations:

$$\begin{aligned} x + y &= 3 \\ y - x &= 5. \end{aligned}$$

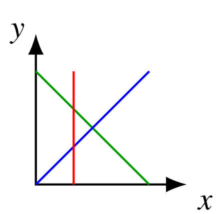
Solution. Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x, y) which satisfy both equations. In this case, there is one point of intersection at $(-1, 4)$ which means we have one unique solution, $x = -1$, $y = 4$.



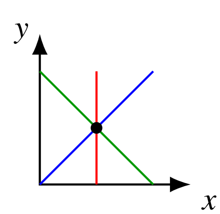
♠

In the above example, we investigated the intersection point of two equations in two variables, x and y. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possibilities for the number of solutions: no solution, one solution, and infinitely many solutions. With three lines, there are more complex ways of achieving these situations. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.



No solution

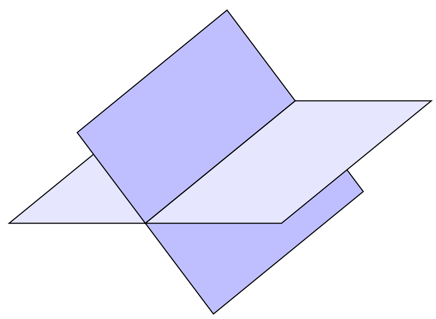


One solution

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x,y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

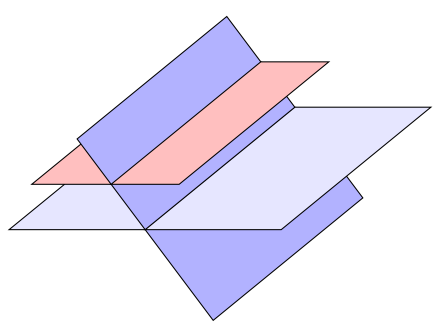
You may recall that equations in three variables, such as $2x+4y-5z=8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x,y,z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.



Notice how these two planes intersect in a line. This means that the points (x,y,z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

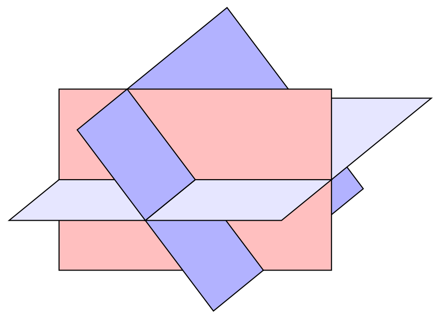
It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the only possibilities for the number of solutions of two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

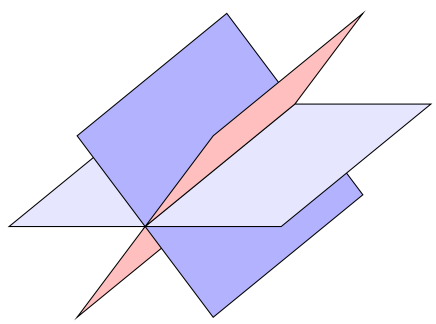


In this diagram, there is no point which lies in all three planes. There is no intersection between **all** three planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.



In this case, the three planes have a single point of intersection. Can you think of other possibilities? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.



We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that all three equations describe the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more possibilities for achieving solutions (or no solutions) than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x+y-2z+4w=8$ with more than three variables are often called **hyperplanes**. You may soon realize that it is tricky to draw the graphs of hyperplanes! In fact, most people cannot visualize more than three dimensions. Fortunately, through the tools of linear algebra, we can examine systems of equations in four variables, five variables, or even hundreds or thousands of variables, without ever needing to graph them. Instead we will use *algebra* to manipulate and solve these systems of equations. We will introduce these algebraic tools in the following sections.

Exercises

Exercise 1.1.1 *Graphically, find the point* (x,y) *which lies on both of the lines* $x+3y=1$ *and* $4x-y=3$. *That is, graph each line and see where they intersect.*

Exercise 1.1.2 *Graphically, find the point of intersection of the two lines* $3x+y=3$ *and* $x+2y=1$. *That is, graph each line and see where they intersect.*

Exercise 1.1.3 *You have a system of* k *equations in two variables,* $k \geq 2$. *Explain the geometric significance of*

(a) *No solution.*

(b) *A unique solution.*

(c) *An infinite number of solutions.*

Exercise 1.1.4 *Draw a picture of three planes such that no two of the planes are parallel, but the three planes have no common intersection.*

1.2 Algebraic view of systems of equations

Outcomes

A. *Recognize the difference between a linear equation and a non-linear equation.*

B. *Determine whether a tuple of real numbers is a solution for a system of linear equations.*

C. *Understand what it means for a system of linear equations to be consistent or inconsistent.*

We have taken an in-depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.2: Linear equation

A ***linear equation*** *is an equation of the form*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b.$$

Here, $a_1,\ldots,a_n$ *are real numbers called the* ***coefficients*** *of the equation,* b *is a real number called the* ***constant term*** *of the equation, and* $x_1,\ldots,x_n$ *are* ***variables****.*

Real numbers, such as the coefficients $a_1, \dots, a_n$, the constant term b, or the values of the variables $x_1, \dots, x_n$, will also be called **scalars**. For now, the word "scalar" is just a synonym for "real number". Later, in Section 1.8, we will discover other kinds of scalars.

Example 1.3: Linear vs. non-linear equation

Which of the following equations are linear?

$$2x + 3y = 5$$
$$2x^2 + 3y = 5$$
$$2\sqrt{x} + 3y = 5$$
$$(\sqrt{2})x + 3y = 5^2$$

Solution. The equation $2x+3y=5$ is linear. The equation $2x^2+3y=5$ is not linear, because it contains the square of a variable instead of a variable. The equation $2\sqrt{x}+3y=5$ is also not linear, because the square root is applied to one of the variables. On the other hand, the equation $(\sqrt{2})x+3y=5^2$ is linear, because $\sqrt{2}$ and 5^2 are real numbers, and can therefore be used as coefficients and constant terms. ♠

We also permit minor notational variants of linear equations. The equation $2x-3y=5$ is linear although Definition 1.2 does not mention subtraction, because it can be regarded as just another notation for $2x+(-3)y=5$. Similarly, the equation $2x=5+3y$ can be regarded as linear, because it can be easily rewritten as $2x-3y=5$ by bringing all the variables (and their coefficients) to the left-hand side. When we need to emphasize that some linear equation is literally of the form $a_1x_1+a_2x_2+\ldots+a_nx_n=b$, we say that the equation is in **standard form**. Thus, the standard form of the equation $2x=5+3y$ is $2x+(-3)y=5$.

A **solution** to a linear equation is an assignment of real numbers to the variables, making the equation true. More precisely, if $r_1, \dots, r_n$ are real numbers, the assignment $x_1 = r_1, \dots, x_n = r_n$ is a solution to the equation in Definition 1.2 if the real number $a_1r_1+a_2r_2+\ldots+a_nr_n$ is equal to the real number b. To save space, we often write solutions in **tuple notation**[1] as $(x_1, \dots, x_n) = (r_1, \dots, r_n)$. When there is no doubt about the order of the variables, we also often simply write the solution as $(r_1, \dots, r_n)$.

Example 1.4: Solutions of a linear equation

Consider the linear equation $2x+3y-4z=5$. *Which of the following are solutions? (a)* $(x,y,z) = (1,1,0)$, *(b)* $(x,y,z)=(0,3,1)$, *(c)* $(x,y,z)=(1,1,1)$.

Solution. The assignment $(x,y,z)=(1,1,0)$ is a solution because $2(1)+3(1)-4(0)=5$. The assignment $(x,y,z)=(0,3,1)$ is also a solution, because $2(0)+3(3)-4(1)=5$. On the other hand, $(x,y,z)=(1,1,1)$ is not a solution, because $2(1)+3(1)-4(1)=1\neq 5$. ♠

A system of linear equations is just several linear equations taken together.

[1]The terminology "tuple" arose as follows. A collection of two items is called a "pair", a collection of three items is called a "triple", followed by "quadruple", "quintuple", "sextuple", and so on. You have to know Latin to know what the next ones are called. To avoid these Latin terms, mathematicians started saying 4-tuple, 5-tuple, 6-tuple and so on, and more generally, n-tuple for an ordered collection of n items. When n doesn't matter or is clear from the context, we often just say "tuple".

Definition 1.5: System of linear equations

*A **system of linear equations** is a list of equations*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

*where a_{ij} and b_i are scalars (i.e., real numbers). The above is a system of m equations in the n variables $x_1, x_2 \ldots, x_n$. As before, the numbers a_{ij} are called the **coefficients** and the numbers b_i are called the **constant terms** of the system of equations.*

The relative size of m and n is not important here. We may have more variables than equations, more equations than variables, or an equal number of equations and variables.

A **solution** to a system of linear equations is an assignment of real numbers to the variables that is a solution to *all* of the equations in the system.

Example 1.6: Solutions of a system of linear equations

Consider the system of linear equations

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ -2x + y + 2z &= -1. \end{aligned}$$

Which of the following are solutions of the system? (a) $(x,y,z) = (1,1,0)$, (b) $(x,y,z) = (6,3,4)$, (c) $(x,y,z) = (0,3,1)$.

Solution. The assignment $(x,y,z) = (1,1,0)$ is a solution of this system of equations, because it is a solution to the first equation and the second equation. Also, $(x,y,z) = (6,3,4)$ is another solution of this system of equations (check this!). On the other hand, $(x,y,z) = (0,3,1)$ is not a solution of the system, because although it is a solution to the first equation, it is not a solution to the second equation. ♠

Recall from Section 1.1 that a system of equations either has a unique solution, infinitely many solutions, or no solution. It is very important to us whether a system of equations has solutions or not. For this reason, we introduce the following terminology:

Definition 1.7: Consistent and inconsistent systems

*A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.*

If we think of each equation as a condition that must be satisfied by the variables, consistent means that there is some choice of values for the variables which can satisfy **all** of the conditions. Inconsistent means that there is no such choice of values for the variables. In the following sections, you will learn a method for determining whether a system of equations is consistent or not, and in case it is consistent, to find all of its solutions.

Exercises

Exercise 1.2.1 *Which of the following equations are linear?*

(a) $2x-3y+4z=-10$

(b) $2.123x_1+5.541x_2-9.101x_3=11.012$

(c) $x^2+y^2+z^2=1$

(d) $\frac{1}{\sqrt{2}}x+4^3y=\sin(\frac{\pi}{3})$

(e) $x+yz=3$

Exercise 1.2.2 *Consider the system of equations*

$$\begin{array}{rcl} x+2y+3z+4w &=& 4 \\ x+y+z+w &=& 2 \\ x+2y+2z+w &=& 2. \end{array}$$

For each of the following tuples (x,y,z,w) of real numbers, determine whether it is a solution of the first equation, second equation, and/or third equation. Which ones are solutions to the system of equations?

(a) $(2,0,-2,2)$ *(b)* $(2,2,-2,0)$ *(c)* $(1,1,-1,1)$ *(d)* $(3,0,-1,1)$ *(e)* $(2,-2,2,0)$

1.3 Elementary operations

Outcomes

A. *Use elementary operations to simplify a system of equations.*

B. *Solve some systems of equations by back substitution.*

C. *Write a system of equations in augmented matrix form.*

D. *Perform elementary row operations on augmented matrices.*

Our strategy for solving systems of linear equations is to successively transform a difficult system of equations into a simpler equivalent system. Here, by an "equivalent" system of equations we mean one that has the same solutions as the original one. We will perform the process of simplifying a system of equations by applying certain basic steps called "elementary operations".

Definition 1.8: Equivalent systems

Two systems of equations are called ***equivalent*** *if they have the same solutions. This means that every solution of the first system is also a solution of the second system, and every solution of the second system is also a solution of the first system.*

How can we know whether two systems of equations are equivalent? It turns out that the following basic operations always transform a system of equations into an equivalent system. In fact, these operations are the *key tool* we use in linear algebra to solve systems of equations.

Definition 1.9: Elementary operations

Elementary operations *are the following operations:*

1. *Interchange the order in which the equations are listed.*
2. *Multiply any equation by a non-zero scalar.*
3. *Add a multiple of one equation to another equation.*

The most important property of the elementary operations is that they do not change the solutions to the system of equations. Before proving that this is true in general, we will first verify it in an example.

Example 1.10: Equivalent systems

Show that the systems

$$\begin{aligned} x+2y &= 7 \\ -2x &= -6 \end{aligned}$$

and

$$\begin{aligned} x+2y &= 7 \\ 4y &= 8 \end{aligned}$$

are equivalent.

Solution. We can see that the second system is obtained from the first one by applying an elementary operation, namely, adding 2 times the first equation to the second equation:

$$-2x+2(x+2y) = -6+2(7)$$

By simplifying, we obtain $4y = 8$.

To verify that the two systems are indeed equivalent, let us first solve the first system. From the second equation, we see that $x = 3$. Substituting $x = 3$ into the first equation, the equation becomes $3+2y = 7$, which we can solve to find $y = 2$. Therefore, the only solution to the first system of equations is $(x,y) = (3,2)$.

Now let us solve the second system. From the second equation, we find that $y = 2$. Substituting $y = 2$ into the first equation, we get $x+4 = 7$, which we can solve to find $x = 3$. Therefore, the only solution to the second system of equations is $(x,y)=(3,2)$. Since the two systems have the same solutions, they are equivalent. ♠

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the set of solutions. The same is true for any size of system in any number of variables. In the following theorem, we use the notation E_i to represent the left-hand side of an equation, while b_i denotes a constant term.

Theorem 1.11: Elementary operations and solutions

Suppose you have a system of two linear equations in any number of variables

$$\begin{array}{r} E_1 = b_1 \\ E_2 = b_2. \end{array} \tag{1.1}$$

Then the following systems are equivalent to (1.1)*:*

1. $$\begin{array}{r} E_2 = b_2 \\ E_1 = b_1. \end{array} \tag{1.2}$$

2. $$\begin{array}{r} E_1 = b_1 \\ kE_2 = kb_2 \end{array} \tag{1.3}$$

   for any scalar k, provided $k \neq 0$.

3. $$\begin{array}{r} E_1 = b_1 \\ E_2 + kE_1 = b_2 + kb_1 \end{array} \tag{1.4}$$

   for any scalar k (including $k=0$).

Proof.

1. By definition, a solution of (1.1) is an assignment of scalars to the variables that is a solution to $E_1=b_1$ and to $E_2=b_2$. But that is exactly the same thing as a solution of (1.2).

2. To prove that the systems (1.1) and (1.3) have the same solution set, let $(x_1,\ldots,x_n)$ be any solution of (1.1). Then $E_1=b_1$ and $E_2=b_2$ are both true. Multiplying both sides of the last equation by k, we know that $kE_2=kb_2$ is true, and so $(x_1,\ldots,x_n)$ is a solution of (1.3). Conversely, let $(x_1,\ldots,x_n)$ be any solution of (1.3). Then $E_1=b_1$ and $kE_2=kb_2$ are true. Because $k\neq 0$, we are allowed to divide both sides of the last equation by k, and therefore $E_2=b_2$ is true. Hence, $(x_1,\ldots,x_n)$ is also a solution of (1.1). Since we have shown that every solution of (1.1) is a solution of (1.3) and vice versa, the two systems are equivalent.

3. To prove that the systems (1.1) and (1.4) have the same solution set, let $(x_1,\ldots,x_n)$ be any solution of (1.1). Then $E_1=b_1$ and $E_2=b_2$ are both true. We multiply both sides of the first equation by k to obtain $kE_1=kb_1$. Then $kE_1+E_2=kb_1+b_2$, and hence $(x_1,\ldots,x_n)$ is a solution of (1.4). For the converse direction, assume $(x_1,\ldots,x_n)$ is a solution of $E_1=b_1$ and $kE_1+E_2=kb_1+b_2$. From the

first equation, we have $kE_1 = kb_1$, and subtracting this from the second equation, we get $E_2 = b_2$, hence $(x_1, \ldots, x_n)$ is a solution of (1.1). Note that unlike in case 2., there was no need to divide by k, and therefore it was not necessary to require $k \neq 0$.

♠

We will now use elementary operations to solve a system of three equations and three variables.

Example 1.12: Solving a system of equations with elementary operations

Solve the system of equations

$$\begin{array}{rcl} x + 3y + 6z &=& 25 \\ 2x + 7y + 14z &=& 58 \\ 2y + 5z &=& 19. \end{array}$$

Solution. By Theorem 1.11, we can do elementary operations on this system without changing the solution set. We will therefore use elementary operations to try to simplify the system of equations. First, we add (-2) times the first equation to the second equation. This yields the system

$$\begin{array}{rcl} x + 3y + 6z &=& 25 \\ y + 2z &=& 8 \\ 2y + 5z &=& 19. \end{array}$$

Next, we add (-2) times the second equation to the third equation. This yields the system

$$\begin{array}{rcl} x + 3y + 6z &=& 25 \\ y + 2z &=& 8 \\ z &=& 3. \end{array} \tag{1.5}$$

At this point, it is easy to find the solution. The last equation tells us that $z = 3$. We can substitute this value of z back into the second equation to get

$$y + 2(3) = 8,$$

which we can simplify and solve for y to find that $y = 2$. Finally, we can substitute the values $z = 3$ and $y = 2$ back into the first equation to get

$$x + 3(2) + 6(3) = 25.$$

Simplifying and solving for x, we find that $x = 1$. Hence, the solution to the system is $(x, y, z) = (1, 2, 3)$.

The process we followed for solving (1.5) by first computing z, then y, then x is called **back substitution**. Alternatively, we could have continued from (1.5) with more elementary operations as follows. Add (-2) times the third equation to the second and then add (-6) times the third to the first. This yields

$$\begin{array}{rcl} x + 3y &=& 7 \\ y &=& 2 \\ z &=& 3. \end{array}$$

Now add (-3) times the second to the first. This yields

$$\begin{array}{ccc} x & & = 1 \\ & y & = 2 \\ & & z = 3, \end{array}$$

a system which has the same solution set as the original system. This second method avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x,y,z) = (1,2,3)$. ♠

Note how we have written each system of equations so that "like" variables line up on columns: one column for x, one column for y, and one column for z. This makes it easier to perform elementary operations. It is often useful to simplify the notation further, writing systems of equations in **augmented matrix** notation. Recall the system of equations from Example 1.12:

$$\begin{array}{rcrcrcr} x & + & 3y & + & 6z & = & 25 \\ 2x & + & 7y & + & 14z & = & 58 \\ & & 2y & + & 5z & = & 19. \end{array}$$

This system can be written as an augmented matrix as follows:

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right].$$

A **matrix** is just a 2-dimensional array of numbers. An augmented matrix has two parts separated by a vertical line. Notice that the augmented matrix notation has exactly the same information as the original system of equations. All the coefficients are written on the left side of the vertical line, and all the constant terms are written on the right side of the vertical line. These two parts of the augmented matrix are also called the **coefficient matrix** and the **constant matrix**. Each row of the augmented matrix corresponds to one linear equation. For example, the top row $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x + 3y + 6z = 25.$$

Each column of the coefficient matrix contains the coefficients for one particular variable. For example, the first column $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$ contains all of the coefficients for the variable x. If a variable does not appear in an equation, the corresponding coefficient is 0. In general, the augmented matrix of a linear system of equations is defined as follows.

Definition 1.13: Augmented matrix of a system of linear equations

*The **augmented matrix** of the system of linear equations*

$$\begin{gathered} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{gathered}$$

is

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

We can consider elementary operations in the context of the augmented matrix. The elementary operations can be used on the rows of an augmented matrix just as we used them on equations previously. For example, instead of adding a multiple of one equation to another, we will now be adding a multiple of one row to another. Note that Theorem 1.11 implies that any elementary row operation used on an augmented matrix will not change the solutions to the corresponding system of equations. For reference, here are the three kinds of elementary row operations, along with a shorthand notation we are going to use for them.

Definition 1.14: Elementary row operations

*The **elementary row operations** are the following:*

1. *Switch two rows. (Notation: $R_i \leftrightarrow R_j$ to switch rows i and j).*
2. *Multiply a row by a non-zero number. (Notation: $R_i \leftarrow kR_i$ to multiply row i by k).*
3. *Add a multiple of one row to another row. (Notation: $R_i \leftarrow R_i + kR_j$ to add k times row j to row i.)*

We write "$\simeq$" to indicate that two augmented matrices are equivalent, i.e., that the corresponding systems of equations have the same set of solutions.

Example 1.15: Elementary row operations

Repeat the calculations of Example 1.12, using the notations we just introduced.

Solution. We have:

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right] \overset{R_3 \leftarrow R_3 - 2R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The final augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3, \end{aligned}$$

which is the same as (1.5). We can solve it by back substitution to obtain the solution $x = 1$, $y = 2$, and $z = 3$.

Alternatively, we can continue with additional row operations:

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{\substack{R_1 \leftarrow R_1 - 6R_3 \\ R_2 \leftarrow R_2 - 2R_3}}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{R_1 \leftarrow R_1 - 3R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Notice how this notation is much more succinct than what we used in Example 1.12. ♠

We end this section with a final word of caution: logically, you can only perform one elementary row operation at a time. For example, it would not be correct to simultaneously add R_1 to R_2 and add R_2 to R_1. What is permitted is to first add R_1 to R_2, then then add the *new* R_2 to R_1. Although we may sometimes try to save space by skipping an intermediate step, as in the last example where we applied the row operations $R_1 \leftarrow R_1 - 6R_3$ and $R_2 \leftarrow R_2 - 2R_3$ in one step, it is important to realize that logically, each row operation must be performed separately before the next one can be done. When in doubt, the only safe course of action is not to skip any steps.

Exercises

Exercise 1.3.1 *Use elementary operations to solve the system of equations*

$$\begin{aligned} 3x + y &= 3 \\ x + 2y &= 1. \end{aligned}$$

Exercise 1.3.2 *Use elementary operations to find the point* (x,y) *that lies on both lines* $x + 3y = 1$ *and* $4x - y = 3$.

Exercise 1.3.3 *Use elementary operations to determine whether the three lines* $x + 2y = 1$, $2x - y = 1$, *and* $4x + 3y = 3$ *have a common point of intersection. If so, find the point, and if not, tell why they don't have such a common point of intersection.*

Exercise 1.3.4 *Do the three planes,* $x + y - 3z = 2$, $2x + y + z = 1$, *and* $3x + 2y - 2z = 0$ *have a common point of intersection? If so, find one and if not, tell why there is no such point.*

Exercise 1.3.5 *Solve the following system of equations by back substitution.*

$$\begin{array}{rcrcrcl} x & + & 3y & - & 2z & = & 5 \\ & & y & + & 3z & = & 4 \\ & & & & z & = & 1. \end{array}$$

Exercise 1.3.6 *Write the following system of linear equations as an augmented matrix. Caution: you first have to simplify and rearrange the equations so that "like" variables are lined up in columns. Write the* *variables in the order x, y, z.*

$$\begin{aligned} x - 3z + 2y &= 5 \\ 6 - x &= 4 + y - z \\ 2x + 3 &= x + 3y. \end{aligned}$$

Exercise 1.3.7 *Four times the weight of Gaston is 150kg more than the weight of Ichabod. Four times the weight of Ichabod is 660kg less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290kg. Brunhilde would balance all three of the others. Find the weights of the four people.*

1.4 Gaussian elimination

Outcomes

A. *Find the echelon form of a matrix.*

B. *Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its echelon form.*

C. *Solve a system of linear equations using Gaussian elimination and back substitution.*

D. *Find the rank of a matrix.*

E. *Determine whether a consistent system of linear equations has a unique solution or an infinite number of solutions from its rank.*

In the previous section, we saw examples of how to solve a system of equations using elementary row operations (and sometimes back substitution). But it is not clear whether every system of equations can be solved this way. How do we know which elementary row operation to apply next? In this section, you will learn a procedure called *Gaussian elimination* by which every system of linear equations can be solved systematically.

Before we start, let's figure out what it means to be "done". At what point should we stop performing row operations? The answer is that we will stop performing row operations when the system of equations is in a special form called *echelon form*, which we now define.

Definition 1.16: Echelon form

An entry of an augmented matrix is called a ***leading entry*** *or* ***pivot entry*** *if it is the leftmost non-zero entry of a row. An augmented matrix is in* ***echelon form*** *(also called* ***row echelon form****) if*

1. *All rows of zeros are below all non-zero rows.*
2. *Each leading entry of a row is in a column to the right of the leading entry of any row above it.*

A column containing a pivot entry is also called a ***pivot column****.*

The word *echelon* comes from French *échelle*, which means ladder. This is because an echelon form looks a bit like a ladder or staircase. Here are some examples of echelon forms.

Example 1.17: Matrices in echelon form

The following augmented matrices are in echelon form. We have circled the pivot entries for clarity.

$$\left[\begin{array}{ccccc|c} 0 & \textcircled{5} & 2 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccccc|c} \textcircled{1} & 4 & 0 & 0 & 5 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 2 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{array}\right], \quad \left[\begin{array}{ccc|c} \textcircled{3} & 0 & 6 & 2 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & \textcircled{2} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 1.18: Not in echelon form

The following augmented matrices are not in echelon form.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \textcircled{1} & 2 & 3 & 3 \\ 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} \textcircled{1} & 2 & 3 \\ \textcircled{2} & 4 & -6 \\ \textcircled{4} & 0 & 7 \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & \textcircled{2} & 3 & 3 \\ \textcircled{1} & 5 & 0 & 2 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In the first matrix, a row of zeros is above a non-zero row. In the second and third matrix, the leading entries of some rows are not to the right of the leading entries of previous rows.

An augmented matrix can always be converted to echelon form by using elementary row operations. The following algorithm shows how to do this.

Algorithm 1.19: Gaussian elimination

This algorithm provides a method for using row operations to take a matrix to its echelon form. We begin with the matrix in its original form.

1. *Starting from the left, find the first non-zero column. This is the first pivot column, and the position at the top of this column will be the position of the first pivot entry. Switch rows if necessary to place a non-zero number in the first pivot position.*
2. *Use row operations to make the entries below the first pivot entry (in the first pivot column) equal to zero.*
3. *Ignoring the row containing the first pivot entry, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more non-zero rows left.*

Most often we will apply this algorithm in order to solve a system of linear equations. This works by first converting the system to echelon form, then using back substitution to find the solutions. The next few examples show how to do this.

Example 1.20: Solving a system of equations: one solution

Solve the following system of equations:

$$\begin{aligned} x+4y+3z &= 11 \\ 2x+10y+7z &= 27 \\ x+y+2z &= 5. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 11 \\ 2 & 10 & 7 & 27 \\ 1 & 1 & 2 & 5 \end{array}\right].$$

In order to find the solution(s) to this system, we first use Algorithm 1.19 to carry the augmented matrix to echelon form. Notice that the first column is non-zero, so this is our first pivot column. The first entry in the first row, 1, is the first pivot entry. We will use row operations to create zeros in the entries below the 1.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 2 & 10 & 7 & 27 \\ 1 & 1 & 2 & 5 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\simeq} \left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 0 & 2 & 1 & 5 \\ 1 & 1 & 2 & 5 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_1}{\simeq} \left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 0 & 2 & 1 & 5 \\ 0 & -3 & -1 & -6 \end{array}\right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column two. Here, the 2 in the second row and second column is in the pivot entry. We could create a zero below the 2 with a single row operation by adding $\frac{3}{2}$ times the second row from the third row. But it is sometimes more convenient not to work with fractions, and therefore we start instead by multiplying the third row by 2.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 0 & \textcircled{2} & 1 & 5 \\ 0 & -3 & -1 & -6 \end{array}\right] \overset{R_3 \leftarrow 2R_3}{\simeq} \left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 0 & \textcircled{2} & 1 & 5 \\ 0 & -6 & -2 & -12 \end{array}\right] \overset{R_3 \leftarrow R_3 + 3R_2}{\simeq} \left[\begin{array}{ccc|c} \textcircled{1} & 4 & 3 & 11 \\ 0 & \textcircled{2} & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The final matrix is our desired echelon form.

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 11 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \tag{1.6}$$

Now we do back substitution to solve for z, y, and then x. The last equation of the echelon form gives us $z = 3$. Substituting this into the second equation, we get $2y + 3 = 5$, which we can solve for y to get $y = 1$. Finally, substituting $y = 1$ and $z = 3$ into the first equation, we have $x + 4(1) + 3(3) = 11$, which we can solve to get $x = -2$. Therefore we have the solution $(x, y, z) = (-2, 1, 3)$.

At this point, it is a good idea to double-check this solution using the original equations

$$\begin{aligned} x + 4y + 3z &= 11 \\ 2x + 10y + 7z &= 27 \\ x + y + 2z &= 5. \end{aligned}$$

For example, we can double-check that $(-2) + 4(1) + 3(3)$ is indeed equal to 11, and similarly for the other two equations. Double-checking the solution against the original equations is an excellent way to guard against any errors that might have happened during the row operations or back substitution.

Finally, we note that $(x, y, z) = (-2, 1, 3)$ is the *only* solution to this system of equations. There cannot be any other solutions, because by Theorem 1.11, any solution of the original system of equations would also have to be a solution of (1.6), and the back substitution leaves us no choice except $z = 3$, $y = 1$, and $x = -2$. ♠

Example 1.21: Solving a system of equations: no solution

Solve the following system of equations:

$$\begin{aligned} y + 2z &= 2 \\ 2x + y - 2z &= 3 \\ 4x - y - 10z &= 4. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 2 & 1 & -2 & 3 \\ 4 & -1 & -10 & 4 \end{array}\right].$$

We use Algorithm 1.19 to carry the augmented matrix to echelon form. The first column is non-zero and will be the first pivot column. We switch the first two rows to move a non-zero number into the pivot position:

$$\overset{R_2 \leftrightarrow R_1}{\simeq} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 2 \\ 4 & -1 & -10 & 4 \end{array}\right].$$

To create zeros below the pivot entry, we subtract 2 times the first row from the third row:

$$\overset{R_3 \leftarrow R_3 - 2R_1}{\simeq} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & -2 \end{array}\right].$$

This finishes the first column. The second pivot column will be column two, with the 1 in the second row and column as the pivot entry. We add 3 times the second row to the third row to create a zero below the pivot:

$$\overset{R_3 \leftarrow R_3+3R_2}{\simeq} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right].$$

This matrix is in echelon form. Note that the final pivot entry is on the right-hand side. The last row corresponds to the equation

$$0x+0y+0z=4.$$

This equation has no solution, because for all x,y,z, the left-hand side will equal 0 and not 4. Therefore, there is no solution to the given system of equations. In other words, the system is inconsistent. ♠

Example 1.22: Solving a system of equations: an infinite set of solutions

Solve the following system of equations:

$$\begin{aligned} 3x-y+5z &= 8 \\ y-10z &= 1 \\ 6x-y &= 17. \end{aligned} \tag{1.7}$$

Solution. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 6 & -1 & 0 & 17 \end{array}\right].$$

We use Gaussian elimination to carry the augmented matrix to echelon form. The first column is the first pivot column, and 3 is the pivot entry. We use row operating to create zeros beneath the pivot entry. We subtract 2 times the first row from the third row and get:

$$\overset{R_3 \leftarrow R_3-2R_1}{\simeq} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 0 & 1 & -10 & 1 \end{array}\right]$$

Now, we have created zeros beneath the pivot entry in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Subtracting the second row from the third row, we get:

$$\overset{R_3 \leftarrow R_3-R_2}{\simeq} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in echelon form. Observe that the first two columns are pivot columns, and the third column is not. We call the corresponding variables x and y **pivot variables**, and the variable z is a **free variable**. The equations corresponding to this echelon form are

$$\begin{aligned} 3x-y+5z &= 8 \\ y-10z &= 1. \end{aligned}$$

Observe that the free variable z is not constrained by any equation. In fact, z can equal any number. We choose t to be any number and let $z = t$. In this context t is called a **parameter**. We then use back substitution to solve for the pivot variables y and x. From the second equation, we have $y = 1 + 10z = 1 + 10t$. From the first equation, we have $3x = 8 + y - 5z = 8 + (1 + 10t) - 5t = 9 + 5t$, and therefore $x = 3 + \frac{5}{3}t$. Therefore, the general solution of this system is

$$\begin{aligned} x &= 3 + \tfrac{5}{3}t \\ y &= 1 + 10t \\ z &= t, \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of the parameter t we select, x, y, and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned} x &= 3 + \tfrac{5}{3}(4) = \tfrac{29}{3} \\ y &= 1 + 10(4) = 41 \\ z &= 4. \end{aligned}$$

♠

In Example 1.22 the solution involved a parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.23: Solving a system of equations: a two-parameter set of solutions

Solve the following system of equations:

$$\begin{aligned} x + 2y - 2z + 2w &= 3 \\ x + 2y - z + 3w &= 5 \\ x + 2y - 3z + w &= 1. \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 3 \\ 1 & 2 & -1 & 3 & 5 \\ 1 & 2 & -3 & 1 & 1 \end{array}\right].$$

We carry this matrix to echelon form using row operations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 3 \\ 1 & 2 & -1 & 3 & 5 \\ 1 & 2 & -3 & 1 & 1 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{\simeq} \left[\begin{array}{cccc|c} \boxed{1} & 2 & -2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 & -2 \end{array}\right] \overset{R_3 \leftarrow R_3 + R_2}{\simeq} \left[\begin{array}{cccc|c} \boxed{1} & 2 & -2 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is in echelon form and we can see that the first and third columns are pivot columns, whereas the second and fourth columns are not. Therefore, x and z are pivot variables and y and w are free variables. We assign parameters $y = s$ and $w = t$ to the free variables. Then we do back substitution to solve for z and x. From the second equation, we have $z = 2 - w = 2 - t$. From the first equation, we have $x = 3 - 2y + 2z - 2w = 3 - 2s + 2(2 - t) - 2t = 7 - 2s - 4t$. Therefore, the general solution is given by

$$\begin{aligned} x &= 7 - 2s - 4t \\ y &= s \\ z &= 2 - t \\ w &= t. \end{aligned}$$

It is customary to write this solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7-2s-4t \\ s \\ 2-t \\ t \end{bmatrix}. \tag{1.8}$$

♠

In Examples 1.20–1.23, we have seem systems of equations with one solution, no solution, and infinitely many solutions with one parameter as well as two parameters. Moreover, in each case, we have been able to determine the number of solutions by looking at the echelon form of the augmented matrix. To summarize, we have the following possibilities for a system of equations:

1. *No solution:* If the echelon form has a row of the form

   $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array}\right],$$

   where $b \neq 0$, then the system is inconsistent and has no solution.

2. *One solution:* For a consistent system of equations: If every column of the coefficient matrix of the echelon form is a pivot column, the system has exactly one solution. The following is an example of an augmented matrix in echelon form for a system of equations with one solution.

   $$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & -2 & 5 \\ 0 & \textcircled{2} & 3 & 0 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

3. *Infinitely many solutions:* For a consistent system of equations: If not all columns of the coefficient matrix of the echelon form are pivot columns, then the system has infinitely many solutions. In this case, each variable corresponding to a non-pivot column is a *free variable* and can be assigned a *parameter*. The remaining variables are *pivot variables* and can be expressed in terms of the parameters. Therefore, the number of parameters in the general solution is equal to the number of non-pivot columns. The following are examples of echelon forms for systems of equations with infinitely many solutions.

   $$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 1 & 5 \\ 0 & \textcircled{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   or

   $$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 5 \\ 0 & 0 & \textcircled{4} & 6 \end{array}\right].$$

There is a special name for the number of pivot variables in a system of equations. It is called the *rank* of the system.

Definition 1.24: Rank of a matrix

Let A be a matrix and consider any echelon form of A. Then, the number r of pivot entries of A does not depend on the echelon form we choose, and is called the **rank** *of A. We denote it by $\operatorname{rank}(A)$. The rank of a system of linear equations is the rank of its coefficient matrix (i.e., the matrix on the left-hand side). It is equal to the number of pivot variables.*

Example 1.25: Finding the rank of a matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution. First, we need to find an echelon form of A. Through the usual algorithm, we find that this is

$$\begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{3} & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we have two pivot entries, and therefore the rank of A is $r = 2$. ♠

Suppose we have a system of m equations in n variables, and suppose that $n > m$. Further assume that the system is consistent. From our above discussion, we know that this system will have infinitely many solutions. This is because there can be at most one pivot entry per row, and therefore at most m variables can be pivot variables. It follows that there are at least $n-m$ free variables. Therefore, the general solution of this system has at least $n-m$ parameters.

Notice that if $n = m$ or $n < m$, it is possible for the system to have a unique solution or infinitely many solutions. In all cases ($n > m$, $n = m$, or $n < m$), it is also possible for the system to be inconsistent (have no solutions).

By refining the above argument, we get the following theorem:

Theorem 1.26: Rank and solutions of consistent system of equations

Consider a system of m equations in n variables, and assume that the coefficient matrix has rank r. Assume further that the system is consistent.

1. *If $r = n$, then the system has a unique solution.*
2. *If $r < n$, then the system has infinitely many solutions, with $n - r$ parameters.*

Here is a final summary of how the rank affects the number of solutions:

1. *No solution.* If the system of equations is inconsistent, then it has no solution, regardless of the rank.

2. *Unique solution.* For a consistent system, suppose $r = n$. Then there is a pivot position in every column of the coefficient matrix of A. Hence, there is a unique solution.

3. *Infinitely many solutions.* For a consistent system, suppose $r < n$. Then there are less pivot positions than columns in the coefficient matrix, meaning that not every column is a pivot column. The columns which are *not* pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters. The system has infinitely many solutions.

Exercises

Exercise 1.4.1 *Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?*

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

Exercise 1.4.2 *Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?*

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$$

Exercise 1.4.3 *Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?*

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

Exercise 1.4.4 *Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?*

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right]$$

Exercise 1.4.5 *Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.*

Exercise 1.4.6 *If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, explain why not.*

Exercise 1.4.7 *Find h such that*

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right]$$

is the augmented matrix of an inconsistent system.

Exercise 1.4.8 *Find h such that*

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right]$$

is the augmented matrix of a consistent system.

Exercise 1.4.9 *Find h such that*

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right]$$

is the augmented matrix of a consistent system.

Exercise 1.4.10 *Choose h and k such that the augmented matrix shown has each of the following:*

(a) one solution

(b) no solution

(c) infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right]$$

Exercise 1.4.11 *Choose h and k such that the augmented matrix shown has each of the following:*

(a) one solution

(b) no solution

(c) infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right]$$

Exercise 1.4.12 *Determine if the system is consistent. If so, is the solution unique?*

$$\begin{array}{c} x+2y+z-w=2 \\ x-y+z+w=1 \\ 2x+y-z=1 \\ 4x+2y+z=5 \end{array}$$

Exercise 1.4.13 *Determine if the system is consistent. If so, is the solution unique?*

$$\begin{aligned} x+2y+z-w&=2\\ x-y+z+w&=0\\ 2x+y-z&=1\\ 4x+2y+z&=3 \end{aligned}$$

Exercise 1.4.14 *Determine which matrices are in echelon form.*

$$(a)\begin{bmatrix} 1 & 1 & 2 & 0\\ 0 & 0 & 3 & 2\\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (b)\begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 7 \end{bmatrix} \quad (c)\begin{bmatrix} 0 & 1 & 0 & 0 & 5\\ 0 & 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Exercise 1.4.15 *Row reduce each of the following matrices to echelon form.*

$$(a)\begin{bmatrix} 2 & -1 & 3 & -1\\ 1 & 0 & 2 & 1\\ 1 & -1 & 1 & -2 \end{bmatrix} \quad (b)\begin{bmatrix} 0 & 0 & -1 & -1\\ 1 & 1 & 1 & 0\\ 1 & 1 & 0 & -1 \end{bmatrix} \quad (c)\begin{bmatrix} 3 & -6 & -7 & -8\\ 1 & -2 & -2 & -2\\ 1 & -2 & -3 & -4 \end{bmatrix}$$

$$(d)\begin{bmatrix} 2 & 4 & 5 & 15\\ 1 & 2 & 3 & 9\\ 1 & 2 & 2 & 6 \end{bmatrix} \quad (e)\begin{bmatrix} 4 & -1 & 7 & 10\\ 1 & 0 & 3 & 3\\ 1 & -1 & -2 & 1 \end{bmatrix} \quad (f)\begin{bmatrix} 3 & 5 & -4 & 2\\ 1 & 2 & -1 & 1\\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$(g)\begin{bmatrix} -2 & 3 & -8 & 7\\ 1 & -2 & 5 & -5\\ 1 & -3 & 7 & -8 \end{bmatrix}$$

Exercise 1.4.16 *Find the general solution of the system whose augmented matrix is*

$$(a)\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2\\ 1 & 3 & 4 & 2\\ 1 & 0 & 2 & 1 \end{array}\right] \quad (b)\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2\\ 2 & 0 & 1 & 1\\ 3 & 2 & 1 & 3 \end{array}\right] \quad (c)\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1\\ 1 & 0 & 4 & 2 \end{array}\right]$$

$$(d)\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2\\ 0 & 1 & 0 & 1 & 2 & 1\\ 1 & 2 & 0 & 0 & 1 & 3\\ 1 & 0 & 1 & 0 & 2 & 2 \end{array}\right] \quad (e)\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2\\ 0 & 1 & 0 & 1 & 2 & 1\\ 0 & 2 & 0 & 0 & 1 & 3\\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right]$$

Exercise 1.4.17 *Solve the system of equations* $7x+14y+15z=22$, $2x+4y+3z=5$, *and* $3x+6y+10z=13$.

Exercise 1.4.18 *Solve the system of equations* $3x-y+4z=6$, $y+8z=0$, *and* $-2x+y=-4$.

Exercise 1.4.19 *Solve the system of equations* $9x-2y+4z=-17$, $13x-3y+6z=-25$, *and* $-2x-z=3$.

Exercise 1.4.20 *Solve the system of equations* $65x+84y+16z=546$, $81x+105y+20z=682$, *and* $84x+110y+21z=713$.

Exercise 1.4.21 *Solve the system of equations* $8x+2y+3z=-3$, $8x+3y+3z=-1$, *and* $4x+y+3z=-9$.

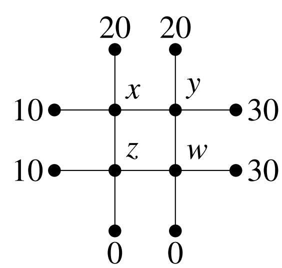
Exercise 1.4.22 *Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.*

Exercise 1.4.23 *Suppose a system of linear equations has an augmented matrix with 2 rows and 4 columns and the last column is a pivot column. Could the system of linear equations be consistent? Explain.*

Exercise 1.4.24 *Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.*

Exercise 1.4.25 *Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?*

Exercise 1.4.26 *The steady state temperature, u, of a plate solves Laplace's equation,* $\Delta u = 0$. *One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x, y, z, and w. One of the equations is* $z=\frac{1}{4}(10+0+w+x)$.



Exercise 1.4.27 *Find the rank of the following matrices.*

$$(a)\ \begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix} \quad (b)\ \begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \quad (c)\ \begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

$$(d)\ \begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (e)\ \begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix}$$

Exercise 1.4.28 *Suppose A is an* $m\times n$*-matrix. Explain why the rank of A is always no larger than* $\min(m,n)$.

Exercise 1.4.29 *State whether each of the following sets of data is possible for a system of equations. If possible, describe the solution set. That is, indicate whether there exists a unique solution, no solution or infinitely many solutions. Here, A is the coefficient matrix, and $[A \mid B]$ denotes the augmented matrix of the system.*

(a) *A is a 5×6-matrix, $\operatorname{rank}(A)=4$ and $\operatorname{rank}[A \mid B]=4$.*

(b) *A is a 3×4-matrix, $\operatorname{rank}(A)=3$ and $\operatorname{rank}[A \mid B]=2$.*

(c) *A is a 4×2-matrix, $\operatorname{rank}(A)=4$ and $\operatorname{rank}[A \mid B]=4$.*

(d) *A is a 5×5-matrix, $\operatorname{rank}(A)=4$ and $\operatorname{rank}[A \mid B]=5$.*

(e) *A is a 4×2-matrix, $\operatorname{rank}(A)=2$ and $\operatorname{rank}[A \mid B]=2$.*

Exercise 1.4.30 *Consider the system $-5x+2y-z=0$ and $-5x-2y-z=0$. Both equations equal zero and so $-5x+2y-z=-5x-2y-z$ which is equivalent to $y=0$. Does it follow that x and z can equal anything? Notice that when $x=1$, $z=-4$, and $y=0$ are plugged in to the equations, the equations do not equal 0. Why?*

1.5 Gauss-Jordan elimination

Outcomes

A. *Find the reduced echelon form of a matrix.*

B. *Solve a system of linear equations using Gauss-Jordan elimination.*

In the previous section, we saw how to solve a system of equations by using Gaussian elimination and back substitution. The back substitution step can be quite confusing and error prone, especially when there are parameters. For example, in Example 1.23, we had to substitute $y=s$, $z=2-t$, and $w=t$ into the equation $x=3-2y+2z-2w$, which required another simplification step.

In this section, you will learn an alternative procedure called *Gauss-Jordan elimination* which eliminates the need for back substitution, at the expense of doing a few additional row operations. The key to this technique is a special kind of echelon form called a *reduced echelon form*.

Definition 1.27: Reduced echelon form

*An augmented matrix is in **reduced echelon form** if*

1. *It is in echelon form.*
2. *Each leading entry is equal to 1.*
3. *All entries above a leading entry are zero.*

Example 1.28: Reduced echelon form

The following augmented matrices are in reduced echelon form. The leading entries have been circled for emphasis. Note how all of the leading entries are equal to 1*, and they have zeros above them.*

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We can carry every augmented matrix to reduced echelon form by doing elementary row operations.

Algorithm 1.29: Gauss-Jordan elimination

This algorithm provides a method for using row operations to take a matrix to its reduced echelon form.

1. *First, use Gaussian elimination (Algorithm 1.19) to reduce the matrix to echelon form.*
2. *Moving from right to left, consider each pivot entry. Without changing the row containing the pivot entry, or any rows below it, use row operations to create zeros in the column above the pivot entry. Finally, divide the row by its pivot entry, to make the pivot entry equal to* 1*.*

Example 1.30: Gauss-Jordan elimination

Solve the system of equations from Example 1.20 using Gauss-Jordan elimination.

$$\begin{aligned} x+4y+3z &= 11 \\ 2x+10y+7z &= 27 \\ x+y+2z &= 5. \end{aligned}$$

Solution. In Example 1.20, we had already reduced the system to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 11 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

We reduce it to reduced echelon form by performing the following row operations:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 11 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - 3R_3}}{\simeq} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{R_1 \leftarrow R_1 - 2R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{R_2 \leftarrow \frac{1}{2}R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The resulting matrix is in reduced echelon form. Note that the final system of equations is especially easy to solve, because the three equations are $x=-2$, $y=1$, and $z=3$. No back substitution is needed. ♠

Example 1.31: Gauss-Jordan elimination

Solve the system of equations from Example 1.23 using Gauss-Jordan elimination.

$$\begin{aligned} x+2y-2z+2w &= 3 \\ x+2y-z+3w &= 5 \\ x+2y-3z+1w &= 1. \end{aligned}$$

Solution. In Example 1.23, we had obtained the following echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We reduce this to reduced echelon form by performing the following additional step:

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{R1 \leftarrow R1+2R2}{\simeq} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 7 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The first equation states that $x = 7-2y-4w$, and the second equation states that $z = 2-w$. Using the free variables y and w as parameters, we obtain the following general solution:

$$\begin{aligned} x &= 7-2y-4w \\ y &= y \\ z &= 2-w \\ w &= w. \end{aligned}$$

Note that we did not really have to do back substitution; all we had to do is to shift parts of the equations to the right-hand side. If the solution looks strange, because it has equations like "$y = y$" in it, keep in mind that this means that y and w are arbitrary numbers, i.e., parameters. We can replace y and w by parameters s and t on the right-hand side, as before:

$$\begin{aligned} x &= 7-2s-4t \\ y &= s \\ z &= 2-t \\ w &= t. \end{aligned}$$

♠

Discussion 1.32: Which procedure is better?

Which one is the better procedure to use, Gaussian elimination with back substitution, or Gauss-Jordan elimination? The answer is: it depends. In certain applications, it is not necessary to completely solve a system of equations. Sometimes it is sufficient just to figure out whether the system is consistent or inconsistent, or whether the solution is unique or not. In those situations, you already get the required information from the echelon form and there is no need to do the additional steps to reduce the system to reduced echelon form. Also, in some situations, Gauss-Jordan elimination can introduce fractions into your augmented matrix, making the matrix more complicated to work with. In such cases, it may sometimes be easier to do back substitution. But in most cases, Gauss-Jordan elimination is simpler to do than back substitution, and therefore I recommend using the Gauss-Jordan method most of the time.

One situation where Gauss-Jordan elimination excels is when you have to solve many systems of equations that all have the same coefficient matrix.

Example 1.33: Multiple systems sharing the same left-hand side

Solve the following two systems of equations.

$$\begin{aligned} x+z &= 1 \\ 2x+y+3z &= 2 \\ 3x+2y+5z &= 4 \end{aligned} \qquad \begin{aligned} x+z &= 2 \\ 2x+y+3z &= 5 \\ 3x+2y+5z &= 8 \end{aligned}$$

Solution. We could certainly solve each system of equations separately. But since the left-hand sides are the same, we will perform exactly the same row operations on both systems. We can save some work by solving both systems together. Instead of a usual augmented matrix with only one constant vector, we create an augmented matrix containing both constant vectors at the same time.

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 5 & 4 & 8 \end{array}\right]$$

Then we row-reduce the coefficient matrix to reduced echelon form as usual. (We do not need to bother reducing the right-hand side to reduced echelon form).

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 5 & 4 & 8 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\simeq} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 2 \end{array}\right] \overset{R_3 \leftarrow R_3 - 2R_2}{\simeq} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

We see that the first system is inconsistent, because it contains a row of the form $[0\ 0\ 0 \mid 1]$. The second system is consistent, and we get the general solution $z = t$, $y = 1 - t$, $x = 2 - t$. ♠

Exercises

Exercise 1.5.1 *Determine which matrices are in reduced echelon form.*

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

Exercise 1.5.2 *Reduce each of the matrices from Exercise 1.4.15 to reduced echelon form.*

Exercise 1.5.3 *Use Gauss-Jordan elimination to solve the system of equations* $-8x+2y+5z=18$, $-8x+3y+5z=13$, *and* $-4x+y+5z=19$.

Exercise 1.5.4 *Use Gauss-Jordan elimination to solve the system of equations* $3x-y-2z=3$, $y-4z=0$, *and* $-2x+y=-2$.

Exercise 1.5.5 *Use Gauss-Jordan elimination to solve the system of equations* $-9x+15y=66$, $-11x+18y=79$, $-x+y=4$, *and* $z=3$.

Exercise 1.5.6 *Use Gauss-Jordan elimination to solve the system of equations* $-19x+8y=-108$, $-71x+30y=-404$, $-2x+y=-12$, $4x+z=14$.

Exercise 1.5.7 *Solve the following two systems of equations simultaneously, by using a single augmented matrix with two constant vectors.*

$$\begin{aligned} x+2y-z&=0 \\ 2x+3y+z&=3 \\ x-y+2z&=3 \end{aligned} \qquad \begin{aligned} x+2y-z&=1 \\ 2x+3y+z&=7 \\ x-y+2z&=4 \end{aligned}$$

1.6 Homogeneous systems

Outcomes

A. *Determine whether a homogeneous system of equations has non-trivial solutions from its rank.*

B. *Find the basic solutions of a homogeneous system of equations.*

C. *Understand the relationship between the general solution of a system of equations and that of its associated homogeneous system.*

There is a special type of system of linear equations that requires additional study. This type of system is called a *homogeneous*[2] system of equations. Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations, and how the solutions of non-homogeneous systems are related to those of their homogeneous counterparts.

[2]The word "homogeneous" has 5 syllables. In scientific usage, it is not the same as the word "homogenous".

Definition 1.34: Homogeneous system of equations

A system of equations is called ***homogeneous*** *if each of the constant terms is equal to* 0*. A homogeneous system therefore has the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0, \end{aligned}$$

where a_{ij} *are coefficients and* x_j *are variables.*

The first thing we note is that a homogeneous system is always consistent. Indeed, it always has the solution $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$. This solution is called the **trivial solution**.

If the system has a solution in which not all of the $x_1, \ldots, x_n$ are equal to zero, then we call this solution **non-trivial**. When working with homogeneous systems of equations, since the trivial solution always exists, we are usually interested in finding whether there are non-trivial solutions.

The following theorem is a special case of Theorem 1.26. Recall that the *rank* of a system is the number of pivot variables in its echelon form.

Theorem 1.35: Rank and solutions of homogeneous system of equations

Consider a homogeneous system of m *equations in* n *variables, and assume that the coefficient matrix has rank* r*. Then the system is consistent, and*

1. *if* $r = n$*, then the system has only the trivial solution;*
2. *if* $r < n$*, then the system has infinitely many solutions.*

Example 1.36: Homogeneous system with more variables than equations

True or false: Suppose a homogeneous system has more variables than equations. Then the system has infinitely many solutions.

Solution. This is true. If the system has m equations and n variables, then the rank can be at most m. Since $m < n$, the system has infinitely many solutions. Note that it is not possible for a homogeneous system to be inconsistent, since there is always the trivial solution. ♠

Example 1.37: Homogeneous system with an equal number of variables and equations

True or false: Suppose a homogeneous system has the same number of variables as equations. Then the system has a unique solution.

Solution. This is false in general. While it is possible for such a system to have a unique solution, it is also possible for it to have infinitely many. Let there be n equations and n variables. Then depending on the echelon form, the rank r could be either equal to n, in which case there is a unique solution, or less than n, in which case there are infinitely many. ♠

We now consider an example of solving a homogeneous system of equations.

Example 1.38: Solutions to a homogeneous system of equations

Find the general solution to the following homogeneous system of equations. Does the system have non-trivial solutions?

$$\begin{aligned} 2x+y+z+4w &= 0 \\ x+2y-z+5w &= 0 \end{aligned}$$

Solution. Notice that this system has $m=2$ equations and $n=4$ variables, so $n>m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions. In particular, it will have non-trivial solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used for non-homogeneous equations. We construct the augmented matrix and reduce it to reduced echelon form.

$$\left[\begin{array}{rrrr|r} 2 & 1 & 1 & 4 & 0 \\ 1 & 2 & -1 & 5 & 0 \end{array}\right] \simeq \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x+z+w &= 0 \\ y-z+2w &= 0. \end{aligned}$$

The free variables are z and w. We set them equal to parameters $z=s$ and $w=t$. Then our general solution has the form

$$\begin{aligned} x &= -s-t \\ y &= s-2t \\ z &= s \\ w &= t. \end{aligned}$$

Hence this system has infinitely many solutions, with two parameters s and t. ♠

Let us write the solution of the last example in another form. Specifically, it can be written as

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}. \tag{1.9}$$

Notice that we have constructed a column from the coefficients of s in each equation, and another column from the coefficients of t. We will discuss this notation more in later chapters. For now, consider what happens when we choose the parameters to be $s=1$ and $t=0$. In this case, we get the solution

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{1.10}$$

which is the same as the column of coefficients for s. This is called a **basic solution** of the homogeneous system of equations. The other basic solution is obtained by setting $s=0$ and $t=1$. In this case,

$$\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}. \tag{1.11}$$

The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. If X_1 and X_2 are the basic solutions (1.10) and (1.11), then the general solution (1.9) is of the form $sX_1 + tX_2$. We say that the general solution of the homogeneous system is a **linear combination** of its basic solutions.

We explore this further in the following example.

Example 1.39: Basic solutions of a homogeneous system

Consider the following homogeneous system of equations.

$$\begin{aligned} x+4y+3z &= 0 \\ 3x+12y+9z &= 0. \end{aligned} \tag{1.12}$$

Find the basic solutions to this system.

Solution. The augmented matrix of this system and the resulting reduced echelon form are

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

When written in equations, this system is given by

$$x+4y+3z=0.$$

Notice that x is the only pivot variable, and y and z are free variables. Let $y=s$ and $z=t$ for parameters s and t. Then the general solution is

$$\begin{aligned} x &= -4s-3t \\ y &= s \\ z &= t, \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t. Therefore, this system has two basic solutions! They are

$$X_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

♠

We can take any non-homogeneous system of equations and get a new homogeneous system by keeping the left-hand sides the same and setting all of the constant terms equal to 0. This is called the **associated homogeneous system** of the system of equations. We end this section by investigating how the solutions of a system of equations are related to the solutions of its associated homogeneous system.

Example 1.40: Non-homogeneous vs. homogeneous system

Solve the system of equations

$$\begin{aligned} x+4y+3z &= 2 \\ 3x+12y+9z &= 6. \end{aligned} \tag{1.13}$$

How are the solutions related to those of the associated homogeneous system in Example 1.39?

Solution. We note that the associated homogeneous system of (1.13) is the system we saw in Example 1.39. We solve the system (1.13) in the usual way by reducing its augmented matrix to reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 3 & 12 & 9 & 6 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and then assigning parameters $y = s$, $z = t$ to the free variables. From the equation $x+4y+3z = 2$, the general solution is

$$\begin{aligned} x &= 2-4s-3t \\ y &= s \\ z &= t, \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \tag{1.14}$$

We see that the general solution is almost exactly the same as that of the homogeneous system in Example 1.39. The only difference is the additional column

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \tag{1.15}$$

♠

Note that the column (1.15), by itself, is a solution of the non-homogeneous system. It is not the most general solution, but rather the particular solution resulting from the parameters $s = 0$ and $t = 0$. We can therefore interpret equation (1.14) as saying that the general solution of the non-homogeneous system is equal to a particular solution of the non-homogeneous system, plus the general solution of the associated homogeneous system. The same is true in general, and we summarize it as a theorem.

Theorem 1.41: Non-homogeneous vs. homogeneous system

Let A be a system of equations, and let B be the associated homogeneous system. Then

$$\text{the general solution of } A = \text{a particular solution of } A + \text{the general solution of } B.$$

Exercises

Exercise 1.6.1 *Find the basic solutions of each of the following homogeneous systems of equations.*

$$(a)\quad \begin{aligned} 2x+3y+4z&=0\\ x-2y+z&=0\\ 4x-y+6z&=0 \end{aligned} \qquad (b)\quad \begin{aligned} x-y+z&=0\\ -x-2y-4z&=0\\ 2x+y+5z&=0 \end{aligned} \qquad (c)\quad \begin{aligned} x+y-z+2w&=0\\ x+3y+z+6w&=0\\ x+2y+4w&=0 \end{aligned}$$

Exercise 1.6.2 *Which of the following homogeneous systems of linear equations have non-trivial solutions?*

(a) 4 equations in 3 variables, rank 3.

(b) 3 equations in 4 variables, rank 3.

(c) 4 equations in 3 variables, rank 2.

(d) 3 equations in 4 variables, rank 2.

Exercise 1.6.3 *My system of equations has a solution* $(x,y,z)=(1,2,4)$. *The associated homogeneous system has basic solutions* $(x,y,z)=(1,0,1)$ *and* $(x,y,z)=(0,1,-1)$. *What is the general solution of my system of equations?*

1.7 Uniqueness of the reduced echelon form

Outcomes

A. *Determine whether two systems of equations are row equivalent, by comparing their reduced echelon form.*

B. *For two homogeneous systems of equations that are not row equivalent, find a solution to one system that is not a solution to the other.*

We have seen in earlier sections that every matrix can be brought into reduced echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced echelon form does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1,x_2,\ldots,x_n$ which is also in reduced echelon form. Recall that the matrix A divides the set of variables in two different types: x_i is a *pivot variable* when column i is a pivot column, and a *free variable* otherwise.

Example 1.42: Pivot and free variables

Find the pivot and free variables in the following system, and find the general solution.

$$\begin{aligned} x+2y-z+w &= 0 \\ x+y-z+w &= 0 \\ x+3y-z+w &= 0 \end{aligned}$$

Solution. The reduced echelon form of the augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From this, we see that columns 1 and 2 are pivot columns. Therefore, x and y are pivot variables and z and w are free variables. We can write the solution to this system as

$$\begin{aligned} x &= s-t \\ y &= 0 \\ z &= s \\ w &= t. \end{aligned}$$

♠

In general, all solutions can be written in terms of the free variables. In such a description, the free variables are written as parameters, while the pivot variables are written as functions of these parameters. Indeed, a pivot variable x_i is a function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 1.43: Pivot and free variables

If x_i is a pivot variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section.

Lemma 1.44: Solutions and the reduced echelon form of a matrix

Let A and B be two augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced echelon form. If A and B are different, then the two systems do not have exactly the same solutions.

Proof. With respect to the linear systems associated with the matrices A and B, there are two cases to consider:

- Case 1: the two systems have the same pivot variables
- Case 2: the two systems do not have the same pivot variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B, where $a \neq b$. Since A is in reduced echelon form, if x_j were a pivot variable for its linear system, we would have $a = 0$. Similarly, if x_j were a pivot variable for the linear system of the matrix B, we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a pivot variable for both linear systems. However, since the systems have the same pivot variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_i = -a$, while the solution of the system for matrix B has $x_i = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a pivot variable for one matrix, let's say A, and a free variable for the other matrix B. The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j. However, by Proposition 1.43 this cannot be a solution of the system for the matrix A. This completes the proof of case 2. ♠

Now, we say that the matrix B is **row equivalent** to the matrix A if B can be obtained from A by performing a sequence of elementary row operations. By Theorem 1.11, we know that row equivalent systems have exactly the same solutions. Now, we can use Lemma 1.44 to prove the main result of this section, which is that each matrix A has a unique reduced echelon form.

Theorem 1.45: Uniqueness of the reduced echelon form

Every matrix A is row equivalent to a unique matrix in reduced echelon form.

Proof. By Gauss-Jordan elimination, we already know that every matrix is row equivalent to some reduced echelon form. What we must show is that the resulting reduced echelon form is unique, i.e., does not depend on the order in which row operations are performed.

Therefore, let A be an $m \times n$-matrix and let B and C be matrices in reduced echelon form, each row equivalent to A. We have to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+. Note that B^+ and C^+ are augmented matrices in reduced echelon form, and that both B^+ and C^+ are row equivalent to A^+, because the addition of a column of zeros does not change the effect of any row operations.

Now, A^+, B^+, and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \ldots, x_n$. Because all three systems are row equivalent, they have exactly the same solutions. By Lemma 1.44, we conclude that $B^+ = C^+$. Omitting the final column of zeros, we must also have $B = C$. ♠

Example 1.46: Row equivalent systems

Determine whether the following two systems of equations are row equivalent.

$$\begin{aligned} 2x+3y+z &= 12 \\ x-2y+4z &= -1 \\ x+2z &= 3, \end{aligned} \qquad \begin{aligned} x+2y &= 7 \\ 3x-y+7z &= 7 \\ y-z &= 2. \end{aligned}$$

Solution. The augmented matrices for the two systems are:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ 1 & -2 & 4 & -1 \\ 1 & 0 & 2 & 3 \end{array}\right], \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 3 & -1 & 7 & 7 \\ 0 & 1 & -1 & 2 \end{array}\right].$$

The reduced echelon forms of the two augmented matrices are:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since both systems have the same reduced echelon form, they are row equivalent. ♠

Example 1.47: Non-row equivalent systems

Determine whether the following two systems of equations are row equivalent. If they are not row equivalent, find a solution to one system that is not a solution to the other.

$$\begin{aligned} x-2y-5z &= 0 \\ x+z &= 0 \\ x+y+4z &= 0, \end{aligned} \qquad \begin{aligned} 2x+2y+z &= 0 \\ x+y+3z &= 0 \\ -x-y+2z &= 0. \end{aligned}$$

Solution. The augmented matrices for the two systems are:

$$\left[\begin{array}{ccc|c} 1 & -2 & -5 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right], \qquad \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right].$$

The reduced echelon forms of the two augmented matrices are:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the two systems have different reduced echelon forms, they are not row equivalent. Following the proof of Lemma 1.44, we see that z is a free variable for the first system, but a pivot variable for the second system. Therefore, there exists a solution of the first system with $z=1$, namely $(x,y,z)=(-1,-3,1)$. But there exists no solution for the second system with $z=1$, and in particular, $(x,y,z)=(-1,-3,1)$ is not a solution of the second system. ♠

We finish this section by pointing out an important consequence of Theorem 1.45, namely that the rank of a matrix is well-defined. Recall that in Definition 1.24, we defined the rank of a matrix A to be the number of pivot entries of "any" echelon form of A. It was not clear, however, why different echelon forms of A could not have different numbers of pivot entries. Now we can answer this question. By the Gauss-Jordan algorithm, we know that every echelon form can be converted to a reduced echelon form without changing the number or position of the pivots. Since the reduced echelon form is unique, it follows that all echelon forms of A have the same number of pivot entries (and in fact the same pivot columns). Therefore, the rank of A is a well-defined quantity.

Exercises

Exercise 1.7.1 *The following are augmented matrices for four systems of equations. Determine which of them, if any, are row equivalent.*

$$\left[\begin{array}{rrrr|r} 1 & 3 & 5 & 1 & 12 \\ -1 & 1 & -1 & 2 & 5 \\ 2 & 0 & 4 & -2 & 0 \end{array}\right] \quad \left[\begin{array}{rrrr|r} 1 & 2 & 4 & -1 & 8 \\ 2 & 4 & 3 & 1 & 15 \\ 3 & 6 & 1 & -1 & 8 \end{array}\right] \quad \left[\begin{array}{rrrr|r} 3 & 6 & -3 & 1 & 6 \\ 2 & 4 & 2 & 1 & 13 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \quad \left[\begin{array}{rrrr|r} 1 & 2 & 4 & 1 & 10 \\ 0 & 1 & 1 & 1 & 5 \\ 2 & 1 & 5 & 0 & 8 \end{array}\right]$$

Exercise 1.7.2 *Find a tuple (x,y,z) that is a solution to one system of equations but not the other.*

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 5 & 0 \\ 2 & 1 & 4 & 3 & 0 \\ 3 & 1 & 5 & 9 & 0 \end{array}\right] \quad \left[\begin{array}{rrrr|r} 1 & 0 & 3 & 6 & 0 \\ 2 & 1 & 8 & 2 & 0 \\ 2 & 0 & 6 & 13 & 0 \end{array}\right]$$

1.8 Fields

Outcomes

A. *Solve systems of equations using scalars from a field other than the real numbers, such as $\mathbb{Z}_2$ or $\mathbb{Z}_5$.*

So far in this chapter, we have worked with real numbers: all of the scalars we used, for coefficients, constant terms, variables, and parameters, were real numbers. But in fact, we have not used very many properties of the real numbers, except for the fact that we can add, subtract, multiply, and divide them. For example, we have never needed to take a square root or to compute a trigonometric function.

In fact, most of linear algebra only requires addition, subtraction, multiplication, and division. This opens the door to doing linear algebra using other kinds of scalars besides the real numbers. For example, we can do linear algebra over the rational numbers, complex numbers, or even over some more exotic number systems that you will learn about in this section. A system of scalars that one can do linear algebra with is called a *field*.

Definition 1.48: Field

A ***field*** *is a set K, together with two operations called* **addition** *and* **multiplication***, and together with two distinct elements 0 and 1, such that addition and multiplication satisfy the following properties:*

(A1) *Commutative law of addition: $a+b=b+a$;*

(A2) *Associative law of addition: $(a+b)+c=a+(b+c)$;*

(A3) *Unit law of addition: $0+a=a$;*

(A4) *Additive inverse: for each $a\in K$, there exists an element $(-a)\in K$ such that $a+(-a)=0$;*

(M1) *Commutative law of multiplication: $ab=ba$;*

(M2) *Associative law of multiplication: $(ab)c=a(bc)$;*

(M3) *Unit law of multiplication: $1a=a$;*

(M4) *Multiplicative inverse: for each non-zero $a\in K$, there exists an element $a^{-1}\in K$ such that $aa^{-1}=1$;*

(D) *Distributive law: $a(b+c)=ab+ac$.*

Properties (A1)–(A4) are about addition, properties (M1)–(M4) are about multiplication, and property (D) is about both addition and multiplication. Here are some examples and non-examples of fields:

Example 1.49: Some fields and some non-fields

(a) *The set $\mathbb{R}$ of real numbers is a field.*

(b) *The set $\mathbb{Q}$ of rational numbers is a field.*

(c) *The set $\mathbb{Z}$ of integers satisfies all field properties except for (M4). It is therefore not a field.*

(d) *The set $\mathbb{N}=\{0,1,2,\dots\}$ of natural numbers satisfies all field properties except (A4) and (M4). It is therefore not a field.*

A field doesn't have to be infinite. The following is an example of a field with only two elements.

Example 1.50: The integers modulo 2

Consider the set of bits *(binary digits)* $\{0,1\}$*. We can multiply them as usual, and add them almost as usual, subject to the alternative rule* $1+1=0$ *(instead of* $1+1=2$*). Here is a summary of the rules for addition and multiplication:*

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

This particular alternative arithmetic is called "arithmetic modulo 2". In computer science, the addition is also called the "logical exclusive or" operation, and multiplication is also called the "logical and" operation. You can also think of 0 *as "even" and* 1 *as "odd", and not that odd plus odd makes even. For example, we can calculate like this:*

$$\begin{aligned} 1\cdot((1+0)+1)+1 &= 1\cdot(1+1)+1 \\ &= 1\cdot 0+1 \\ &= 0+1 \\ &= 1. \end{aligned}$$

The binary digits form a field $\mathbb{Z}_2=\{0,1\}$*, also called* ***the field of integers modulo 2****.*

You can convince yourself that the 9 properties of fields are all satisfied by the integers modulo 2. This is a bit tedious, but it can be checked by calculations. For example, to verify (A1), we have to check that $0+0=0+0$, $0+1=1+0$, $1+0=0+1$, and $1+1=1+1$. Perhaps the most interesting properties are (A4) and (M4). For (A4), we can set $(-0)=0$ and $(-1)=1$. It may be surprising that $(-1)=1$, but you can check for yourself that $1+(-1)=1+1=0$ when calculating modulo 2. For (M4), we can set $1^{-1}=1$.

When solving systems of linear equations, we only used addition, subtraction, multiplication, and division. Therefore, we can solve systems of equations using the elements of any field as the scalars, instead of the real numbers.

Example 1.51: Solving a system of equations over $\mathbb{Z}_2$

Solve the following system in the integers modulo 2:

$$\begin{aligned} x+y&=0 \\ x+z&=1 \\ y+z&=1. \end{aligned}$$

Solution. As usual, we write the augmented matrix of the system of equations, then reduce it to reduced echelon form using elementary row operations. The only difference is that we will perform all arithmetic operations modulo 2, rather than in the real numbers. The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

The first pivot entry is the 1 in the upper left. We use a row operation to create a zero below it. Note that, because we are working modulo 2, adding 1 and subtracting 1 is the same thing, so $0-1=1$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\simeq} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

The next pivot entry is in row 2 and column 2. We create a zero below it by subtracting row 2 from row 3, and a zero above it by subtracting row 2 from row 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \leftarrow R_1 - R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The resulting system is in reduced echelon form. We can see that the system is consistent, because there is no row whose left-hand side is zero and whose right-hand side is non-zero. We also see that there are two pivot columns, and therefore two pivot variables, x and y. On the other hand, z is a free variable, so we set it equal to a parameter: $z=t$. Notice that this time, the parameter t is not a real number, but an element of $\mathbb{Z}_2$. From the equation $x+z=1$, we get $x=1-z=1+t$. Can you guess why I have written $1+t$ instead of $1-t$? This is because $(-1)=1$ in the integers modulo 2. So $1-t=1+(-1)t=1+1t=1+t$. Similarly, from the equation $y+z=1$, we get that $y=1+t$. Therefore, the general solution to the system of equations is

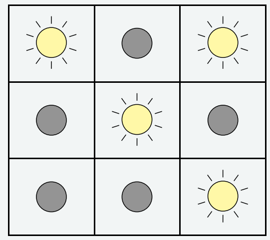
$$\begin{aligned} x &= 1+t \\ y &= 1+t \\ z &= t, \end{aligned}$$

where $t\in\{0,1\}$ is an arbitrary parameter. Recall that this means that each time we plug in a particular value for t, we get a solution.

There is one difference between solving equations in the real numbers and solving equations in $\mathbb{Z}_2$. In the real numbers, a system of equations has either no solution, a unique solution, or infinitely many solutions. This is because when there is a parameter, we automatically get infinitely many solutions. By contrast, in $\mathbb{Z}_2$, there are only two scalars, and therefore only two possible values for the parameter t, namely $t=0$ and $t=1$. For $t=0$ we get the solution $(x,y,z)=(1,1,0)$, and for $t=1$ we get the solution $(x,y,z)=(0,0,1)$. Thus, when the general solution has one parameter in $\mathbb{Z}_2$, there are only two solutions, instead of infinitely many. ♠

Example 1.52: A game with buttons and lights

Consider a game with 9 lights arranged in a square:



Each light is also a button. When a button is pressed, its own light, and all the lights neighboring it (i.e., above, below, to the left and to the right) are toggled (i.e., any light that was off is turned on and vice versa). Figure out which buttons to press to turn off all the lights if the starting position is as shown above.

Solution. We number the lights and buttons from top to bottom, left to right, like this:

1	2	3
4	5	6
7	8	9

.

Let x_i be a variable in $\mathbb{Z}_2$, corresponding to the event "button i is pressed" (or more precisely, "button i is pressed an odd number of times", because pressing a button twice is the same as not pressing it at all. That is why we are working modulo 2). The light in position 1 is initially on. It is toggled each time buttons 1, 2, and 4 are pressed, i.e., it is toggled $x_1+x_2+x_4$ times. We want this light to be off in the end. So we must have $1+x_1+x_2+x_4=0$. Similarly, the light in position 2 is initially off. To ensure that it stays off, we must have $0+x_1+x_2+x_3+x_5=0$. In this way, we obtain 9 linear equations in 9 variables:

$$\begin{array}{rcll}
1 & + & x_1+x_2+x_4 & = & 0 \\
0 & + & x_1+x_2+x_3+x_5 & = & 0 \\
1 & + & x_2+x_3+x_6 & = & 0 \\
0 & + & x_1+x_4+x_5+x_7 & = & 0 \\
1 & + & x_2+x_4+x_5+x_6+x_8 & = & 0 \\
0 & + & x_3+x_5+x_6+x_9 & = & 0 \\
0 & + & x_4+x_7+x_8 & = & 0 \\
0 & + & x_5+x_7+x_8+x_9 & = & 0 \\
1 & + & x_6+x_8+x_9 & = & 0.
\end{array}$$

If we write this system in standard form, we obtain the following augmented matrix:

$$\left[\begin{array}{ccccccccc|c}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}\right].$$

We solve this system of equations by doing Gauss-Jordan elimination with scalars in $\mathbb{Z}_2$. The reduced echelon form is

$$\left[\begin{array}{ccccccccc|c} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&1&0 \end{array}\right].$$

The unique solution is $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) = (0,0,1,1,1,0,0,1,0)$. This means that we must press buttons 3, 4, 5, and 8. ♠

Example 1.53: The integers modulo 5

Consider the set $\mathbb{Z}_5 = \{0,1,2,3,4\}$*, called the* ***integers modulo 5****. We define their addition and multiplication by computing the usual addition and multiplication, then "reducing" the answer modulo 5. Here, "reducing" a number means repeatedly subtracting 5 until the answer is between* 0 *and* 4*. Imagine a clock showing 5 numbers instead of the usual 12:*

If we want to calculate 3 o'clock plus 4 hours, we get 2 o'clock, because whenever the clock reaches 5, it resets to 0. This is how addition and multiplication modulo 5 are defined:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

We note that the integers modulo 5 form a field. Most of the properties are tedious but easy to verify. Perhaps the most interesting of the field properties is (M4). It says that for each non-zero element a, there is another element a^{-1} such that $aa^{-1} = 1$. By looking at the multiplication table, we see that $1 \cdot 1 = 1$, $2 \cdot 3 = 1$, $3 \cdot 2 = 1$, and $4 \cdot 4 = 1$. Therefore we can set $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, and $4^{-1} = 4$.

Example 1.54: Division in $\mathbb{Z}_5$

What is 2 divided by 3 in $\mathbb{Z}_5$*?*

Solution. There are no fractions in $\mathbb{Z}_5$. The key to dividing is this: instead of dividing by a, multiply by a^{-1}. So we have:

$$2/3 = 2 \cdot 3^{-1} = 2 \cdot 2 = 4.$$

So 2 divided by 3 equals 4. This makes sense, because 4 times 3 equals 2, when calculating modulo 5. ♠

Example 1.55: Solving a system of equations over $\mathbb{Z}_5$

Solve the following system of linear equations over $\mathbb{Z}_5$:

$$\begin{aligned} 2x+z &= 1 \\ x+4y+z &= 3 \\ x+2y+3z &= 2. \end{aligned}$$

Solution. We perform the usual Gauss-Jordan algorithm on the augmented matrix. The only thing to keep in mind is that, instead of dividing a row by a, we should multiply it by a^{-1}. And of course, we should reduce all intermediate results modulo 5. For example, to change the first pivot entry from 2 to 1, we multiply by $2^{-1} = 3$, instead of dividing by 2.

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 4 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{array}\right] \overset{R_1 \leftarrow 3R_1}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 1 & 4 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 4 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right] \overset{R_2 \leftarrow 4R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - 2R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \overset{\substack{R_1 \leftarrow R_1 - 3R_3 \\ R_2 \leftarrow R_2 - 2R_3}}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

The final matrix is in reduced echelon form, and we see that the system has the unique solution $(x,y,z) = (1,2,4)$. Please double-check the solution with respect to the original equations. ♠

Example 1.56: The integers modulo 6 are not a field

Do the integers modulo 6 form a field?

Solution. The integers modulo 6 are the set $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$, with the addition and multiplication modulo 6:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Then $\mathbb{Z}_6$ satisfies all of the field axioms except (M4). To see why (M4) fails, let $a = 2$, and note, by looking at the multiplication table, that there is no $b \in \mathbb{Z}_6$ such that $ab = 1$. Therefore, $\mathbb{Z}_6$ is not a field. ♠

We conclude this section with a fact that we will not prove.

Theorem 1.57: The integers modulo a prime

Let n be a positive integer. Then the set $\mathbb{Z}_n = \{0, \ldots, n-1\}$ of integers modulo n, with addition and multiplication modulo n, forms a field if and only if n is prime. Thus, for example, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, and $\mathbb{Z}_{11}$ are fields, whereas $\mathbb{Z}_4$, $\mathbb{Z}_6$, and $\mathbb{Z}_9$ are not.

Another field that is very useful in mathematics and the natural sciences is the field $\mathbb{C}$ of **complex numbers**. You can read about the complex numbers in Appendix A.

Example 1.58: Solving a system of equations over $\mathbb{C}$

Solve the following system of equations over the complex numbers:

$$\begin{aligned} x - y + z &= -1 + i, \\ x + iy + 3z &= 1 + 3i. \end{aligned}$$

Solution. We perform Gauss-Jordan elimination on the augmented matrix. When multiplying or dividing, we have to use complex number arithmetic. For example,

$$\frac{2}{1+i} = \frac{2}{1+i} \cdot \frac{1-i}{1-i} = \frac{2-2i}{2} = 1 - i.$$

The row operations are:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -1+i \\ 1 & i & 3 & 1+3i \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\simeq} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1+i \\ 0 & 1+i & 2 & 2+2i \end{array}\right] \overset{R_2 \leftarrow R_2/(1+i)}{\simeq} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1+i \\ 0 & 1 & 1-i & 2 \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 + R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 2-i & 1+i \\ 0 & 1 & 1-i & 2 \end{array}\right].$$

Therefore the general solution is $z = t$, $y = 2 - (1-i)t$, $x = 1 + i - (2-i)t$, where $t \in \mathbb{C}$ is a parameter, i.e., t is any complex number. In vector form, the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2+i \\ -1+i \\ 1 \end{bmatrix}.$$

♠

Exercises

Exercise 1.8.1 *Solve each of the following systems of equations with scalars in $\mathbb{Z}_2$. If there is more than one solution, write the general solution in parametric form and also write down all of the solutions individually. How many solutions are there?*

$$(a)\left[\begin{array}{cccc|c} 0&1&1&1&0\\ 1&0&0&1&1\\ 1&1&1&0&1 \end{array}\right] \quad (b)\left[\begin{array}{ccc|c} 0&1&1&1\\ 1&0&1&1\\ 1&1&0&1 \end{array}\right] \quad (c)\left[\begin{array}{cccc|c} 0&1&1&0&1\\ 1&0&1&0&0\\ 1&1&0&1&0 \end{array}\right] \quad (d)\left[\begin{array}{ccc|c} 0&1&1&1\\ 1&0&1&1\\ 1&1&1&0 \end{array}\right]$$

Exercise 1.8.2 *Solve each of the following systems of equations with scalars in $\mathbb{Z}_3$. How many solutions does each system have?*

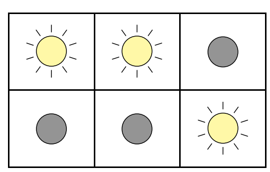
$$(a)\left[\begin{array}{ccc|c} 0&1&1&1\\ 1&2&0&1\\ 2&0&2&1 \end{array}\right] \quad (b)\left[\begin{array}{cccc|c} 2&1&2&1&0\\ 0&2&1&1&2\\ 1&0&0&1&1 \end{array}\right] \quad (c)\left[\begin{array}{ccc|c} 1&2&0&2\\ 1&0&1&1\\ 1&1&1&0 \end{array}\right]$$

Exercise 1.8.3 *Solve each of the following systems of equations with scalars in $\mathbb{Z}_5$.*

$$(a)\left[\begin{array}{cccc|c} 0&2&1&4&0\\ 1&1&2&3&2\\ 2&4&0&0&4 \end{array}\right] \quad (b)\left[\begin{array}{ccc|c} 1&2&4&1\\ 3&0&1&1\\ 2&4&3&2 \end{array}\right]$$

Exercise 1.8.4 *In $\mathbb{Z}_7$, calculate 1^{-1}, 2^{-1}, 3^{-1}, 4^{-1}, 5^{-1}, and 6^{-1}. Hint: write down the multiplication table.*

Exercise 1.8.5 *Consider a game similar to Example 1.52, with 6 lights arranged in a rectangle:*



Again, each light doubles as a button. Pressing it toggles its own light, as well as all of its neighbors. Which buttons do you have to press to turn off all the light from the starting position shown above? Is the answer unique? Is every pattern of lights reachable from this starting position?

Exercise 1.8.6 *Solve each of the following systems of equations with scalars in the complex numbers.*

$$(a)\left[\begin{array}{ccc|c} 1&1&1+i&2\\ 1+i&2+i&3i&3+2i \end{array}\right] \quad (b)\left[\begin{array}{ccc|c} 1&i&1+i&-1+2i\\ 1+i&2&1-i&4+4i\\ 1&-1+i&-i&0 \end{array}\right]$$

1.9 Application: Balancing chemical reactions

The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions. Consider the chemical reaction

$$SnO_2 + H_2 \to Sn + H_2O.$$

Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs that transforms a combination of tin dioxide (SnO_2) and hydrogen (H_2) into a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each substance we began with and how much of each substance is involved in the result.

An important theory we will use here is the mass balance theory. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of atoms of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x,y,z,w such that

$$x\,SnO_2 + y\,H_2 \to z\,Sn + w\,H_2O,$$

where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x,y,z,w. Thus we need

$$\begin{array}{rl} Sn: & x = z \\ O: & 2x = w \\ H: & 2y = 2w. \end{array}$$

We can rewrite these equations as

$$\begin{array}{rl} Sn: & x - z = 0 \\ O: & 2x - w = 0 \\ H: & 2y - 2w = 0. \end{array}$$

The augmented matrix for this system of equations is given by

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right].$$

The reduced echelon form of this matrix is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right],$$

and the solution is given in parametric form as

$$\begin{aligned} x &= \tfrac{1}{2}t \\ y &= t \\ z &= \tfrac{1}{2}t \\ w &= t. \end{aligned}$$

For example, let $t = 2$ and this would yield $x = 1$, $y = 2$, $z = 1$, and $w = 2$. We can put these values back into the expression for the reaction which yields

$$SnO_2 + 2H_2 \to Sn + 2H_2O.$$

Observe that each side of the expression contains the same number of atoms of each element. This means that the chemical reaction is balanced. Of course, because it is a homogeneous system of equations, any multiple of a solution is also a solution. For example,

$$2SnO_2 + 4H_2 \to 2Sn + 4H_2O$$

is also correct. It just means that we have just doubled the amount of every substance involved. In chemistry, the numbers you are finding would typically be the number of mols of the molecules on each side. Thus one mol of SnO_2 added two mols of H_2 yields one mol of Sn and two mols of H_2O.

Here is another example.

Example 1.59: Balancing a chemical reaction

Potassium is denoted by K, oxygen by O, phosphorus by P and hydrogen by H. Consider the reaction given by

$$KOH + H_3PO_4 \to K_3PO_4 + H_2O.$$

Balance this chemical reaction.

Solution. We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that

$$x\, KOH + y\, H_3PO_4 \to z\, K_3PO_4 + w\, H_2O.$$

preserves the total number of atoms of each element. Finding these values can be done by finding the solution to the following system of equations.

$$\begin{array}{ll} K: & x = 3z \\ O: & x + 4y = 4z + w \\ H: & x + 3y = 2w \\ P: & y = z. \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right],$$

and the reduced echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is given in terms of the parameter t as

$$\begin{aligned} x &= t \\ y &= \tfrac{1}{3}t \\ z &= \tfrac{1}{3}t \\ w &= t \end{aligned}$$

Choose a value for t, say 3. This yields $x = 3$, $y = 1$, $z = 1$, and $w = 3$. It follows that the balanced reaction is given by

$$3KOH + H_3PO_4 \rightarrow K_3PO_4 + 3H_2O$$

Note that this results in the same number of atoms of each element on both sides. ♠

Exercises

Exercise 1.9.1 *Balance the following chemical reactions.*

(a) $KNO_3 + H_2CO_3 \rightarrow K_2CO_3 + HNO_3$

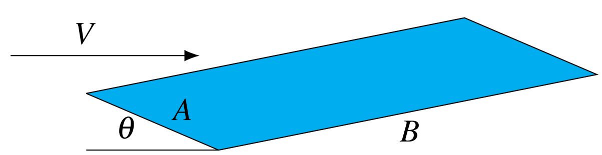
(b) $AgI + Na_2S \rightarrow Ag_2S + NaI$

(c) $Ba_3N_2 + H_2O \rightarrow Ba(OH)_2 + NH_3$

(d) $CaCl_2 + Na_3PO_4 \rightarrow Ca_3(PO_4)_2 + NaCl$

1.10 Application: Dimensionless variables

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown.



The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$\mathrm{m}^0\,\mathrm{kg}^0\,\mathrm{s}^0$
speed of wind	V	$\mathrm{m\,s}^{-1}$
speed of sound	V_0	$\mathrm{m\,s}^{-1}$
density of air	ρ	$\mathrm{kg\,m}^{-3}$
viscosity	μ	$\mathrm{kg\,s}^{-1}\,\mathrm{m}^{-1}$
lift	l	$\mathrm{kg\,s}^{-2}\,\mathrm{m}$

Here m denotes meters, s refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ, which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how “sticky" the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu(\text{area})(\text{velocity gradient}) = \text{ tangential force}$$

Hence

$$(\text{units of } \mu)\,\mathrm{m}^2\left(\frac{\mathrm{m}}{\mathrm{s\,m}}\right) = \mathrm{kg\,s}^{-2}\,\mathrm{m}$$

Thus the units of μ are

$$\mathrm{kg\,s}^{-1}\,\mathrm{m}^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to centimeters would result in an incorrect value for l. The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f(g_1, \ldots, g_k)\rho V^2 AB$$

where the units of $\rho V^2 AB$ are

$$\frac{\mathrm{kg}}{\mathrm{m}^3}\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \mathrm{m}^2 = \frac{\mathrm{kg}\times\mathrm{m}}{\mathrm{s}^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \tag{1.16}$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$\mathrm{m}^{x_1}\,\mathrm{m}^{x_2}(\mathrm{m}^{x_4}\,\mathrm{s}^{-x_4})(\mathrm{m}^{x_5}\,\mathrm{s}^{-x_5})(\mathrm{kg}\,\mathrm{m}^{-3})^{x_6}(\mathrm{kg}\,\mathrm{s}^{-1}\,\mathrm{m}^{-1})^{x_7} = \mathrm{m}^0\,\mathrm{kg}^0\,\mathrm{s}^0$$

Notice that there are no units of θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i.

$$\begin{array}{rc} \mathrm{m}: & x_1+x_2+x_4+x_5-3x_6-x_7=0 \\ \mathrm{s}: & -x_4-x_5-x_7=0 \\ \mathrm{kg}: & x_6+x_7=0 \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

The reduced echelon form is given by

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and so the solutions are of the form

$$\begin{array}{rcl} x_1 & = & -x_2-x_7 \\ x_3 & = & x_3 \\ x_4 & = & -x_5-x_7 \\ x_6 & = & -x_7 \end{array}$$

Thus, in terms of vectors, the solution is

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array}\right] = \left[\begin{array}{c} -x_2-x_7 \\ x_2 \\ x_3 \\ -x_5-x_7 \\ x_5 \\ -x_7 \\ x_7 \end{array}\right]$$

Thus the free variables are x_2, x_3, x_5, x_7. By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula (1.16). For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^1$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR. Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ. Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^1$. However, it is written as V/V_0. This is called the Mach number $\mathscr{M}$. Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $\mathrm{Re} = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(\mathrm{Re}, AR, \theta, \mathscr{M})\,\mathrm{kg} \times \mathrm{m}/\mathrm{s}^2$$

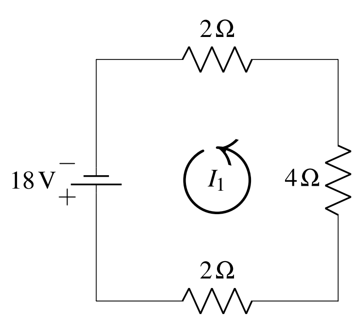
This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ. Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments, to get a formula for the lift that would be reasonable. You could also consider more variables and more complicated situations in the same way.

Exercises

Exercise 1.10.1 *In this section, we observed that $\rho V^2 AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables that will yield something that has the units of force.*

1.11 Application: Resistor networks

The tools of linear algebra can be used to study the application of resistor networks. An example of an electrical circuit is below.



The jagged lines (—/\/\/\/—) denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The voltage source (—|⊢—) causes the current to flow in the direction from the longer of the two lines toward the shorter[3]. Voltage is measured in volts, written as V. The current for a circuit is labelled I_k, and is measured in amperes, written as A.

[3]By *current*, we always mean the *conventional current*, which flows from plus to minus. It is the opposite of the electron flow, which goes from minus to plus.

In the above figure, the current I_1 has been labelled with an arrow in the counterclockwise direction. This is an entirely arbitrary decision and we could have chosen to label the current in the counterclockwise direction. With our choice of direction here, we define a positive current to flow in the counterclockwise direction and a negative current to flow in the clockwise direction.

The goal of this section is to use the values of resistors and voltage sources in a circuit to determine the current. An essential theorem for this application is Kirchhoff's law.

Theorem 1.60: Kirchhoff's law

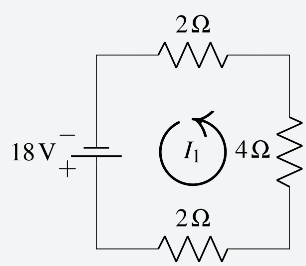
The sum of the resistance (R) times the amperes (I) in the counterclockwise direction around a loop equals the sum of the voltage sources (V) in the same direction around the loop.

Kirchhoff's law allows us to set up a system of linear equations and solve for any unknown variables. When setting up this system, it is important to trace the circuit in the counterclockwise direction. If a resistor or voltage source is crossed against this direction, the related term must be given a negative sign.

We will explore this in the next example where we determine the value of the current in the initial diagram.

Example 1.61: Solving for current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.



Solution. Begin in the bottom left corner, and trace the circuit in the counterclockwise direction. At the first resistor, multiplying resistance and current gives $2I_1$. Continuing in this way through all three resistors gives $2I_1 + 4I_1 + 2I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source matches the counterclockwise direction specified, so the voltage is positive.

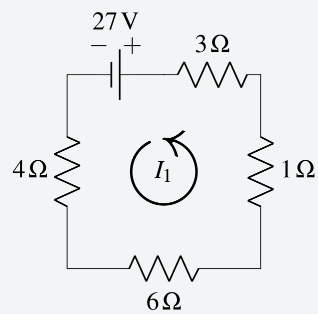
Therefore the equation and solution are given by

$$\begin{aligned} 2I_1 + 4I_1 + 2I_1 &= 18, \\ 8I_1 &= 18, \\ I_1 &= \frac{9}{4}\,\text{A}. \end{aligned}$$

Since the answer is positive, this confirms that the current flows counterclockwise. ♠

Example 1.62: Solving for current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.



Solution. Begin in the top left corner this time, and trace the circuit in the counterclockwise direction. At the first resistor, multiplying resistance and current gives $4I_1$. Continuing in this way through the four resistors gives $4I_1+6I_1+1I_1+3I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source is opposite to the counterclockwise direction, so the voltage is negative.

Therefore the equation and solution are given by

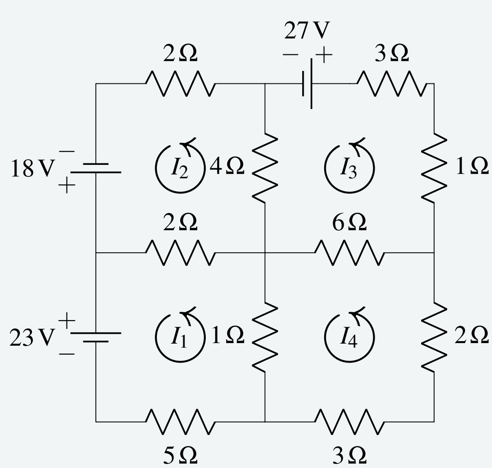
$$\begin{aligned} 4I_1+6I_1+1I_1+3I_1 &= -27, \\ 14I_1 &= -27, \\ I_1 &= -\frac{27}{14}\,\text{A}. \end{aligned}$$

Since the answer is negative, this tells us that the current flows clockwise. ♠

A more complicated example follows. Two of the circuits below may be familiar; they were examined in the examples above. However as they are now part of a larger system of circuits, the answers will differ.

Example 1.63: Unknown currents

The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchhoff's Law, write an equation for each circuit and solve for each current.



Solution. Starting with the top left circuit, multiply the resistance by the current and sum the resulting products. Specifically, consider the resistor labelled $2\,\Omega$ that is part of the circuits of I_1 and I_2. Notice that current I_2 runs through this in a positive (counterclockwise) direction, and I_1 runs through in the opposite (negative) direction. The product of resistance and current is then $2(I_2 - I_1) = 2I_2 - 2I_1$. Continue in this way for each resistor, and set the sum of the products equal to the voltage source to write the equation:

$$2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 = 18.$$

The above process is used on each of the other three circuits, and the resulting equations are:

Upper right circuit:

$$4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + 3I_3 = -27.$$

Lower right circuit:

$$3I_4 + 2I_4 + 6I_4 - 6I_3 + I_4 - I_1 = 0.$$

Lower left circuit:

$$5I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -23.$$

Notice that the voltage for the upper right and lower left circuits are negative due to the clockwise direction they indicate. The resulting system has four equations in four variables. Simplifying and rearranging with variables in order, we have:

$$\begin{aligned} -2I_1 + 8I_2 - 4I_3 &= 18, \\ -4I_2 + 14I_3 - 6I_4 &= -27, \\ -I_1 - 6I_3 + 12I_4 &= 0, \\ 8I_1 - 2I_2 - I_4 &= -23. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{rrrr|r} -2 & 8 & -4 & 0 & 18 \\ 0 & -4 & 14 & -6 & -27 \\ -1 & 0 & -6 & 12 & 0 \\ 8 & -2 & 0 & -1 & -23 \end{array}\right].$$

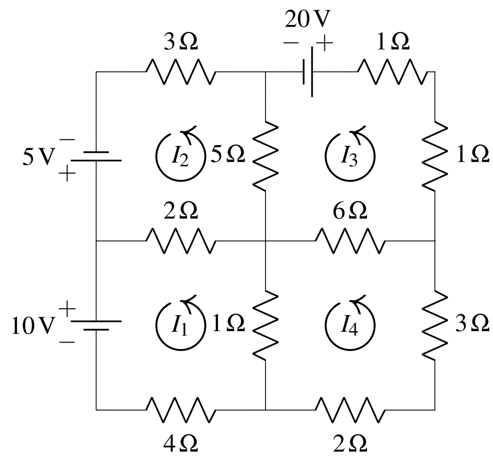
The solution to this system of equations is

$$\begin{aligned} I_1 &= -3\,\text{A}, \\ I_2 &= \frac{1}{4}\,\text{A}, \\ I_3 &= -\frac{5}{2}\,\text{A}, \\ I_4 &= -\frac{3}{2}\,\text{A}. \end{aligned}$$

This tells us that currents I_1, I_3, and I_4 travel clockwise while I_2 travels counterclockwise. ♠

Exercises

Exercise 1.11.1 *Consider the following diagram of four circuits.*

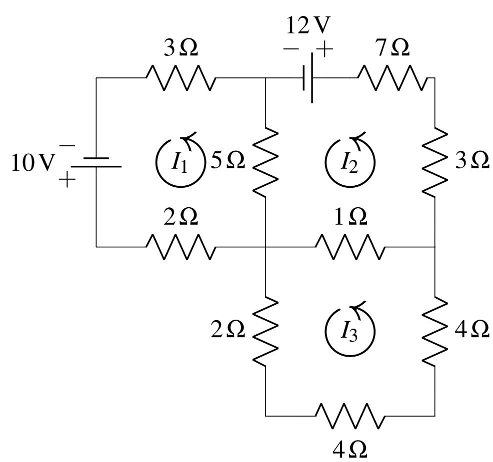


The current in amperes in the four circuits is denoted by I_1, I_2, I_3, and I_4. It is understood that a positive current means a current flowing in the counterclockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction. In the above diagram, the top left circuit should give the equation

$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5.$$

Write equations for each of the other three circuits and then give a solution to the resulting system of equations.

Exercise 1.11.2 *Find I_1, I_2, and I_3, the counterclockwise currents in amperes in the three circuits of the following diagram.*



2. Vectors in $\mathbb{R}^n$

2.1 Points and vectors

Outcomes

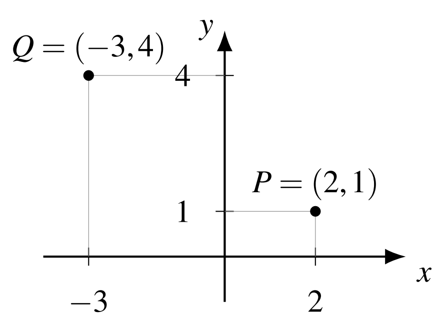
A. Understand the geometric and algebraic meaning of points and vectors in $\mathbb{R}^n$.

B. Find the position vector of a point in $\mathbb{R}^n$.

C. Determine whether two vectors are equal.

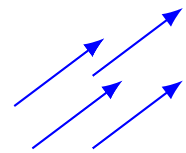
In this section, we define points and vectors in n-dimensional space, and discuss some of their interpretations. We start with a brief review of Cartesian coordinate systems.

Points in n-dimensional space. You are probably already familiar with Cartesian coordinates, which let you describe points in 2- or 3-dimensional space. Consider the familiar coordinate plane, with an x-axis and a y-axis. Any point within this coordinate plane is identified by its x- and y-coordinates. For example, the point P in the following diagram has x-coordinate 2 and y-coordinate 1. We write these coordinates as an ordered pair $P = (2,1)$. Here, "ordered" means that the x-coordinate comes first, and then the y-coordinate, i.e., $(1,2)$ is not the same point as $(2,1)$. Coordinates can be positive, negative, or zero. The special point with coordinates $(0,0)$ is called the **origin** of the coordinate system, and also written as 0.

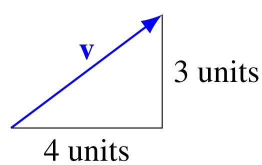


The situation in 3 dimensions is analogous. Here, the coordinate system has three axes, and each point is described by a triple of coordinates, which we can write as (x,y,z). We can extend these ideas beyond $n = 3$. A coordinate system for n-dimensional space has n axes, which we may call $x_1,\ldots,x_n$ (as there are not enough letters in the alphabet to continue after z). A point of n-dimensional space is described by an ordered n-tuple $(x_1,\ldots,x_n)$ of coordinates. For example, $P = (2,1,0,-1)$ is a point in 4-dimensional space which has x_1-coordinate 2, x_2-coordinate 1, and so on. While most people cannot really picture space beyond 3 dimensions, it is easy to imagine tuples of n real numbers. Thus, although we may not be able to "see" the points in higher dimensions, we can still talk about their coordinates.

Vectors in n-dimensional space. Unlike a point, which describes a location in a coordinate system, a vector describes an *offset* or a *distance and direction*. We usually picture a vector as an arrow, starting at one point (called the **tail** of the arrow) and ending at another point (called the **tip** of the arrow).



Two vectors are considered equal if they have the same direction and length. Thus, all four blue arrows in the above image describe exactly the same vector. Mathematically, a vector in 2-dimensional space is described as an offset in the x-direction and an offset in the y-direction. For example, a certain vector $\mathbf{v}$ may be described by the instruction: "move 4 units in the direction parallel to the x-axis, and move 3 units in the direction parallel to the y-axis". This situation is pictured here:



The numbers 4 and 3 are also called the x-component and the y-component of the vector. Notice that a point has "coordinates", but a vector has "components". We write the components of a vector as an ordered column within square brackets: $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Note that components can also be negative; for example, a negative x-component indicates to move left instead of right, and a negative y-component indicates to move down instead of up. The vector with all components equal to 0 is called the **zero vector**, and is written $\mathbf{0}$.

The situation in 3 dimensions is similar. Here, a vector is described by three components, namely, its x-component, y-component, and z-component. The three components are written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The same idea generalizes to n-dimensional vectors when n is greater than 3.

Definition 2.1: Column vectors and $\mathbb{R}^n$

*A n-dimensional **column vector**, often simply called a **vector**, is an ordered list of n real numbers, written as a column within square brackets:*

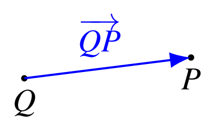
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

*We write $\mathbb{R}^n$ for the set of all n-dimensional column vectors. It is also known as **n-dimensional Euclidean space**.*

Vectors are usually denoted by boldface lower-case letters such as $\mathbf{v}$, $\mathbf{w}$, $\mathbf{a}$, $\mathbf{b}$. Some people write a small arrow above the vector, but we do not do this here.

Points vs. vectors. What is the relationship between points and vectors? Algebraically, they seem to be almost the same thing, because a point (x,y) and a vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ are both an ordered pair of real numbers, written in a slightly different way. On the other hand, geometrically, a point is a location in space, and has neither a length nor direction, whereas a vector has length and direction, but is not fixed at any particular location. Indeed, to convince yourself that despite the similarity in their notation, points and vectors are different kinds of objects, imagine that we moved the origin of the coordinate system to a different location. Then the coordinates of all the points would change, whereas the components of all the vectors would remain the same. To describe the components of a vector, we require axes and a scale, but no origin. To describe the coordinates of a point, we require axes, a scale, and an origin.

Vectors from points. If Q and P are two points in n-dimensional space, we can define a **vector from** Q **to** P. This vector is written $\overrightarrow{QP}$, and is described by the arrow whose tail is at Q and whose tip is at P, as in the following picture:



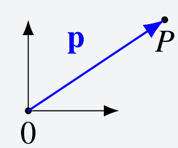
If the point Q has coordinates $(q_1,\ldots,q_n)$ and the point P has coordinates $(p_1,\ldots,p_n)$, then the components of $\overrightarrow{QP}$ are

$$\overrightarrow{QP} = \begin{bmatrix} p_1 - q_1 \\ \vdots \\ p_n - q_n \end{bmatrix}.$$

An important special case of this is the case when the point Q is the origin. The following definition is concerned with that situation.

Definition 2.2: The position vector of a point

*Let P be a point in n-dimensional space. The **position vector** of P is the vector $\mathbf{p} = \overrightarrow{0P}$ whose tail is at the origin and whose tip is at P.*

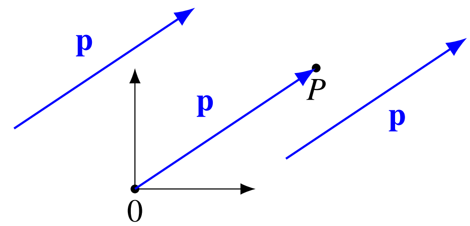


If the point P has coordinates $(p_1,\ldots,p_n)$, then the components of the position vector are

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}.$$

*Thus, the coordinates of a point are the same as the components of its position vector. For this reason, the position vector is also sometimes called the **coordinate vector** of P.*

Points from vectors. Conversely, given any vector $\mathbf{p}$, we may find a point P that has $\mathbf{p}$ as its position vector. To do so geometrically, we first have the move the vector $\mathbf{p}$ around until its tail is at the origin. The point P will then be located at its tip.



Algebraically, if the vector $\mathbf{p}$ has components

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix},$$

then the point P will have coordinates $(p_1, \ldots, p_n)$. This is just the opposite process of Definition 2.2.

So although we went to some lengths to point out that vectors and points are different geometric objects, as soon as an origin of a coordinate system has been fixed, we can always talk about a point by talking about its coordinate vector. We will systematically do so, and eventually the distinction between a point and its coordinate vector will become blurred, so that we will be able to talk about $\mathbb{R}^n$ as "a set of points" or "a set of vectors" interchangeably.

Equality of vectors. Two vectors are equal precisely when all corresponding components are equal. In symbols, if

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$ and ... and $u_n = v_n$.

Notation. In the text, it is often awkward to write column vectors, because they take up so much space. To save space, we sometimes use a superscript "T" to denote a column vector. For example, we write $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$, or sometimes $[1,2,3]^T$, to denote the vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The letter "T" stands for "transpose". To transpose a vector means to turn a row into a column or vice versa.

Exercises

Exercise 2.1.1 *What is an element of* $\mathbb{R}^1$*?*

Exercise 2.1.2 *Given the points* $P = (2,0,-4)$ *and* $Q = (5,-2,1)$*, find* $\overrightarrow{PQ}$ *and* $\overrightarrow{QP}$.

Exercise 2.1.3 *Find* x *and* y *so that* $\mathbf{u} = [5x-3y,4]^T$ *and* $\mathbf{v} = [2x-2y,2y]^T$ *are equal in* $\mathbb{R}^2$.

2.2 Addition

Outcomes

A. *Compute sums and differences of vectors algebraically and geometrically.*

B. *Use the laws of vector addition to prove equalities between vector expressions.*

Addition of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 2.3: Addition of vectors in $\mathbb{R}^n$

For vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$*, the sum* $\mathbf{u}+\mathbf{v} \in \mathbb{R}^n$ *is defined by*

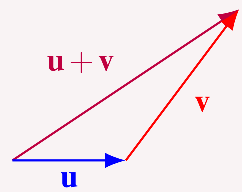
$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}.$$

To add vectors, we simply add corresponding components. Therefore, in order to add vectors, they must be the same size. For example, $[1,2,3]^T + [4,5,6]^T = [1+4,2+5,3+6]^T = [5,7,9]^T$.

The geometric significance of vector addition in $\mathbb{R}^n$ is given in the following proposition.

Proposition 2.4: Geometry of vector addition

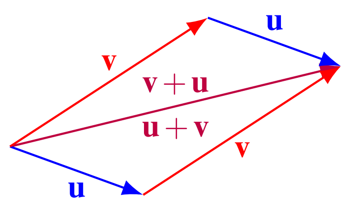
Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in* $\mathbb{R}^n$*. Slide* $\mathbf{v}$ *so that the tail of* $\mathbf{v}$ *is on the tip of* $\mathbf{u}$*. Then draw the arrow which goes from the tail of* $\mathbf{u}$ *to the tip of* $\mathbf{v}$*. This arrow represents the vector* $\mathbf{u}+\mathbf{v}$*.*



Example 2.5: Commutative law

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors. Using the geometry of vector addition, explain why* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$*.*

Solution. In the following diagram, the vectors $\mathbf{u}$ and $\mathbf{v}$ form a parallelogram. Therefore, whether we line up the tail of $\mathbf{u}$ with the tip of $\mathbf{v}$ or vice versa, we obtain the same vector, which is both $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}+\mathbf{u}$.

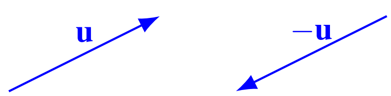


♠

Definition 2.6: Negative

The ***negative*** *of a vector* $\mathbf{u}=\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ *is defined by* $-\mathbf{u}=\begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix}$.

Geometrically, the vector $-\mathbf{u}$ has the same magnitude as $\mathbf{u}$, but the opposite direction.



To define the **subtraction** of two vectors, we simply regard $\mathbf{u}-\mathbf{v}$ as an abbreviation for $\mathbf{u}+(-\mathbf{v})$, exactly as we do with real numbers. Algebraically, this just amounts to componentwise subtraction:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} - \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1-v_1 \\ \vdots \\ u_n-v_n \end{bmatrix}.$$

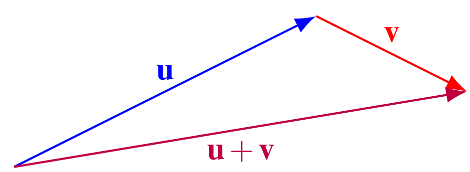
The following example illustrates how to subtract vectors geometrically.

Example 2.7: Graphing vector addition

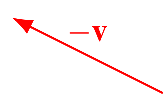
Consider the following picture of vectors $\mathbf{u}$ *and* $\mathbf{v}$*.*

Sketch a picture of $\mathbf{u}+\mathbf{v}$ *and* $\mathbf{u}-\mathbf{v}$*.*

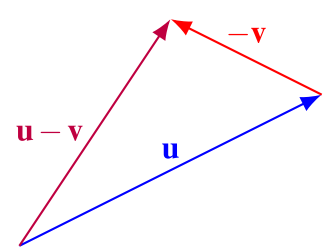
Solution. We will first sketch $\mathbf{u}+\mathbf{v}$. Begin by drawing $\mathbf{u}$ and then at the point of $\mathbf{u}$, place the tail of $\mathbf{v}$ as shown. Then $\mathbf{u}+\mathbf{v}$ is the vector which results from drawing a vector from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$.



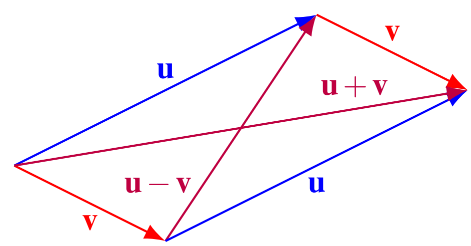
Next consider $\mathbf{u}-\mathbf{v}$. This means $\mathbf{u}+(-\mathbf{v})$. From the above geometric description of vector addition, $-\mathbf{v}$ is the vector which has the same length but which points in the opposite direction to $\mathbf{v}$. Here is a picture of $-\mathbf{v}$



The following picture fully represents $\mathbf{u}-\mathbf{v}$:



Given any two vectors $\mathbf{u}$ and $\mathbf{v}$ one can create a parallelogram with sides these vectors and diagonals $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}-\mathbf{u}$:



♠

Addition of vectors satisfies some important properties which are outlined in the following proposition. Recall that $\mathbf{0}$ is the **zero vector**, the vector from $\mathbb{R}^n$ in which all components are equal to 0.

Proposition 2.8: Properties of vector addition

The following properties hold for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

- *The commutative law of addition*
$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}.$$
- *The associative law of addition*
$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$$
- *The existence of an additive unit*
$$\mathbf{u}+\mathbf{0}=\mathbf{u}.$$
- *The existence of an additive inverse*
$$\mathbf{u}+(-\mathbf{u})=\mathbf{0}.$$

Exercises

Exercise 2.2.1 *Find* $\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]+\left[\begin{array}{c} 1 \\ 5 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 2 \\ -4 \end{array}\right]$.

Exercise 2.2.2 *Use the properties of vector addition from Proposition 2.8 to show the following equalities. Justify every step.*

(a) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=(\mathbf{v}+\mathbf{w})+\mathbf{u}$.

(b) $(\mathbf{u}+\mathbf{0})+(\mathbf{v}+(-\mathbf{u}))=\mathbf{v}$.

2.3 Scalar multiplication

Outcomes

A. *Multiply a scalar by a vector algebraically and geometrically.*

B. *Use the laws of scalar multiplication to prove equalities between vector expressions.*

Scalar multiplication of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 2.9: Scalar multiplication of vectors in $\mathbb{R}^n$

If $k \in \mathbb{R}$ is a scalar and $\mathbf{u} \in \mathbb{R}^n$ is a vector, then their ***scalar multiplication*** *$k\mathbf{u} \in \mathbb{R}^n$ is defined by*

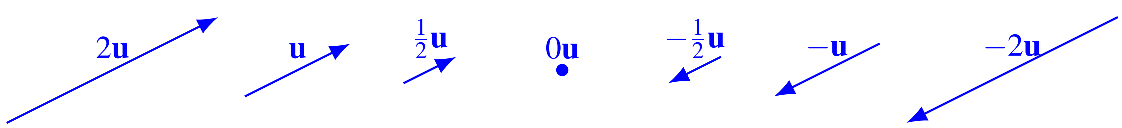
$$k\mathbf{u} = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}.$$

For example $3\,[1,2,3]^T = [3,6,9]^T$ and $-2\,[1,2,3]^T = [-2,-4,-6]^T$.

Example 2.10: Geometric meaning of scalar multiplication

Let $\mathbf{u} = [2,1]^T$, and draw the following vectors to scale: $2\mathbf{u}$, $\mathbf{u}$, $\frac{1}{2}\mathbf{u}$, $0\mathbf{u}$, $-\frac{1}{2}\mathbf{u}$, $-\mathbf{u}$, and $-2\mathbf{u}$. What is the geometric meaning of scalar multiplication?

Solution. Here is a picture of the seven vectors. We draw their tails in different places to make their relationship easier to see.



We see that the vector $k\mathbf{u}$ has the same direction as $\mathbf{u}$ when k is positive, and the opposite direction when k is negative. Further, the length of the vector is scaled by a factor of $|k|$. It increases if $|k| > 1$ and decreases if $|k| < 1$. For example, the vector $2\mathbf{u}$ is exactly twice as long as $\mathbf{u}$. (It is because of this scaling property that scalars are called scalars). ♠

Just as with addition, scalar multiplication of vectors satisfies several important properties. These are outlined in the following proposition.

Proposition 2.11: Properties of scalar multiplication

The following properties hold for vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$ *and* k,ℓ *scalars.*

- *The distributive law over vector addition*

$$k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}.$$

- *The distributive law over scalar addition*

$$(k+\ell)\mathbf{u} = k\mathbf{u}+\ell\mathbf{u}.$$

- *The associative law for scalar multiplication*

$$k(\ell\mathbf{u}) = (k\ell)\mathbf{u}.$$

- *The rule for multiplication by* 1

$$1\mathbf{u} = \mathbf{u}.$$

Proof. We will show the proof of:

$$k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}.$$

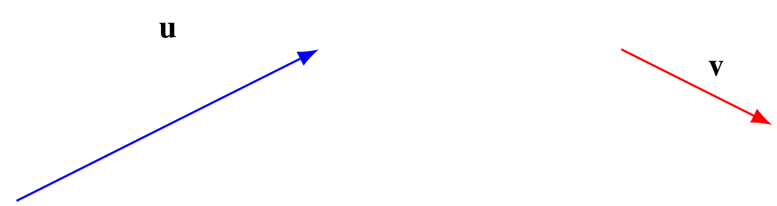
Assume $\mathbf{u} = [u_1,\ldots,u_n]^T$ and $\mathbf{v} = [v_1,\ldots,v_n]^T$. We have:

$$\begin{aligned} k(\mathbf{u}+\mathbf{v}) &= k\,[u_1+v_1,\ldots,u_n+v_n]^T \\ &= [k(u_1+v_1),\ldots,k(u_n+v_n)]^T \\ &= [ku_1+kv_1,\ldots,ku_n+kv_n]^T \\ &= [ku_1,\ldots,ku_n]^T+[kv_1,\ldots,kv_n]^T \\ &= k\mathbf{u}+k\mathbf{v}. \end{aligned}$$

♠

Exercises

Exercise 2.3.1 *Consider the vectors* **u** *and* **v** *drawn below.*



Draw $-\mathbf{u}$, $2\mathbf{v}$, *and* $-\frac{1}{2}\mathbf{v}$.

Exercise 2.3.2 *Find* $-3\begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$.

Exercise 2.3.3 *Use the properties of scalar multiplication from Proposition 2.11 and the properties of vector addition from Proposition 2.8 to prove the following equalities. Justify every step.*

(a) $(k+\ell)(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}+\ell\mathbf{u}+\ell\mathbf{v}$.

(b) $0\mathbf{u} = \mathbf{0}$.

(c) $(-1)\mathbf{u} = -\mathbf{u}$.

(d) $-(k\mathbf{u}) = k(-\mathbf{u}) = (-k)\mathbf{u}$.

2.4 Linear combinations

Outcomes

A. *Compute linear combinations of vectors algebraically and geometrically.*

B. *Determine whether a vector is a linear combination of given vectors.*

C. *Find the coefficients of one vector as a linear combination of other vectors.*

Now that we have studied both vector addition and scalar multiplication, we can combine the two operations. You may remember that when we talked about the solutions to homogeneous systems of equations in Section 1.6, we briefly mentioned that the general solution of a homogeneous system is a linear combination of its basic solutions. We now return to the concept of a linear combination.

Definition 2.12: Linear combination

A vector $\mathbf{v}$ *is said to be a* ***linear combination*** *of the vectors* $\mathbf{u}_1,\ldots,\mathbf{u}_n$ *if there exist scalars* $a_1,\ldots,a_n$ *such that*

$$\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n.$$

The numbers $a_1,\ldots,a_n$ *are called the* ***coefficients*** *of the linear combination.*

Example 2.13: Linear combination

We have

$$3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}.$$

Thus we can say that

$$\mathbf{v} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

For the specific case of $\mathbb{R}^3$, there are three special vectors which we often use. They are given by

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can write any vector $\mathbf{u} = [a_1, a_2, a_3]^T$ as a linear combination of these vectors, namely

$$\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

We will use this notation from time to time.

Example 2.14: Determining if linear combination

Can $\mathbf{v} = [1,3,5]^T$ *be written as a linear combination of* $\mathbf{u}_1 = [2,2,6]^T$, $\mathbf{u}_2 = [1,6,8]^T$, *and* $\mathbf{u}_3 = [3,8,18]^T$*? If yes, find the coefficients.*

Solution. This question can be rephrased as: can we find scalars x, y, z such that

$$x\mathbf{u}_1 + y\mathbf{u}_2 + z\mathbf{u}_3 = \mathbf{v}?$$

Multiplying out produces the system of linear equations

$$\begin{aligned} 2x + y + 3z &= 1 \\ 2x + 6y + 8z &= 3 \\ 6x + 8y + 18z &= 5. \end{aligned}$$

Now we row reduce the corresponding augmented matrix to solve.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{array}\right] \overset{\substack{R_1 \leftarrow R_1 - R_2 \\ R_3 \leftarrow R_3 - 3R_1}}{\simeq} \left[\begin{array}{ccc|c} 0 & -5 & -5 & -2 \\ 2 & 6 & 8 & 3 \\ 0 & 5 & 9 & 2 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_3}{\simeq} \left[\begin{array}{ccc|c} 0 & 0 & 4 & 0 \\ 2 & 6 & 8 & 3 \\ 0 & 5 & 9 & 2 \end{array}\right] \overset{R_2 \leftrightarrow R_1}{\simeq}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 8 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 5 & 9 & 2 \end{array}\right] \overset{\substack{\frac{1}{2}R_1 \\ R_2 \leftrightarrow R_3}}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 4 & \frac{3}{2} \\ 0 & 5 & 9 & 2 \\ 0 & 0 & 4 & 0 \end{array}\right] \overset{\frac{1}{4}R_3}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 4 & \frac{3}{2} \\ 0 & 5 & 9 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{\substack{R_1 \leftarrow R_1 - 4R_3 \\ R_2 \leftarrow R_2 - 9R_3}}{\simeq}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & \frac{3}{2} \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{\frac{1}{5}R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 \leftarrow R_1 - 3R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{10} \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \end{array}\right].$$

We are in the case where we have a unique solution:

$$x = \frac{3}{10}$$
$$y = \frac{2}{5}$$
$$z = 0.$$

This means that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$:

$$\mathbf{v} = \frac{3}{10}\mathbf{u}_1 + \frac{2}{5}\mathbf{u}_2 + 0\mathbf{u}_3.$$

The coefficients are $\frac{3}{10}$, $\frac{2}{5}$, and 0. In fact, $\mathbf{v}$ is also a linear combination of just $\mathbf{u}_1$ and $\mathbf{u}_2$. ♠

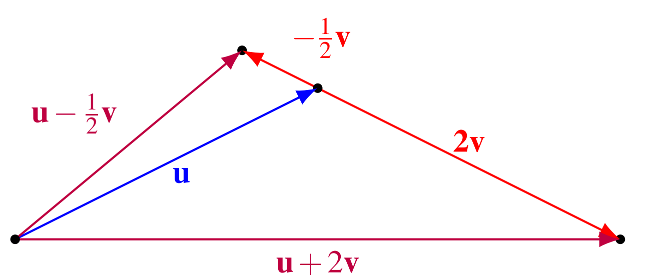
In the following example, we examine the geometric meaning of linear combinations.

Example 2.15: Graphing a linear combination of vectors

Consider the following picture of the vectors $\mathbf{u}$ *and* $\mathbf{v}$

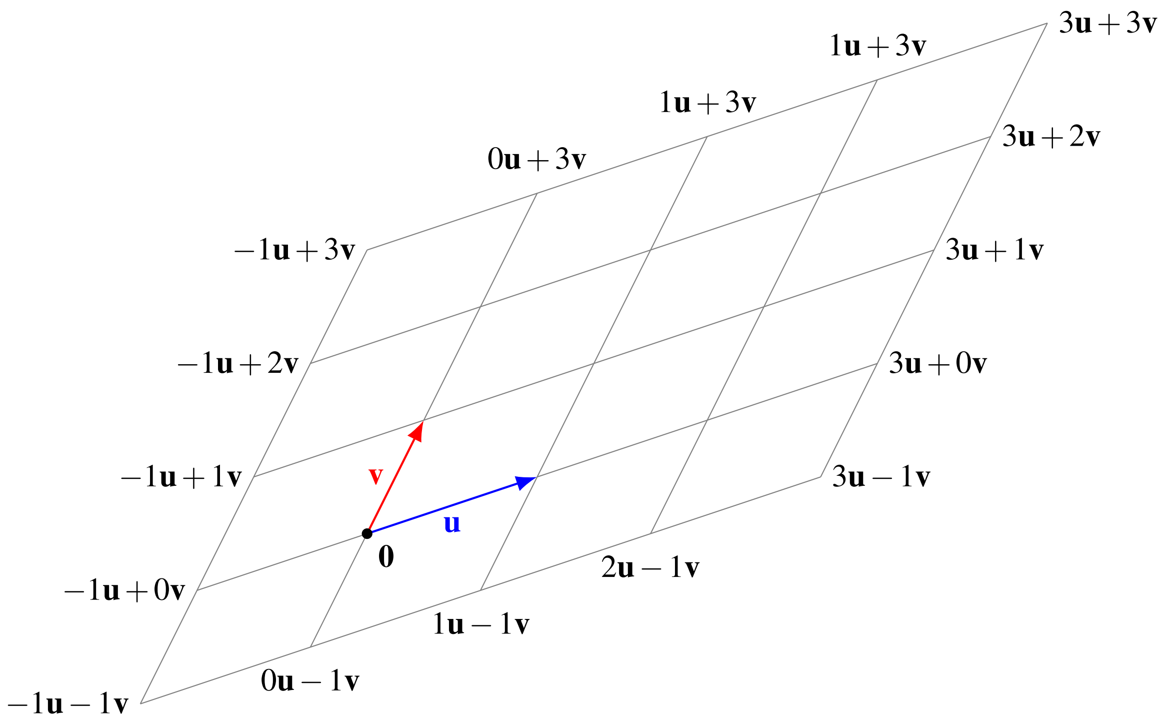
Sketch a picture of $\mathbf{u} + 2\mathbf{v}$ *and* $\mathbf{u} - \frac{1}{2}\mathbf{v}$.

Solution. Both vectors are shown below.



♠

Given any two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, we can create a grid of their linear combinations. The integer ones are pictured below. From this we can see that all vectors in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.



Exercises

Exercise 2.4.1 *Find* $-7\begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6\begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$.

Exercise 2.4.2 *Decide whether*

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

If yes, find the coefficients.

Exercise 2.4.3 *Decide whether*

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \textit{and} \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

If yes, find the coefficients.

2.5 Length of a vector

Outcomes

A. *Compute the distance between points in n-dimensional space.*

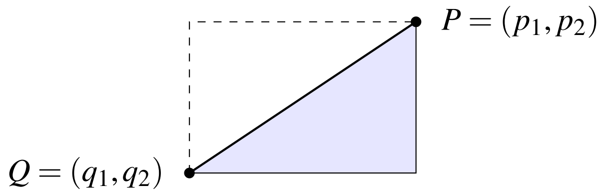
B. *Compute the length of a vector algebraically and geometrically.*

C. *Find vectors that are a given distance from other vectors.*

D. *Use algebraic properties of the length operation to prove equalities.*

E. *Normalize a vector.*

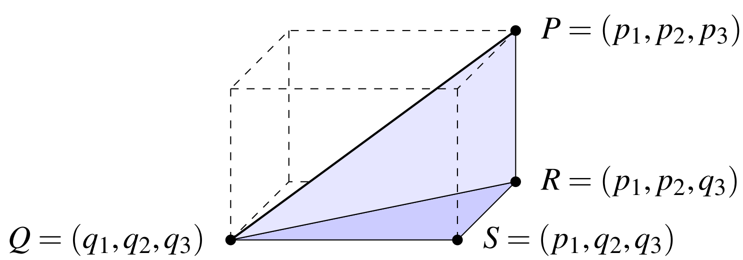
In this section, we explore what is meant by the length of a vector in $\mathbb{R}^n$. We develop this concept by first looking at the distance between two points in $\mathbb{R}^n$. Consider two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in the plane, as in the following picture.



The distance between P and Q is shown in the picture as a solid line, which is the hypotenuse of a right triangle. The lengths of the two other sides of this triangle are $|p_1 - q_1|$ and $|p_2 - q_2|$. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$d(P,Q) = \sqrt{|p_1 - q_1|^2 + |p_2 - q_2|^2} = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$

Now consider two points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ in 3-dimensional space.



We will use the Pythagorean Theorem twice to find the length of the solid line connecting P and Q. First, by the Pythagorean Theorem applied to the right triangle QSR, the length of the line joining R and Q equals

$$d(R,Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$

Second, by the Pythagorean Theorem applied to the triangle QRP, the length of the line joining P and Q equals

$$d(P,Q) = \sqrt{d(R,Q)^2 + (p_3 - q_3)^2} = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$$

This discussion motivates the following definition for the distance between points in $\mathbb{R}^n$.

Definition 2.16: Distance between points

*Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ be two points in $\mathbb{R}^n$. Then the **distance** between these points is defined as*

$$d(P,Q) = \sqrt{(p_1 - q_1)^2 + \ldots + (p_n - q_n)^2}.$$

*This formula is also called the **distance formula**. We may also write $|PQ|$ for the distance between P and Q.*

In the following example, we use Definition 2.16 to find the distance between two points in $\mathbb{R}^4$.

Example 2.17: Distance between points

Find the distance between the points $P = (1,2,-4,6)$ and $Q = (2,3,-1,0)$ in $\mathbb{R}^4$.

Solution. Using the distance formula, we have

$$d(P,Q) = \sqrt{(1-2)^2 + (2-3)^2 + (-4-(-1))^2 + (6-0)^2} = \sqrt{1^2 + 1^2 + 3^2 + 6^2} = \sqrt{47}.$$

♠

Example 2.18: The plane between two points

Describe the points in $\mathbb{R}^3$ that are equally distant from the two points $Q = (1,2,3)$ and $R = (0,1,2)$.

Solution. Let $P = (p_1, p_2, p_3)$ be such a point. Then P is the same distance from Q and R, thus $d(P,Q) = d(P,R)$. By the distance formula, we have

$$\sqrt{(p_1 - 1)^2 + (p_2 - 2)^2 + (p_3 - 3)^2} = \sqrt{(p_1 - 0)^2 + (p_2 - 1)^2 + (p_3 - 2)^2}.$$

Squaring both sides, we obtain

$$(p_1 - 1)^2 + (p_2 - 2)^2 + (p_3 - 3)^2 = p_1^2 + (p_2 - 1)^2 + (p_3 - 2)^2,$$

and so

$$(p_1^2 - 2p_1 + 1) + (p_2^2 - 4p_2 + 4) + (p_3^2 - 6p_3 + 9) = p_1^2 + (p_2^2 - 2p_2 + 1) + (p_3^2 - 4p_3 + 4).$$

Simplifying, this becomes

$$-2p_1 - 4p_2 - 6p_3 + 14 = -2p_2 - 4p_3 + 5,$$

which can finally be written as

$$2p_1 + 2p_2 + 2p_3 = 9. \tag{2.1}$$

Therefore, the points $P = (p_1, p_2, p_3)$ that are the same distance from Q and R form a plane whose equation is given by (2.1). ♠

We can now use our understanding of the distance between two points to define what is meant by the length of a vector.

Definition 2.19: Length of a vector

Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

be a vector in $\mathbb{R}^n$. *Then the* ***length*** *of* $\mathbf{u}$, *written* $\|\mathbf{u}\|$, *is given by*

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}.$$

The length of a vector is also sometimes called its ***magnitude*** *or its* ***norm***.

This definition corresponds to Definition 2.16, if we consider the vector $\mathbf{u}$ to have its tail at the point $0 = (0, \ldots, 0)$ and its tip at the point $U = (u_1, \ldots, u_n)$. Then the length of $\mathbf{u}$ is equal to the distance between 0 and U. In general, $\left\|\overrightarrow{PQ}\right\| = d(P, Q)$.

Reconsider Example 2.17. We could have also computed the distance between P and Q as the length of the vector connecting them. This vector is $\overrightarrow{PQ} = [1, 1, 3, -6]^T$, and its length is

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{1^2 + 1^2 + 3^2 + 6^2} = \sqrt{47}.$$

The following proposition states a few important properties of the length of vectors.

Proposition 2.20: Properties of length

The following hold for all vectors $\mathbf{u}, \mathbf{v}$ *and scalars* k.

- $\|\mathbf{u}\| \geq 0$
- $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.
- $\|k\mathbf{u}\| = |k| \, \|\mathbf{u}\|$.

We conclude this section by giving a special name to vectors of length 1.

Definition 2.21: Unit vector

A vector $\mathbf{u} \in \mathbb{R}^n$ *is called a* ***unit vector*** *if it has length* 1*, that is, if*

$$\|\mathbf{u}\| = 1.$$

Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$. Then there is a unit vector $\mathbf{u}$ that points in the same direction as $\mathbf{v}$, but has length 1. This vector is given by

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

We often use the term **normalize** to refer to this process. When we normalize a vector, we find the corresponding unit vector.

Example 2.22: Normalizing a vector

Consider the vector $\mathbf{v} = [1, -3, 4]^T$. *Find the unit vector* $\mathbf{u}$ *that has the same direction as* $\mathbf{v}$

Solution. We have $\|\mathbf{v}\| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$, and therefore

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{26}}[1, -3, 4]^T = \left[\frac{1}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right]^T.$$

♠

Exercises

Exercise 2.5.1 *Find the distance between the points* $P = (0, 1, 3)$ *and* $Q = (2, -1, 0)$ *in* $\mathbb{R}^3$.

Exercise 2.5.2 *Find the distance between the points* $P = (1, 3, -1, 0)$ *and* $Q = (2, 2, 3, 3)$ *in* $\mathbb{R}^4$.

Exercise 2.5.3 *Describe the points in* $\mathbb{R}^3$ *that are equally distant from the two points* $Q = (1, 1, 1)$ *and* $R = (-1, -1, -1)$.

Exercise 2.5.4 *Describe the points in* $\mathbb{R}^3$ *that have distance* 1 *from the origin.*

Exercise 2.5.5 *Find the length of each of the following vectors.*

$$\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}.$$

Exercise 2.5.6 *Prove the properties of Proposition 2.20.*

Exercise 2.5.7 *Prove that for all vectors* $\mathbf{u} \in \mathbb{R}^n$*, we have* $\|-\mathbf{u}\| = \|\mathbf{u}\|$.

Exercise 2.5.8 *Which of the following are unit vectors?*

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 2.5.9 *Normalize the following vectors.*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 1 \\ -1 \end{bmatrix}.$$

2.6 The dot product

Outcomes

A. *Compute the dot product of vectors geometrically and algebraically.*

B. *Use properties of the dot product, including the Cauchy-Schwarz inequality and the triangle inequality, to prove further equalities and inequalities.*

C. *Determine whether two vectors are orthogonal.*

D. *Compute the scalar and vector projection of one vector onto another.*

E. *Decompose a vector into orthogonal components.*

There are two ways of multiplying vectors that are useful in applications. The first of these is called the **dot product**, and the second is called the **cross product**. We will consider the dot product here, and the cross product in the next section.

2.6.1. Definition and properties

When we take the dot product of two vectors, the result is a scalar. For this reason, the dot product is also called the **scalar product**. Sometimes it is also called the **inner product**. The definition is as follows.

Definition 2.23: Dot product

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ *be two vectors in* $\mathbb{R}^n$*. We define their* ***dot product*** *as*

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

Example 2.24: Compute a dot product

Find $\mathbf{u} \bullet \mathbf{v}$ *for* $\mathbf{u} = [1,2,0,-1]^T$ *and* $\mathbf{v} = [0,1,2,3]^T$.

Solution. We have

$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\ &= 0 + 2 + 0 + -3 \\ &= -1. \end{aligned}$$

♠

The dot product satisfies a number of important properties.

Proposition 2.25: Properties of the dot product

The dot product satisfies the following properties, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are vectors and* k, ℓ *are scalars.*

- $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$.
- $\mathbf{u} \bullet \mathbf{u} \geq 0$, *and* $\mathbf{u} \bullet \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.
- $(k\mathbf{u} + \ell\mathbf{v}) \bullet \mathbf{w} = k(\mathbf{u} \bullet \mathbf{w}) + \ell(\mathbf{v} \bullet \mathbf{w})$.
- $\mathbf{u} \bullet (k\mathbf{v} + \ell\mathbf{w}) = k(\mathbf{u} \bullet \mathbf{v}) + \ell(\mathbf{u} \bullet \mathbf{w})$.
- $\mathbf{u} \bullet \mathbf{u} = \|\mathbf{u}\|^2$.

The proof is left as an exercise. Note that, by the last part of the proposition, we can also use the dot product to find the length of a vector.

Example 2.26: Length of a vector

Use a dot product to find $\|\mathbf{u}\|$*, where*

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \end{bmatrix}.$$

Solution. By the last part of Proposition 2.25, we have $\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}}$. We have $\mathbf{u} \bullet \mathbf{u} = 2^2 + 1^2 + 4^2 + 2^2 = 25$, and therefore $\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{25} = 5$. ♠

2.6.2. The Cauchy-Schwarz and triangle inequalities

The **Cauchy-Schwarz inequality** is a fundamental inequality satisfied by the dot product. It is given in the following proposition.

Proposition 2.27: Cauchy-Schwarz inequality

The dot product satisfies the inequality

$$|\mathbf{u} \bullet \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \tag{2.2}$$

Furthermore equality is obtained if and only if one of $\mathbf{u}$ *or* $\mathbf{v}$ *is a scalar multiple of the other.*

Proof. First note that if $\mathbf{u} = \mathbf{0}$, then both sides of (2.2) are equal to zero, and so the inequality holds in this case. Therefore, we will assume in what follows that $\mathbf{u} \neq \mathbf{0}$. Define a function of $t \in \mathbb{R}$ by

$$f(t) = (t\mathbf{u} + \mathbf{v}) \bullet (t\mathbf{u} + \mathbf{v}).$$

Then by Proposition 2.25, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Also from Proposition 2.25, we have

$$\begin{aligned} f(t) &= t\mathbf{u} \bullet (t\mathbf{u} + \mathbf{v}) + \mathbf{v} \bullet (t\mathbf{u} + \mathbf{v}) \\ &= t^2 \mathbf{u} \bullet \mathbf{u} + t(\mathbf{u} \bullet \mathbf{v}) + t\mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\ &= t^2 \|\mathbf{u}\|^2 + 2t(\mathbf{u} \bullet \mathbf{v}) + \|\mathbf{v}\|^2. \end{aligned}$$

This means the graph of $y = f(t)$ is a parabola which opens upwards and is never negative. It follows that this function has at most one root. From the quadratic formula, we know that a quadratic function $at^2 + bt + c$ has one or zero roots if and only if $b^2 - 4ac \leq 0$. Applying this reasoning to the function $f(t)$, we obtain

$$(2(\mathbf{u} \bullet \mathbf{v}))^2 - 4\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \leq 0,$$

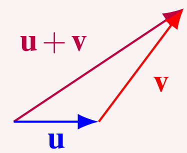
which is equivalent to $|\mathbf{u} \bullet \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$. ♠

An important consequence of the Cauchy-Schwarz inequality is the so-called **triangle inequality**, which states that the length of one side of a triangle is less than or equal the sum of the lengths of the two other sides.

Proposition 2.28: Triangle inequality

For $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, *we have*

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{2.3}$$



Proof. By properties of the dot product and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) \\ &= (\mathbf{u}\bullet\mathbf{u}) + (\mathbf{u}\bullet\mathbf{v}) + (\mathbf{v}\bullet\mathbf{u}) + (\mathbf{v}\bullet\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\bullet\mathbf{v}) + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\bullet\mathbf{v}| + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

Therefore,

$$\|\mathbf{u}+\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.$$

Taking square roots of both sides, we obtain (2.3). ♠

Example 2.29: Triangle inequality

Use the triangle inequality to show

$$\|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\|$$

holds for all vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$.

Solution. We have

$$\|\mathbf{u}\| = \|(\mathbf{u}-\mathbf{v})+\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|,$$

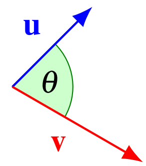
where we have used the triangle inequality in the last step. Note that this is an inequality between real numbers. Bringing $\|\mathbf{v}\|$ to the other side of the equation, we have

$$\|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\|.$$

♠

2.6.3. The geometric significance of the dot product

The **included angle** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is the angle θ between the vectors such that $0 \leq \theta \leq \pi$.



The dot product can be used to determine the included angle between two vectors.

Proposition 2.30: The dot product and the included angle

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in* $\mathbb{R}^n$, *and let* θ *be the included angle. Then the following equation holds.*

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

In words, the dot product of two vectors equals the product of the magnitude (or length) of the two vectors multiplied by the cosine of the included angle. Note that this gives a geometric description of the dot product that does not depend explicitly on the coordinates of the vectors.

Example 2.31: Find the angle between two vectors

Find the angle between the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Solution. By Proposition 2.30,

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

Hence,

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

First, we compute $\mathbf{u} \bullet \mathbf{v} = (2)(0) + (2)(3) = 6$. Then,

$$\|\mathbf{u}\| = \sqrt{2^2 + 2^2} = \sqrt{8},$$
$$\|\mathbf{v}\| = \sqrt{0^2 + 3^2} = 3.$$

Therefore, we have

$$\cos\theta = \frac{6}{3\sqrt{8}} = \frac{1}{\sqrt{2}}.$$

Taking the inverse cosine of both sides of the equation, we find that $\theta = \frac{\pi}{4}$ radians, or 45 degrees. ♠

Example 2.32: Computing a dot product from an angle

Let $\mathbf{u}, \mathbf{v}$ *be vectors with* $\|\mathbf{u}\| = 3$ *and* $\|\mathbf{v}\| = 4$. *Suppose the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *is* $\pi/3$. *Find* $\mathbf{u} \bullet \mathbf{v}$.

Solution. From Proposition 2.30, we have

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta = 3 \cdot 4 \cdot \cos\left(\frac{\pi}{3}\right) = 3 \cdot 4 \cdot \frac{1}{2} = 6.$$

♠

2.6.4. Orthogonal vectors

Two non-zero vectors are said to be **orthogonal**, sometimes also called **perpendicular**, if the included angle is $\pi/2$ radians (90°). By convention, we also say that the zero vector is orthogonal to all vectors.

Proposition 2.33: Orthogonal vectors

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$. *Then* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if*

$$\mathbf{u} \bullet \mathbf{v} = 0.$$

We also write $\mathbf{u} \perp \mathbf{v}$ *to indicate that* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal.*

Proof. If $\mathbf{u}$ or $\mathbf{v}$ is zero, the vectors are orthogonal by definition, and the dot product is 0 in that case, so the proposition holds. Now assume $\mathbf{u}$ and $\mathbf{v}$ are both non-zero. Then by Proposition 2.30, we have $\mathbf{u} \bullet \mathbf{v} = 0$ if and only if $\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$ if and only if $\cos\theta = 0$. Recall that the included angle is between 0 and π. Therefore, $\cos\theta = 0$ if and only if $\theta = \pi/2$. ♠

Example 2.34: Determine whether two vectors are orthogonal

Determine whether the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

are orthogonal.

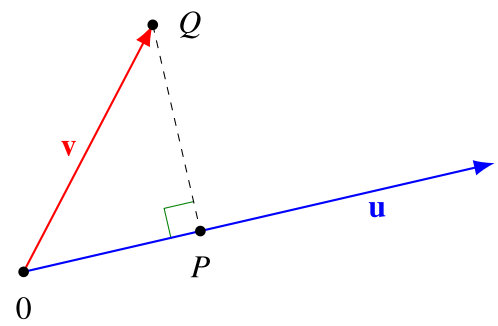
Solution. In order to determine if these two vectors are orthogonal, we compute the dot product. We have

$$\mathbf{u} \bullet \mathbf{v} = (2)(1) + (1)(3) + (-1)(5) = 0,$$

and therefore, by Proposition 2.33, the two vectors are orthogonal. ♠

2.6.5. Projections

It is sometimes important to find the component of a vector in a particular direction. Consider the following picture:



Here, $\mathbf{u}$ is a non-zero vector specifying a *direction*, and $\mathbf{v}$ is any vector. We have given the label Q to the tip of $\mathbf{v}$. The point P lies at the place along $\mathbf{u}$ that is closest to Q, or equivalently, such that $(0,P,Q)$ forms a right triangle. The distance from 0 to P (measured positively in the direction of $\mathbf{u}$) is called the **component** of $\mathbf{v}$ in the direction of $\mathbf{u}$, and is denoted $\operatorname{comp}_{\mathbf{u}}(\mathbf{v})$. The vector $\overrightarrow{0P}$ is called the **projection** of $\mathbf{v}$ onto $\mathbf{u}$, and is denoted $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$. We wish to find formulas for these quantities.

Let θ be the included angle between $\mathbf{v}$ and $\mathbf{u}$. From trigonometry, considering the right triangle $(0,P,Q)$, we know that

$$\cos\theta = \frac{|0P|}{|0Q|} = \frac{|0P|}{\|\mathbf{v}\|},$$

and therefore

$$|0P| = \|\mathbf{v}\|\cos\theta. \tag{2.4}$$

On the other hand, from Proposition 2.30, we have

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta,$$

and therefore

$$\|\mathbf{v}\|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}. \tag{2.5}$$

Putting equations (2.4) and (2.5) together, we obtain the desired formula for the component of $\mathbf{v}$ in the direction of $\mathbf{u}$:

$$\operatorname{comp}_{\mathbf{u}}(\mathbf{v}) = |0P| = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}. \tag{2.6}$$

Note that it is possible for this quantity to be negative; this happens when the angle between $\mathbf{v}$ and $\mathbf{u}$ is obtuse. In this case, $\mathbf{v}$ will have a negative component along $\mathbf{u}$.

The vector $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \overrightarrow{0P}$ can now be computed by re-scaling $\mathbf{u}$ to the correct length. Specifically, we first normalize $\mathbf{u}$ by dividing it by its own length, and then multiply by $|0P|$. In formulas:

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \overrightarrow{0P} = \frac{|0P|}{\|\mathbf{u}\|}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}. \tag{2.7}$$

The following definition summarizes what we have just found.

Definition 2.35: Vector projection

Let $\mathbf{u}$ *be a non-zero vector and* $\mathbf{v}$ *any vector. Then the* ***component of*** $\mathbf{v}$ ***in the direction of*** $\mathbf{u}$ *is defined to be the scalar*

$$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|}.$$

The ***projection of*** $\mathbf{v}$ ***onto*** $\mathbf{u}$ *is defined to be the vector*

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{u} \bullet \mathbf{u}}\mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

These two operations are also called the ***scalar projection*** *and* ***vector projection****, respectively.*

Example 2.36: Find the projection of one vector onto another

Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ *if*

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Solution. We can use the formula provided in Definition 2.35 to find $\text{proj}_{\mathbf{u}}(\mathbf{v})$. First, compute $\mathbf{u} \bullet \mathbf{v}$. This is given by

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (2)(1) + (3)(-2) + (-4)(1) = -8.$$

Similarly, $\mathbf{u} \bullet \mathbf{u}$ is given by

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = 2^2 + 3^2 + (-4)^2 = 29.$$

Therefore, the projection is equal to

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = -\frac{8}{29}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -16/29 \\ -24/29 \\ 32/29 \end{bmatrix}.$$

♠

An important application of projections is that every vector $\mathbf{v}$ can be uniquely written as a sum of two orthogonal vectors, one of which is a scalar multiple of some given non-zero vector $\mathbf{u}$, and the other of which is orthogonal to $\mathbf{u}$.

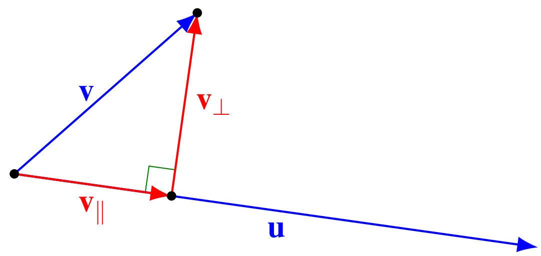
Theorem 2.37: Decomposition into components

Let $\mathbf{u}$ *be a non-zero vector, and let* $\mathbf{v}$ *be any vector. Then there exist unique vectors* $\mathbf{v}_{||}$ *and* $\mathbf{v}_{\perp}$ *such that*

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp} \tag{2.8}$$

where $\mathbf{v}_{||}$ *is a scalar multiple of* $\mathbf{u}$, *and* $\mathbf{v}_{\perp}$ *is orthogonal to* $\mathbf{u}$.

Proof.



To show that such a decomposition exists, let

$$\mathbf{v}_{||} = \text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u},$$

and define $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{||}$. By definition, (2.8) is satisfied, and $\mathbf{v}_{||}$ is a scalar multiple of $\mathbf{u}$. We must show that $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$. For this, we verify that their dot product equals zero:

$$\begin{aligned} \mathbf{u}\bullet\mathbf{v}_{\perp} &= \mathbf{u}\bullet(\mathbf{v}-\mathbf{v}_{||}) \\ &= \mathbf{u}\bullet\mathbf{v} - \mathbf{u}\bullet\mathbf{v}_{||} \\ &= \mathbf{u}\bullet\mathbf{v} - \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u}\bullet\mathbf{u} \\ &= \mathbf{u}\bullet\mathbf{v} - \mathbf{u}\bullet\mathbf{v} \\ &= 0. \end{aligned}$$

To show uniqueness, suppose that (2.8) holds and $\mathbf{v}_{||} = k\mathbf{u}$. Taking the dot product of both sides of (2.8) with $\mathbf{u}$ and using $\mathbf{u}\bullet\mathbf{v}_{\perp} = 0$, this yields

$$\begin{aligned} \mathbf{u}\bullet\mathbf{v} &= \mathbf{u}\bullet(\mathbf{v}_{||}+\mathbf{v}_{\perp}) \\ &= \mathbf{u}\bullet k\mathbf{u} + \mathbf{u}\bullet\mathbf{v}_{\perp} \\ &= k\,\|\mathbf{u}\|^2, \end{aligned}$$

which implies $k = \mathbf{u}\bullet\mathbf{v}/\|\mathbf{u}\|^2$. Thus there can be no more than one such vector $\mathbf{v}_{||}$. Since $\mathbf{v}_{\perp}$ must equal $\mathbf{v}-\mathbf{v}_{||}$, it follows that there can be no more than one choice for both $\mathbf{v}_{||}$ and $\mathbf{v}_{\perp}$, proving their uniqueness. ♠

Example 2.38: Decomposition into components

Decompose the vector $\mathbf{v}$ *into* $\mathbf{v} = \mathbf{a} + \mathbf{b}$ *where* $\mathbf{a}$ *is parallel to* $\mathbf{u}$ *and* $\mathbf{b}$ *is orthogonal to* $\mathbf{u}$.

$$\mathbf{v} = \begin{bmatrix} -5 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Solution. We can let $\mathbf{a} = \mathbf{v}_{\|}$ and $\mathbf{b} = \mathbf{v}_{\perp}$ as in the proof of Theorem 2.37. Then

$$\mathbf{a} = \mathbf{v}_{\|} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{u} \bullet \mathbf{u}}\mathbf{u} = \frac{11}{9}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 11/9 \\ 22/9 \\ -22/9 \end{bmatrix}$$

and

$$\mathbf{b} = \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|} = \begin{bmatrix} -5 \\ 3 \\ -5 \end{bmatrix} - \begin{bmatrix} 11/9 \\ 22/9 \\ -22/9 \end{bmatrix} = \begin{bmatrix} -56/9 \\ 5/9 \\ -23/9 \end{bmatrix}.$$

♠

Exercises

Exercise 2.6.1 *Find* $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

Exercise 2.6.2 *Let* $\mathbf{a}, \mathbf{b}$ *be vectors. Show that* $(\mathbf{a} \bullet \mathbf{b}) = \frac{1}{4}(\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2)$.

Exercise 2.6.3 *Using the properties of the dot product, prove the parallelogram identity:*

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$$

Exercise 2.6.4 *Find* $\cos\theta$ *where* θ *is the angle between the vectors*

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Exercise 2.6.5 *Find* $\cos\theta$ *where* θ *is the angle between the vectors*

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

Exercise 2.6.6 *Use the formula given in Proposition 2.30 to verify the Cauchy Schwarz inequality and to show that equality occurs if and only if one of the vectors is a scalar multiple of the other.*

Exercise 2.6.7 *Show that the triangle with vertices* $A = (2,0,-3)$, $B = (5,-2,1)$ *and* $C = (7,5,3)$ *is a right triangle.*

Exercise 2.6.8 *Find* $\text{proj}_{\mathbf{v}}(\mathbf{w})$ *where* $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Exercise 2.6.9 *Find* $\text{proj}_{\mathbf{v}}(\mathbf{w})$ *where* $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Exercise 2.6.10 *Find* $\text{proj}_{\mathbf{v}}(\mathbf{w})$ *where* $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

Exercise 2.6.11 *Find* $\text{comp}_{\mathbf{v}}(\mathbf{w})$ *where* $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$.

Exercise 2.6.12 *Find* $\text{comp}_{\mathbf{v}}(\mathbf{w})$ *where* $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Exercise 2.6.13 *Does it make sense to speak of* $\text{proj}_{\mathbf{0}}(\mathbf{w})$*?*

Exercise 2.6.14 *Decompose the vector* $\mathbf{v}$ *into* $\mathbf{v} = \mathbf{a} + \mathbf{b}$ *where* $\mathbf{a}$ *is parallel to* $\mathbf{u}$ *and* $\mathbf{b}$ *is orthogonal to* $\mathbf{u}$.

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Exercise 2.6.15 *Prove the Cauchy Schwarz inequality in* $\mathbb{R}^n$ *as follows. For* $\mathbf{u}, \mathbf{v}$ *vectors, consider*

$$(\mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u})) \bullet (\mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u})) \geq 0$$

Simplify using the axioms of the dot product and then put in the formula for the projection. Notice that this expression equals 0 *and you get equality in the Cauchy Schwarz inequality if and only if* $\mathbf{u} = \text{proj}_{\mathbf{v}}(\mathbf{u})$. *What is the geometric meaning of* $\mathbf{u} = \text{proj}_{\mathbf{v}}(\mathbf{u})$*?*

Exercise 2.6.16 *Let* $\mathbf{v}, \mathbf{w}\ \mathbf{u}$ *be vectors. Show that* $(\mathbf{w} + \mathbf{u})_{\perp} = \mathbf{w}_{\perp} + \mathbf{u}_{\perp}$, *where* $\mathbf{w}_{\perp} = \mathbf{w} - \text{proj}_{\mathbf{v}}(\mathbf{w})$.

Exercise 2.6.17 *Show that*

$$\mathbf{u} \bullet (\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) = 0$$

and conclude every vector in $\mathbb{R}^n$ *can be written as the sum of two vectors, one which is orthogonal and one which is parallel to the given vector.*

2.7 The cross product

Outcomes

A. *Compute the cross product of vectors algebraically and geometrically.*

B. *Compute the box product of three vectors in $\mathbb{R}^3$.*

C. *Determine whether a system of three vectors in $\mathbb{R}^3$ is right-handed, algebraically and geometrically.*

D. *Find the areas of parallelograms and triangles in $\mathbb{R}^3$.*

E. *Find the volume of a parallelepiped determined by three vectors in $\mathbb{R}^3$.*

F. *Use properties of the cross product and the dot product to prove algebraic equalities.*

Unlike the dot product, the cross product is only defined in $\mathbb{R}^3$, i.e., only in 3-dimensional space. The cross product of two vectors is a vector.

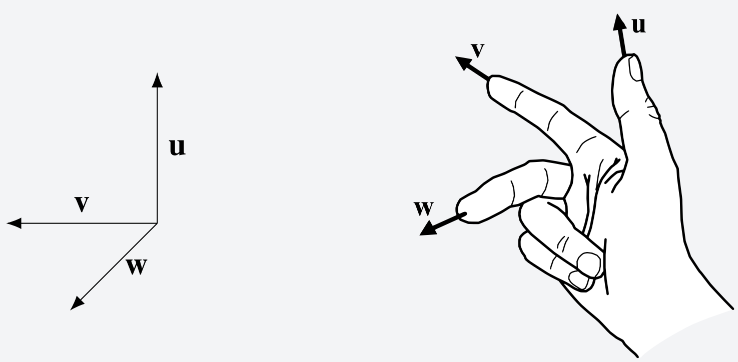
We will first discuss the geometric meaning of the cross product, and then give an algebraic description. Both descriptions are equally important: the geometric description is essential for applications in physics and geometry, whereas the algebraic description is necessary for computing.

2.7.1. Right-handed systems of vectors

We begin with a discussion of right-handed systems of vectors in 3-dimensional space.

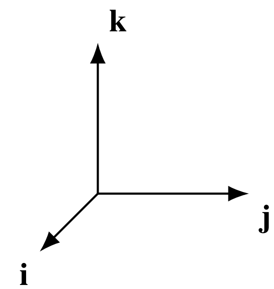
Definition 2.39: Right-handed system of vectors

Three vectors, $\mathbf{u},\mathbf{v},\mathbf{w}$ form a right-handed system if when you extend the thumb of your right hand in the direction of $\mathbf{u}$ and your index finger in the direction of $\mathbf{v}$, your relaxed middle finger points roughly in the direction of $\mathbf{w}$.



You should consider how a right-handed system would differ from a left-handed system. Try using your left hand and you will see that the vector **w** would need to point in the opposite direction.

Recall the special vectors $\mathbf{i} = [1,0,0]^T$, $\mathbf{j} = [0,1,0]^T$, and $\mathbf{k} = [0,0,1]^T$ we saw in Section 2.4. We always assume that our coordinate system is drawn in such a way that the vectors **i**, **j**, **k** form a right-handed system. Thus, if the thumb of your right hand points along the x-axis and your index finger points along the y-axis, your middle finger should point along the z-axis.



When all three vectors lie in a plane, then we say that the vectors are **coplanar**. In this case, the system is neither right-handed nor left-handed.

2.7.2. Geometric description of the cross product

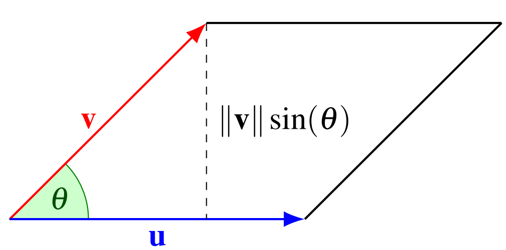
The following is the geometric description of the cross product. Recall that the dot product of two vectors results in a scalar. In contrast, the cross product results in a vector, as the cross product gives a direction as well as a magnitude.

Definition 2.40: Geometric definition of cross product

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in* $\mathbb{R}^3$. *Their* ***cross product****, written* $\mathbf{u} \times \mathbf{v}$, *is the vector defined by the following three rules.*

1. *Its length is* $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$, *where* θ *is the included angle between* $\mathbf{u}$ *and* $\mathbf{v}$.
2. *It is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. *The vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{u} \times \mathbf{v}$, *in that order, form a right-handed system.*

We note that the length of the cross product, $\|\mathbf{u} \times \mathbf{v}\|$, given by the formula $\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$, is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, as shown in the following picture.



2.7.3. Algebraic definition of the cross product

From its geometric description, we can prove that the cross product satisfies the following properties.

Proposition 2.41: Properties of the cross product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then the following hold.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
2. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
3. $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$.
4. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
5. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$.

Proof. Formula 1. follows immediately from the definition. The vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have the same magnitude, $\|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$, and an application of the right hand rule shows they have opposite direction.

Formula 2. is proven as follows. If k is a non-negative scalar, the direction of $(k\mathbf{u}) \times \mathbf{v}$ is the same as the direction of $\mathbf{u} \times \mathbf{v}, k(\mathbf{u} \times \mathbf{v})$ and $\mathbf{u} \times (k\mathbf{v})$. The magnitude is k times the magnitude of $\mathbf{u} \times \mathbf{v}$ which is the same as the magnitude of $k(\mathbf{u} \times \mathbf{v})$ and $\mathbf{u} \times (k\mathbf{v})$. Using this yields equality in 2. In the case where $k < 0$, everything works the same way except the vectors are all pointing in the opposite direction and you must multiply by $|k|$ when comparing their magnitudes.

The distributive laws, 3. and 4., are harder to establish. For now, we will content ourselves with noticing that if we know that 3. is true, 4. follows. Namely, assuming 3., and using 1., we have

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \times \mathbf{u} &= -\mathbf{u} \times (\mathbf{v} + \mathbf{w}) \\ &= -(\mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}) \\ &= \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}.\end{aligned}$$

♠

In turn, we can use the properties from Proposition 2.41 to get an algebraic description of the cross product. We begin by determining the cross products of the special vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. They are as follows:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{i} = -\mathbf{k}, & \mathbf{i} \times \mathbf{i} = \mathbf{0}, \\ \mathbf{k} \times \mathbf{i} = \mathbf{j}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j}, & \mathbf{j} \times \mathbf{j} = \mathbf{0}, \\ \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i}, & \mathbf{k} \times \mathbf{k} = \mathbf{0}. \end{array}$$

With this information and the laws of Proposition 2.41, we can compute the cross product of any two vectors from their coordinates. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then we have:

$$\mathbf{u} \times \mathbf{v} \quad = \quad (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$\begin{aligned}
&= \quad u_1v_1(\mathbf{i}\times\mathbf{i}) + u_1v_2(\mathbf{i}\times\mathbf{j}) + u_1v_3(\mathbf{i}\times\mathbf{k}) \\
&\quad + u_2v_1(\mathbf{j}\times\mathbf{i}) + u_2v_2(\mathbf{j}\times\mathbf{j}) + u_2v_3(\mathbf{j}\times\mathbf{k}) \\
&\quad + u_3v_1(\mathbf{k}\times\mathbf{i}) + u_3v_2(\mathbf{k}\times\mathbf{j}) + u_3v_3(\mathbf{k}\times\mathbf{k}) \\
&= \quad u_1v_1\mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} \\
&\quad - u_2v_1\mathbf{k} + u_2v_2\mathbf{0} + u_2v_3\mathbf{i} \\
&\quad + u_3v_1\mathbf{j} - u_3v_2\mathbf{i} + u_3v_3\mathbf{0} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

The resulting formula for the cross product is summarized in the following Proposition.

Proposition 2.42: Coordinate description of cross product

The cross product can be computed as follows:

$$\left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right] \times \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}\right] = \left[\begin{array}{c} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{array}\right].$$

We will now look at an example of how to compute a cross product.

Example 2.43: Find a cross product

Find $\mathbf{u}\times\mathbf{v}$ *for the vectors*

$$\mathbf{u} = \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right] \quad \textit{and} \quad \mathbf{v} = \left[\begin{array}{r} 3 \\ -2 \\ 1 \end{array}\right].$$

Solution. Using Proposition 2.42, we compute

$$\left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right] \times \left[\begin{array}{r} 3 \\ -2 \\ 1 \end{array}\right] = \left[\begin{array}{c} (-1)(1) - (2)(-2) \\ (2)(3) - (1)(1) \\ (1)(-2) - (-1)(3) \end{array}\right] = \left[\begin{array}{c} 3 \\ 5 \\ 1 \end{array}\right].$$

♠

We use this concept in the following examples.

Example 2.44: Area of a parallelogram

Find the area of the parallelogram determined by the following vectors $\mathbf{u}$ *and* $\mathbf{v}$*:*

$$\mathbf{u} = \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right], \quad \mathbf{v} = \left[\begin{array}{r} 3 \\ -2 \\ 1 \end{array}\right].$$

Solution. Notice that these vectors are the same as the ones given in Example 2.43. Recall from the geometric description of the cross product that the area of the parallelogram is the magnitude of $\mathbf{u}\times\mathbf{v}$.

From Example 2.43, $\mathbf{u}\times\mathbf{v} = [3,5,1]^T$. Thus the area of the parallelogram is

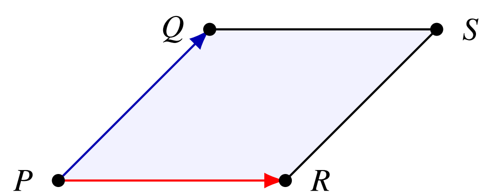
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{3^2+5^2+1^2} = \sqrt{35}.$$

♠

Example 2.45: Area of a parallelogram

Find the area of the parallelogram with vertices $(1,0,1)$, $(2,2,3)$, $(-1,1,3)$, *and* $(0,3,5)$.

Solution. Let $P=(1,0,1)$, $Q=(2,2,3)$, $R=(-1,1,3)$, and $S=(0,3,5)$.



First, we check that this really is a parallelogram. We have to have $\overrightarrow{PQ} = \overrightarrow{RS}$. Indeed, this is the case, as $\overrightarrow{PQ} = [2-1, 2-0, 3-1]^T = [1,2,2]^T$ and $\overrightarrow{RS} = [0-(-1), 3-1, 5-3]^T = [1,2,2]^T$. We also compute $\overrightarrow{PR} = \overrightarrow{QS} = [-2,1,2]^T$. The area of the parallelogram is

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \left\|\begin{bmatrix}1\\2\\2\end{bmatrix}\times\begin{bmatrix}-2\\1\\2\end{bmatrix}\right\| = \left\|\begin{bmatrix}2\\-6\\5\end{bmatrix}\right\| = \sqrt{2^2+(-6)^2+5^2} = \sqrt{65}.$$

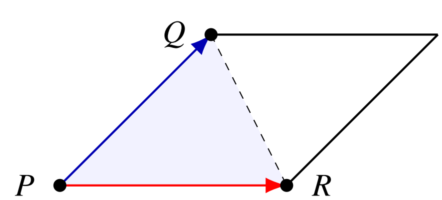
♠

We can also use this concept to find the area of a triangle, as in the following example.

Example 2.46: Area of triangle

Find the area of the triangle determined by the points $(1,2,3)$, $(0,2,5)$, *and* $(5,1,2)$.

Solution. Let $P=(1,2,3)$, $Q=(0,2,5)$, and $R=(5,1,2)$. The area of the triangle is exactly half of the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.



We have $\overrightarrow{PQ} = [-1,0,2]^T$ and $\overrightarrow{PR} = [4,-1,-1]^T$. The area of the parallelogram is the magnitude of the cross product:

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \left\|\begin{bmatrix}-1\\0\\2\end{bmatrix}\times\begin{bmatrix}4\\-1\\-1\end{bmatrix}\right\| = \left\|\begin{bmatrix}2\\7\\1\end{bmatrix}\right\| = \sqrt{2^2+7^2+1^2} = \sqrt{54}.$$

Hence the area of the triangle is $\frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6}$. ♠

In general, the area of the triangle determined by three points P, Q, R in $\mathbb{R}^3$ is given by

$$\frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|.$$

2.7.4. The box product

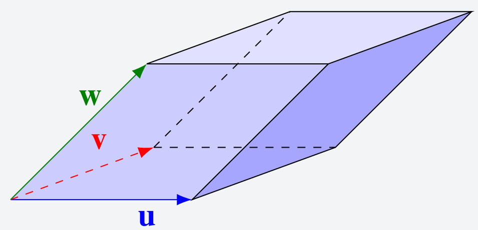
In this section, we explore another application of the cross product. Recall that we can use the cross product to find the the area of a parallelogram. As we will now show, we can also use the cross product together with the dot product to find the volume of a parallelepiped. We begin with a definition.

Definition 2.47: Parallelepiped

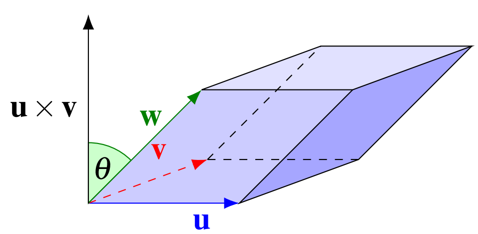
The parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *consists of the set of points of the form*

$$r\mathbf{u} + s\mathbf{v} + t\mathbf{w},$$

where r, s, t *are real numbers between* 0 *and* 1, *inclusive. The parallelepiped is a* 3*-dimensional body bounded by parallelograms as shown in this picture.*



Notice that the base of the parallelepiped is the parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$. Therefore, its area is equal to $\|\mathbf{u} \times \mathbf{v}\|$. The height of the parallelepiped is $\|\mathbf{w}\| \cos\theta$, where θ is the angle between $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$, as shown in this picture.



The volume of this parallelepiped is the area of the base times the height which is just

$$\|\mathbf{u} \times \mathbf{v}\| \, \|\mathbf{w}\| \cos\theta = (\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}.$$

This expression is known as the **box product** and is sometimes written as $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$.

Consider what happens if you interchange $\mathbf{v}$ with $\mathbf{w}$ or $\mathbf{u}$ with $\mathbf{w}$. Geometrically, we can see that this merely introduces a minus sign. We find that the box product of three vectors equals the volume of the parallelepiped determined by the three vectors if the three vectors form a right-handed system, and the negative of the volume if the vectors form a left-handed system. We summarize this in the following proposition:

Proposition 2.48: The box product

Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be three vectors in $\mathbb{R}^3$ that define a parallelepiped. The box product $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}$ is equal to:

- *The volume of the parallelepiped, if $\mathbf{u},\mathbf{v},\mathbf{w}$ form a right-handed system.*
- *The negative of the volume of the parallelepiped, if $\mathbf{u},\mathbf{v},\mathbf{w}$ form a left-handed system.*

In any case, the volume of the parallelepiped can be computed as the absolute value of the box product, given by $|(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}|$.

Example 2.49: Volume of a parallelepiped

Find the volume of the parallelepiped determined by the vectors

$$\mathbf{u}=\left[\begin{array}{r}1\\2\\-5\end{array}\right],\quad \mathbf{v}=\left[\begin{array}{r}1\\3\\-6\end{array}\right],\quad \mathbf{w}=\left[\begin{array}{r}3\\2\\3\end{array}\right].$$

Solution. According to the above discussion, we can take the cross product of any two of these vectors, and then the dot product with the third vector. The result will be either plus or minus the desired volume. Therefore we can obtain the volume by taking the absolute value.

We first compute the cross product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}\times\mathbf{v}=\left[\begin{array}{r}1\\2\\-5\end{array}\right]\times\left[\begin{array}{r}1\\3\\-6\end{array}\right]=\left[\begin{array}{c}(2)(-6)-(-5)(3)\\(-5)(1)-(1)(-6)\\(1)(3)-(2)(1)\end{array}\right]=\left[\begin{array}{r}3\\1\\1\end{array}\right]$$

Then we take the dot product of this vector with $\mathbf{w}$:

$$(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}=\left[\begin{array}{r}3\\1\\1\end{array}\right]\bullet\left[\begin{array}{r}3\\2\\3\end{array}\right]=9+2+3=14.$$

Thus, the volume of the parallelepiped is 14 cubic units. ♠

The following is a consequence of Proposition 2.48:

Corollary 2.50: Right- and left-handed systems of vectors

The box product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ *is:*

- *Positive, if* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *form a right-handed system.*
- *Negative, if* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *form a left-handed system.*
- *Zero, if* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are coplanar.*

Example 2.51: Right- and left-handed systems of vectors

Which of the following systems of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is right-handed? Which one is left-handed? Which one is coplanar?*

(a) $\mathbf{u} = [1,2,0]^T$, $\mathbf{v} = [0,0,1]^T$, $\mathbf{w} = [1,-1,1]^T$.

(b) $\mathbf{u} = [1,1,1]^T$, $\mathbf{v} = [1,2,3]^T$, $\mathbf{w} = [0,1,1]^T$.

(c) $\mathbf{u} = [0,1,2]^T$, $\mathbf{v} = [1,2,2]^T$, $\mathbf{w} = [1,1,0]^T$.

Solution.

(a) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = [2,-1,0]^T \cdot [1,-1,1]^T = 3$, so the box product is positive and the system of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed.

(b) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = [1,-2,1]^T \cdot [0,1,1]^T = -1$, so the box product is negative and the system is left-handed.

(c) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = [-2,2,-1]^T \cdot [1,1,0]^T = 0$, so the box product is zero and the vectors are coplanar.

♠

We finish this section with a law involving the dot product and the cross product. It represents a fundamental observation that comes directly from the geometric definition of the box product.

Proposition 2.52: Box product law

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors. Then* $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Proof. This follows from observing that both $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ compute the same box product, i.e., they either both give the volume of the parallelepiped or they both give the negative of the volume.

Alternatively, we can calculate each product explicitly:

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3, \\
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3.
\end{aligned}$$

In Chapter 7, you will learn that these expressions are a special case of a determinant. ♠

Exercises

Exercise 2.7.1 *Which of the following systems of vectors are right-handed? Which are left-handed?*

$$(a)\ \mathbf{i},\mathbf{k},\mathbf{j}, \qquad (b)\ \mathbf{j},\mathbf{k},\mathbf{i}, \qquad (c)\ \mathbf{k},\mathbf{i},\mathbf{j}, \qquad (d)\ \mathbf{k},\mathbf{j},\mathbf{i}.$$

Exercise 2.7.2 *Show that if* $\mathbf{a} \times \mathbf{u} = \mathbf{0}$ *for every unit vector* $\mathbf{u}$, *then* $\mathbf{a} = \mathbf{0}$.

Exercise 2.7.3 *Find the area of the parallelogram determined by the vectors* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Exercise 2.7.4 *Find the area of the parallelogram determined by the vectors* $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$.

Exercise 2.7.5 *Find the area of the parallelogram with vertices* $(-2,3,1)$, $(2,1,1)$, $(1,2,-1)$, *and* $(5,0,-1)$.

Exercise 2.7.6 *Find the area of the triangle determined by the three points,* $(1,0,3),(4,1,0)$ *and* $(-3,1,1)$.

Exercise 2.7.7 *Find the area of the triangle determined by the three points,* $(1,2,3),(2,3,4)$ *and* $(3,4,5)$. *Did something interesting happen here? What does it mean geometrically?*

Exercise 2.7.8 *Is* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$? *What is the meaning of* $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$? *Explain.* ***Hint:*** *Try* $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$.

Exercise 2.7.9 *Verify directly from the coordinate description of the cross product that* $\mathbf{u} \times \mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$. *Then show by direct computation that this coordinate description satisfies*

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \bullet \mathbf{v})^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2(\theta)), \end{aligned}$$

where θ *is the angle included between the two vectors. Explain why* $\|\mathbf{u} \times \mathbf{v}\|$ *has the correct magnitude.*

Exercise 2.7.10 *Prove the following formula by direct calculation:* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$.

Exercise 2.7.11 *Use the formula from Exercise 2.7.10 to prove that the cross product satisfies the so-called* ***Jacobi identity****:*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

Exercise 2.7.12 *Find the volume of the parallelepiped determined by the vectors* $\left[\begin{array}{r}1\\-7\\-5\end{array}\right]$, $\left[\begin{array}{r}1\\-2\\-6\end{array}\right]$, *and* $\left[\begin{array}{r}3\\2\\3\end{array}\right]$.

Exercise 2.7.13 *Which of the following systems of vectors* $\mathbf{u},\mathbf{v},\mathbf{w}$ *are right-handed? Which are left-handed? Which are coplanar?*

(a) $\mathbf{u}=[1,0,1]^T$, $\mathbf{v}=[1,2,0]^T$, $\mathbf{w}=[0,0,1]^T$.

(b) $\mathbf{u}=[0,1,1]^T$, $\mathbf{v}=[-1,2,0]^T$, $\mathbf{w}=[1,1,2]^T$.

(c) $\mathbf{u}=[1,-1,0]^T$, $\mathbf{v}=[1,0,1]^T$, $\mathbf{w}=[3,1,4]^T$.

(d) $\mathbf{u}=[1,0,0]^T$, $\mathbf{v}=[1,2,0]^T$, $\mathbf{w}=[2,0,-1]^T$.

Exercise 2.7.14 *Suppose* $\mathbf{u},\mathbf{v}$, *and* $\mathbf{w}$ *are three vectors whose components are all integers. Can you conclude the volume of the parallelepiped determined from these three vectors will always be an integer?*

Exercise 2.7.15 *What does it mean geometrically if the box product of three vectors equals zero?*

Exercise 2.7.16 *Show that*

$$(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}=(\mathbf{v}\times\mathbf{w})\bullet\mathbf{u}=(\mathbf{w}\times\mathbf{u})\bullet\mathbf{v}.$$

Exercise 2.7.17 *Use the formula from Exercise 2.7.10 to show that*

$$(\mathbf{u}\times\mathbf{v})\bullet((\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{z}))=((\mathbf{v}\times\mathbf{w})\bullet\mathbf{z})((\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}).$$

Exercise 2.7.18 *Simplify* $\|\mathbf{u}\times\mathbf{v}\|^2+(\mathbf{u}\bullet\mathbf{v})^2-\|\mathbf{u}\|^2\|\mathbf{v}\|^2$.

Exercise 2.7.19 *This problem uses calculus. For* $\mathbf{u},\mathbf{v},\mathbf{w}$ *functions of* t, *prove that the derivative satisfies the following product rules:*

$$\begin{array}{rcl}(\mathbf{u}\times\mathbf{v})' & = & \mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}'\\ (\mathbf{u}\bullet\mathbf{v})' & = & \mathbf{u}'\bullet\mathbf{v}+\mathbf{u}\bullet\mathbf{v}'\end{array}$$

3. Lines and planes in $\mathbb{R}^n$

3.1 Lines

Outcomes

A. *Find the vector, parametric, and symmetric equations of a line.*

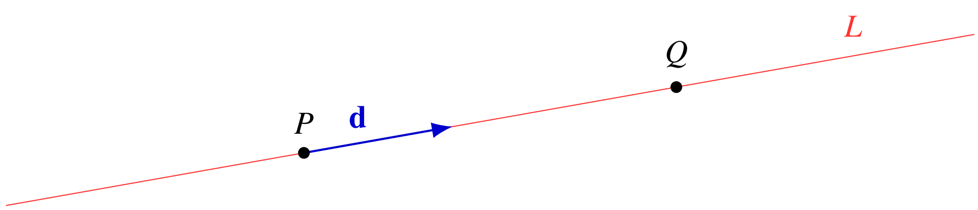
B. *Determine whether a point is on a given line.*

C. *Determine whether two lines intersect.*

D. *Find the angle between two lines.*

E. *Find the projection of a point onto a line.*

We can use the concept of vectors and points to find equations for lines in $\mathbb{R}^n$. Consider a straight line L that passes through a point P in the direction given by a non-zero vector $\mathbf{d}$.



The line L is infinitely long in both directions, although the picture only shows a finite part of it. To find an equation for this line, first suppose that Q is an arbitrary point on L. Then the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{d}$. In other words, there exists some real number t such that

$$\overrightarrow{PQ} = t\,\mathbf{d}.$$

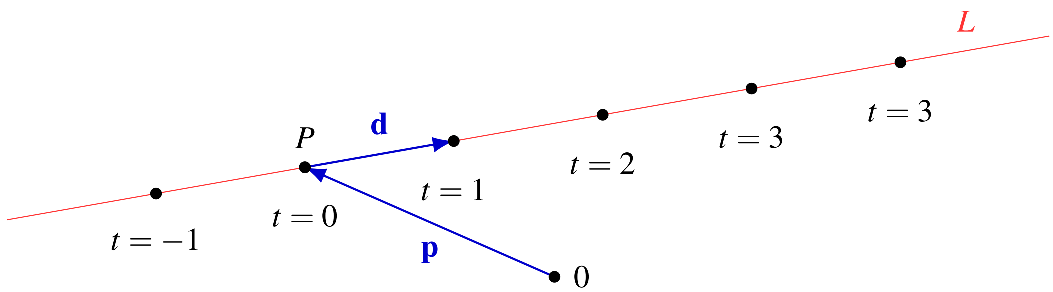
If $\mathbf{p}$ is the position vector of P and $\mathbf{q}$ is the position vector of Q, we can write

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}.$$

Putting together the last two equations, we get $\mathbf{q} - \mathbf{p} = t\,\mathbf{d}$, which we can write as

$$\mathbf{q} = \mathbf{p} + t\,\mathbf{d}.$$

This is called the **vector equation** of the line L. The vector $\mathbf{d}$ is called the **direction vector**, and t is called a **parameter**. The parameter t can be any real number; each time we plug in a different number for t, we get a different point Q on the line. The following picture shows the effect of the parameter:



The following definition summarizes the above.

Definition 3.1: Vector equation of a line

Let $\mathbf{p}$ *be a vector and* $\mathbf{d}$ *a non-zero vector. Then*

$$\mathbf{q} = \mathbf{p} + t\,\mathbf{d}$$

is the ***vector equation*** *of a straight line* L*. Specifically, as the parameter* t *ranges over the real numbers,* $\mathbf{q}$ *ranges over the position vectors of all the points* Q *on the line* L*. The vector* $\mathbf{d}$ *is called the* ***direction vector*** *of the line.*

Example 3.2: A line from a point and a direction vector

Find a vector equation for the line which contains the point $P = (2,0,3)$ *and has direction vector* $\mathbf{d} = [1,2,1]^T$.

Solution. The position vector of the point P is $\mathbf{p} = [2,0,3]^T$. The equation of the line is $\mathbf{q} = \mathbf{p} + t\,\mathbf{d}$, which we can write as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

♠

Example 3.3: A line from two points

Find a vector equation for the line through the points $P = (1,2,0,1)$ *and* $R = (2,-4,6,3)$.

Solution. We can use P as the base point; its position vector is $\mathbf{p} = [1,2,0,1]^T$. We can use $\mathbf{d} = \overrightarrow{PR} = [1,-6,6,2]$ as the direction vector. Then a vector equation of the line is $\mathbf{q} = \mathbf{p} + t\,\mathbf{d}$, which we can also write as

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -6 \\ 6 \\ 2 \end{bmatrix}.$$

♠

When we write a vector equation in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix},$$

it is also called the **component form** of the vector equation.

Notice that the vector equation of a line is not unique. In fact, there are infinitely many vector equations for the same line. For example, we can replace the parameter t with another parameter, say $3s$ or $1-r$.

Example 3.4: Change of parameter

Consider the vector equation from Example 3.2,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find two other equations for the same line, by changing the parameter to $3s$ and to $1-r$.

Solution. If we let $t = 3s$, we get

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 3s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}. \end{aligned}$$

If we let $t = 1-r$, we get

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + (1-r) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - r \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + r \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}. \end{aligned}$$

♠

Definition 3.5: Parametric equations of a line

A line with vector equation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

can also be written as a set of n scalar equations:

$$\begin{aligned} x_1 &= p_1 + t\,d_1, \\ x_2 &= p_2 + t\,d_2, \\ &\vdots \\ x_n &= p_n + t\,d_n, \end{aligned}$$

When written in this form, they are called the ***parametric equations*** *of the line.*

Example 3.6: Parametric equations

Find parametric equations for the line through the points $P = (1,2,0,1)$ and $R = (2,-4,6,3)$.

Solution. This is a same line as in Example 3.3. We can easily convert the vector equation

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -6 \\ 6 \\ 2 \end{bmatrix}$$

to a set of parametric equations:

$$\begin{aligned} x &= 1 + t, \\ y &= 2 - 6t, \\ z &= 6t, \\ w &= 1 + 2t. \end{aligned}$$

♠

Example 3.7: Determine whether a point is on a line

Determine whether the point $P = (5,8,4)$ is on the line L given by the vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Solution. The point P is on the line L if and only if there exists some $t \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix}.$$

Subtracting $[1,2,1]^T$ from both sides of the equation, this is equivalent to

$$t\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}.$$

We can write this as a set of parametric equations:

$$\begin{aligned} 2t &= 4, \\ 3t &= 6, \\ t &= 3. \end{aligned}$$

This is a system of three linear equations in one variable, and we quickly see that it is inconsistent. Therefore, the point P does not lie on the line L. ♠

Example 3.8: Determine whether two lines intersect

Determine whether the lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

intersect. If yes, find the point of intersection.

Solution. The two lines intersect if and only if there exist $t, s \in \mathbb{R}$ such that

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Bringing s to the left-hand side, and subtracting $[3,1,0]^T$ from both sides of the equation, this is equivalent to

$$t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - s\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix}.$$

If we write this vector equation as a set of three parametric equations, it is a system of 3 linear equations in 2 variables. The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 2 & -2 & -2 \\ 0 & -1 & -2 \\ 1 & 2 & 5 \end{array}\right].$$

This system has reduced echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right],$$

and has the unique solution $t = 1$ and $s = 2$. Therefore, the lines intersect. (Other possible cases are: If the system is inconsistent, the lines do not intersect. If the system has more than one solution, the lines are identical). We find the point of intersection by plugging the parameter $t = 1$ into the equation of the first line (or equivalently, but plugging $s = 2$ into the equation of the second line - doing it both ways is a good way to double-check your answer). Therefore, the point of intersection is

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

♠

There is one other form for a line which is useful, which is the **symmetric form**. Consider the line given by

$$\begin{aligned} x &= 1 + 2t, \\ y &= 1 - t, \\ z &= 3 + 2t. \end{aligned}$$

We can solve each equation for t:

$$\begin{aligned} t &= \tfrac{x-1}{2}, \\ t &= \tfrac{y-1}{-1}, \\ t &= \tfrac{z-3}{2}. \end{aligned}$$

Finally, we can eliminate t from the equations by setting all three equations equal to one another:

$$\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z-3}{2}.$$

The latter is really a system of 2 equations in 3 variables. This is the **symmetric form** of the equation of the line. In the following example, we look at how to convert the equation of a line from symmetric form to parametric form.

Example 3.9: Change symmetric form to parametric form

Consider the line whose equations are given in ***symmetric form*** *as*

$$\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+3}{1}.$$

Find parametric and vector equations for this line.

Solution. We set all three quantities equal to t:

$$t = \frac{x-2}{3}, \quad t = \frac{y-1}{2}, \quad t = \frac{z+3}{1}.$$

Solving these equations for x, y, z yields

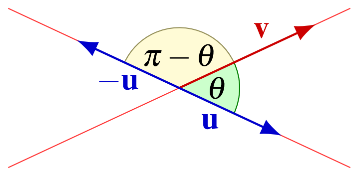
$$\begin{aligned} x &= 2 + 3t, \\ y &= 1 + 2t, \\ z &= -3 + t. \end{aligned}$$

These are the parametric equations for the line. The vector equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

♠

We can use the dot product to find the angle between two intersecting lines. This is simply the smallest angle between (any of) their direction vectors. The only subtlety here is that if **u** is a direction vector for a line, then so is $-\mathbf{u}$, and thus we will find pairs of complementary angles. We will take the smaller of the two angles.



Example 3.10: Find the angle between two lines

Find the angle between the two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

and

$$L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Solution. The direction vectors are

$$\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

The answer is given by

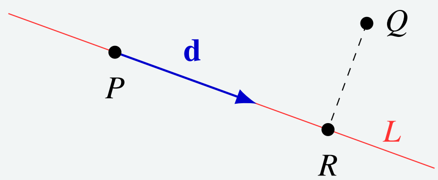
$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = -\frac{1}{2},$$

which gives $\theta = \frac{2\pi}{3}$. Now the angles between any two direction vectors for these lines will either be $\frac{2\pi}{3}$ or its complement $\phi = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. We choose the smaller angle, and therefore conclude that the angle between the two lines is $\frac{\pi}{3}$. ♠

Finally, we will show how to use projections to find the shortest distance from a point to a line.

Example 3.11: Shortest distance from a point to a line

Let L be the line which goes through the point $P = (0,4,-2)$ with direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and let $Q = (1,3,5)$. Find the shortest distance from Q to the line L, and find the point R on L that is closest to Q.



Solution. In order to determine the shortest distance from Q to L, we will first find the vector $\overrightarrow{PQ}$ and then find the projection of this vector onto L. The vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}.$$

Then, if R is the point on L closest to Q, it follows that

$$\overrightarrow{PR} = \mathrm{proj}_{\mathbf{d}} \overrightarrow{PQ} = \frac{\mathbf{d} \bullet \overrightarrow{PQ}}{\|\mathbf{d}\|^2} \mathbf{d} = \frac{15}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Now, the distance from Q to L is given by

$$\left\| \overrightarrow{RQ} \right\| = \left\| \overrightarrow{PQ} - \overrightarrow{PR} \right\| = \sqrt{26}.$$

The point R is found by adding the vector $\overrightarrow{PR}$ to the position vector $\overrightarrow{0P}$ for P as follows

$$\begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 17/3 \\ 4/3 \end{bmatrix}.$$

Therefore, $R = (\frac{10}{3}, \frac{17}{3}, \frac{4}{3})$. ♠

Exercises

Exercise 3.1.1 *Find the vector equation for the line through $(-7,6,0)$ and $(-1,1,4)$. Then, find the parametric equations for this line.*

Exercise 3.1.2 *Find parametric equations for the line through the point $(7,7,1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.*

Exercise 3.1.3 *Parametric equations of the line are*

$$\begin{aligned} x &= t+2, \\ y &= 6-3t, \\ z &= -t-6. \end{aligned}$$

Find a direction vector for the line and a point on the line.

Exercise 3.1.4 *The equation of a line in two dimensions is written as $y = x - 5$. Find a vector equation for this line.*

Exercise 3.1.5 *Find parametric equations for the line through $(6,5,-2,3)$ and $(5,1,2,1)$.*

Exercise 3.1.6 *Consider the following vector equation for a line in $\mathbb{R}^3$:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find a new vector equation for the same line by doing the change of parameter $t = 2 - s$.

Exercise 3.1.7 *Consider the line given by the following parametric equations:*

$$\begin{aligned} x &= 2t+2, \\ y &= 5-4t, \\ z &= -t-3. \end{aligned}$$

Find symmetric equations for the line.

Exercise 3.1.8 *Find the point on the line segment from $P = (-4,7,5)$ to $Q = (2,-2,-3)$ which is $\frac{1}{7}$ of the way from P to Q.*

Exercise 3.1.9 *Suppose a triangle in $\mathbb{R}^n$ has vertices at P, Q, and R. Consider the lines which are drawn from a vertex to the mid point of the opposite side. Show these three lines intersect in a point and find the coordinates of this point.*

Exercise 3.1.10 *Determine whether the lines*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

intersect. If yes, find the point of intersection.

Exercise 3.1.11 *Determine whether the lines*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

intersect. If yes, find the point of intersection.

Exercise 3.1.12 *Find the angle between the two lines*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Exercise 3.1.13 *Let $P = (1,2,3)$ be a point in $\mathbb{R}^3$. Let L be the line through the point $P_0 = (1,4,5)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find the shortest distance from P to L, and find the point Q on L that is closest to P.*

Exercise 3.1.14 *Let $P = (0,2,1)$ be a point in $\mathbb{R}^3$. Let L be the line through the points $P_0 = (1,1,1)$ and $P_1 = (4,1,2)$. Find the shortest distance from P to L, and find the point Q on L that is closest to P.*

Exercise 3.1.15 *When we computed the angle between two lines in Example 3.10, we calculated two different angles and took the smaller of the two. Show that one can get the same answer by taking the absolute value of the dot product, i.e., by solving*

$$\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

3.2 Planes

Outcomes

A. *Find the vector and parametric equations of a plane in $\mathbb{R}^n$.*

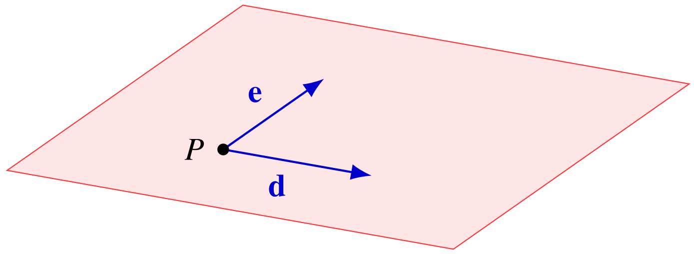
B. *Find the normal and standard equations of a plane in $\mathbb{R}^3$.*

C. *Find the intersection of two planes, or of a line and a plane.*

D. *Find the angle between two planes, or between a line and a plane.*

E. *Find the shortest distance between a point and a plane.*

Much like the above discussion with lines, vectors can be used to determine planes in $\mathbb{R}^n$. Consider a point P and two direction vectors $\mathbf{d}$ and $\mathbf{e}$ that are not parallel to each other. Then there is a unique plane passing through P and containing $\mathbf{d}$ and $\mathbf{e}$:



The plane is infinite in each direction, although in the picture, we have only shown a small part of it. If $\mathbf{p}$ is the position vector of P and $\mathbf{q}$ is the position vector of some other point in the plane, we have

$$\mathbf{q} = \mathbf{p} + t\,\mathbf{d} + s\,\mathbf{e}$$

for some real numbers t and s. This is called the **vector equation** of the plane.

Definition 3.12: Vector equation of a plane

Let $\mathbf{p}$ *be a vector and let* $\mathbf{d}, \mathbf{e}$ *be non-zero, non-parallel vectors. Then*

$$\mathbf{q} = \mathbf{p} + t\,\mathbf{d} + s\,\mathbf{e}$$

is the ***vector equation*** *of a plane.*

The vector equation of a plane can also be written in **component form**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}, + s \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

and in **parametric form**

$$\begin{aligned} x_1 &= p_1 + t\,d_1 + s\,e_1, \\ x_2 &= p_2 + t\,d_2 + s\,e_2, \\ &\vdots \\ x_n &= p_n + t\,d_n + s\,e_n. \end{aligned}$$

The latter set of equations are also called the **parametric equations** of the plane.

Example 3.13: Vector and parametric equations

Find vector and parametric equations for the plane through the points $P = (1,2,0,0)$, $Q = (2,2,0,1)$, *and* $R = (0,1,1,0)$.

Solution. We can use P as the base point and $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as the direction vectors. We have

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{PR} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vector equation is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

We can also write this as a system of parametric equations:

$$\begin{aligned} x &= 1+t-s, \\ y &= 2-s, \\ z &= s, \\ w &= t. \end{aligned}$$

♠

Note that the vector and parametric equations of a plane are not unique. For example, in Example 3.13, we could have equally used Q or R as the base point, and/or used $\overrightarrow{QR}$ as one of the direction vectors. In each case we would have obtained a different equation for the same plane.

Example 3.14: Determine whether a point is on a plane

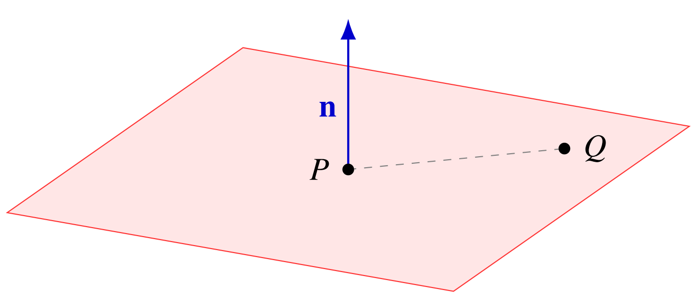
Determine whether the point $S = (4,4,-2,1)$ lies on the plane through the points $P = (1,2,0,0)$, $R = (2,2,0,1)$, and $Q = (0,1,1,0)$.

Solution. We already found the parametric equations for this plane in Example 3.13. To determine whether the point $S = (4,4,-2,1)$ lies on this plane, we must substitute its coordinates into the parametric equations:

$$\begin{aligned} 4 &= 1+t-s, \\ 4 &= 2-s, \\ -2 &= s, \\ 1 &= t. \end{aligned}$$

This is a system of linear equations. We solve it to find that it has the unique solution $(t,s) = (1,-2)$. Therefore, the point S lies on the given plane, and more specifically, it is the point that corresponds to the parameters $t = 1$ and $s = -2$. ♠

In the special case of 3 dimensions, a plane can also be described by a point and a normal vector. A **normal vector** of a plane is a vector that is perpendicular to the plane.



Given a non-zero vector $\mathbf{n}$ in $\mathbb{R}^3$ and a point P, there exists a unique plane that contains P and has $\mathbf{n}$ as a normal vector. We wish to find an equation for this plane. If Q is an arbitrary point on the plane, then by definition, the normal vector is orthogonal to the vector $\overrightarrow{PQ}$. Writing this as a formula, we have $\mathbf{n}\bullet\overrightarrow{PQ} = 0$. If $\mathbf{p}$ and $\mathbf{q}$ are the position vectors of P and Q, respectively, we have $\overrightarrow{PQ} = \mathbf{q}-\mathbf{p}$, and therefore the equation of the plane can be written as

$$\mathbf{n}\bullet(\mathbf{q}-\mathbf{p}) = 0,$$

or equivalently,

$$\mathbf{n}\bullet\mathbf{q} = \mathbf{n}\bullet\mathbf{p}.$$

This is called the **normal equation** of the plane. Note that in this equation, $\mathbf{n}$ and $\mathbf{p}$ are given and fixed, whereas $\mathbf{q}$ is a variable ranging over the position vectors of all points on the plane.

Definition 3.15: Normal equation of a plane in $\mathbb{R}^3$

Let $\mathbf{n}$ be a non-zero vector in $\mathbb{R}^3$, and let P be a point with position vector $\mathbf{p}$. Then there is a unique plane through P with normal vector $\mathbf{n}$. It is described by the equation

$$\mathbf{n}\bullet\mathbf{q} = \mathbf{n}\bullet\mathbf{p}.$$

*This equation is called the **normal equation** of the plane.*

Example 3.16: Finding the normal equation of a plane

Find the normal equation of the plane through the point $P = (1,3,0)$ and orthogonal to $\mathbf{n} = [2,1,1]^T$.

Solution. Let $\mathbf{p} = [1,3,0]^T$ be the position vector of P, and let $\mathbf{q} = [x,y,z]^T$ be the position vector of some arbitrary point Q in the plane. The normal equation is $\mathbf{n}\bullet\mathbf{q} = \mathbf{n}\bullet\mathbf{p}$, which we can write in component form:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

We can pre-compute the dot product on the right-hand side: $\mathbf{n}\bullet\mathbf{p} = 1(2)+3(1) = 0(1) = 5$. Therefore, the normal equation can also be written as

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5.$$

♠

Notice that the last equation in Example 3.16 can also be written in the form

$$2x+y+z = 5.$$

This last form is called the **standard equation** of the plane.

Definition 3.17: Standard equation of a plane in $\mathbb{R}^3$

*Let $\mathbf{n} = [a,b,c]^T$ be the normal vector for a plane that contains the point $P = (x_0, y_0, z_0)$. The **standard equation** of the plane is given by*

$$ax + by + cz = d,$$

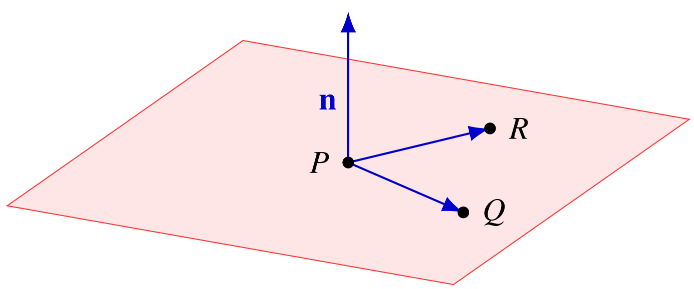
where $a,b,c,d \in \mathbb{R}$ and $d = ax_0 + by_0 + cz_0$.

Example 3.18: Normal and standard equations

Find normal and standard equations for the plane through the points $P = (0,1,3)$, $Q = (2,-1,0)$, and $R = (1,2,2)$.

Solution. We first need to find a normal vector for the plane. Since the normal vector must be perpendicular to the plane, it must be orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. We can therefore use the cross product to compute a normal vector for the plane:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$



Now we can easily obtain the normal equation from any point on the plane (say P) and the normal vector we just calculated:

$$\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

We get the standard equation by computing the dot products on the left- and right-hand sides:

$$5x - y + 4z = 11.$$

It is worthwhile to double-check the answer by substituting each of the three original points P, Q, and R into this equation. For example, for $Q = (2,-1,0)$, we obtain $5(2) - (-1) + 4(0)$, which is indeed 11. ♠

Example 3.19: Find the normal vector of a plane

Find a normal vector for the plane $2x + 3y - z = 7$.

Solution. The standard equation $2x+3y-z=7$ can be rewritten as a normal equation

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7.$$

Therefore,

$$\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

is a normal vector for the plane. ♠

Example 3.20: Determine whether a point is on a plane

Let $\mathbf{n} = [1,2,3]^T$ be the normal vector for a plane which contains the point $P = (2,1,4)$. Determine if the point $Q = (5,4,1)$ is in this plane.

Solution. By Definition 3.15, Q is a point in the plane if and only if

$$\mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot \mathbf{p},$$

where $\mathbf{p}$ and $\mathbf{q}$ are the position vectors of P and Q, respectively. Given $\mathbf{n}$, P, and Q as above, this equation becomes

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Since both sides of the equation are equal to 16, the equation is true. So the point Q is indeed in the plane determined by $\mathbf{n}$ and P. ♠

Example 3.21: Vector equation from normal equation

Find a vector equation for the plane $x+3y-2z=7$.

Solution. This is the same thing as finding the general solution of a system of one linear equation in 3 variables. Since there is only a single equation $x+3y-2z=7$, it is already in echelon form. The variables y and z are free, so we set them equal to parameters: $z=t$ and $y=s$. The variable x is a pivot variable, and we get $x=7+2t-3s$. So the general solution of the equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

This is also a vector equation for the plane. ♠

Example 3.22: Intersection of two planes

Find the intersection of the planes $x-2y+z=0$ *and* $2x-3y-z=4$.

Solution. Finding the intersection means finding all of the points (x,y,z) that are on both planes simultaneously. This is the same as solving the system of equations

$$\begin{aligned} x-2y+z &= 0, \\ 2x-3y-z &= 4. \end{aligned}$$

We solve the system by Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & 4 \end{array}\right] \underset{\simeq}{R_2 \leftarrow R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 4 \end{array}\right] \underset{\simeq}{R_1 \leftarrow R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 8 \\ 0 & 1 & -3 & 4 \end{array}\right].$$

Therefore, the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix},$$

where t is a parameter. This is the parametric equation of a line. Therefore, the two planes intersect in a line. Specifically, the intersection is the line through the point $(8,4,0)$ with direction vector $[5,3,1]^T$. ♠

Example 3.23: Intersection of a line and a plane

Find the intersection of the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

and the plane $2x+2y-z=2$.

Solution. Let us write

$$\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then the equation of the line is $\mathbf{q} = \mathbf{p} + t\,\mathbf{d}$ and the equation of the plane is $\mathbf{n} \bullet \mathbf{q} = 2$. Substituting the first equation into the second one, we get $\mathbf{n} \bullet (\mathbf{p} + t\,\mathbf{d}) = 2$. Using distributivity of the dot product, we can write this last equation as $\mathbf{n} \bullet \mathbf{p} + t(\mathbf{n} \bullet \mathbf{d}) = 2$. By computing the dot products $\mathbf{n} \bullet \mathbf{p} = 6$ and $\mathbf{n} \bullet \mathbf{d} = -2$, this equation simplifies to $6-2t=2$, or $t=2$. Therefore, the line intersects the plane when $t=2$, or at the point

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}.$$

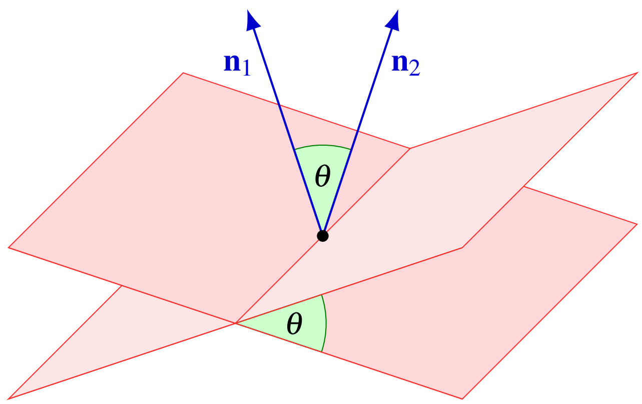
An alternative method is to directly substitute the parametric equation of the line, $x = 1-t$, $y = 2+t$, and $z = 2t$, into the equation of the plane, $2x+2y-z = 2$. In this case, we get $2(1-t)+2(2+t)-(2t) = 2$, which we can solve for t to obtain $t = 2$. ♠

The next few examples are concerned with calculating angles between planes, angles between lines and planes, and finding the distance between points and planes.

Example 3.24: Find the angle between two planes

Find the angle between the planes $7x-y = 5$ and $4x+3y+5z = 3$.

Solution. The angle between two planes is the same thing as the angle between their normal vectors.



The normal vectors are $\mathbf{n}_1 = [7,-1,0]^T$ and $\mathbf{n}_2 = [4,3,5]^T$. The angle between them is given by

$$\cos\theta = \frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{25}{50} = \frac{1}{2}.$$

Therefore, the angle is $\arccos(\frac{1}{2}) = \pi/3$, or 60 degrees. ♠

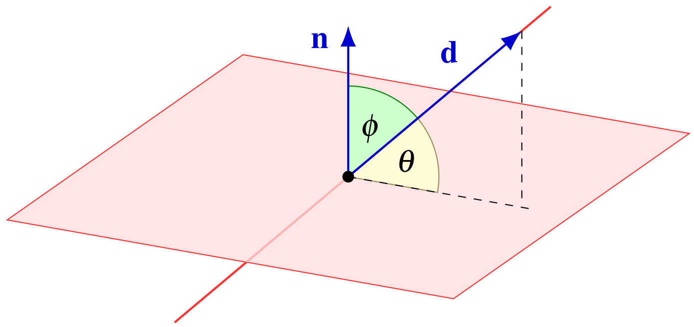
Example 3.25: Find the angle between a line and a plane

Find the angle between the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

and the plane $2x+2y-z = 2$.

Solution. To get the angle θ between the plane and the line, we can compute the angle ϕ between the direction vector of the line and the normal vector of the plane, and then take $\theta = \frac{\pi}{2} - \phi$.



The direction vector of the line is $[2,-1,-2]^T$ and the normal vector of the plane is $[2,2,-1]$. We have

$$\cos\phi = \frac{\mathbf{n}\cdot\mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|} = \frac{4}{9},$$

and therefore $\phi = \arccos(\frac{4}{9}) \approx 1.11$ radians. We have $\theta = \frac{\pi}{2} - \phi \approx 0.46$ radians, or about 26.4 degrees. ♠

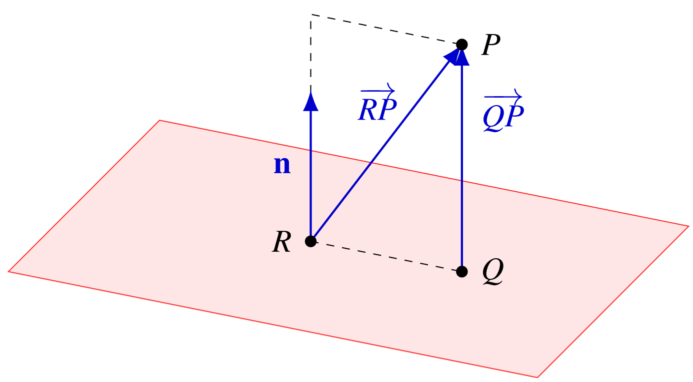
Example 3.26: Shortest distance from a point to a plane

Find the shortest distance from the point $P = (3,2,3)$ to the plane given by $2x+y+2z = 2$, and find the point Q on the plane that is closest to P.

Solution. In this problem, we are going to use the projection of one vector onto another, which was introduced in Section 2.6.5. Pick an arbitrary point R on the plane. Then, it follows that

$$\overrightarrow{QP} = \mathrm{proj}_{\mathbf{n}}\,\overrightarrow{RP}$$

and $\left\|\overrightarrow{QP}\right\|$ is the shortest distance from P to the plane. Further, the position vector of the point Q can be computed as $\mathbf{q} = \mathbf{p} - \overrightarrow{QP}$, where $\mathbf{p}$ is the position vector of P.



From the above scalar equation, we have that $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Now, choose any point on the plane, for example, $R = (1,0,0)$ (notice that this satisfies $2x+y+2z = 2$). Then,

$$\overrightarrow{RP} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Next, compute $\overrightarrow{QP} = \text{proj}_{\mathbf{n}} \overrightarrow{RP}$.

$$\overrightarrow{QP} = \text{proj}_{\mathbf{n}} \overrightarrow{RP} = \left(\frac{\mathbf{n} \bullet \overrightarrow{RP}}{\|\mathbf{n}\|^2} \right) \mathbf{n} = \frac{12}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Then, $\left\| \overrightarrow{QP} \right\| = 4$ so the shortest distance from P to the plane is 4. To find the point Q on the plane that is closest to P, we have

$$\mathbf{q} = \mathbf{p} - \overrightarrow{QP} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, $Q = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$. ♠

Exercises

Exercise 3.2.1 *Find vector and parametric equations for the plane through the points* $P = (0,1,1)$, $Q = (-1,2,1)$, *and* $R = (1,1,2)$.

Exercise 3.2.2 *Consider the following vector equation for a plane in* $\mathbb{R}^4$:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Find a new vector equation for the same plane by doing the change of parameters $t = 1 - r_1$, $s = r_1 + r_2$.

Exercise 3.2.3 *Determine which of the following points lie on the plane through the points* $P = (2,6,1)$, $R = (1,4,1)$, *and* $Q = (1,2,-1)$.

(a) $S_1 = (1,2,4)$.

(b) $S_2 = (1,5,2)$.

(c) $S_3 = (0,0,0)$.

Exercise 3.2.4 *Use cross products to find the normal vector to the plane going through the points* $P = (1,2,3)$, $Q = (-2,1,8)$ *and* $R = (2,2,2)$.

Exercise 3.2.5 *Find normal and standard equations of the plane through the point* $P = (1,1,2)$ *and orthogonal to* $\mathbf{n} = [1,0,-1]^T$.

Exercise 3.2.6 *Find normal and standard equations for the plane through the points* $P = (2,1,0)$, $Q = (1,-1,0)$, *and* $R = (1,1,-1)$.

Exercise 3.2.7 *Find a vector equation for the plane* $2x+y-z=1$.

Exercise 3.2.8 *The chapter mentions that the normal equation and standard equation of a plane only work in* $\mathbb{R}^3$*, and not in general* $\mathbb{R}^n$*. Why does the equation* $ax+by+cz+dw=e$ *not describe a plane in* $\mathbb{R}^4$*?*

Exercise 3.2.9 *Find the intersection between the planes* $x+3y+4z=3$ *and* $2x+5y-z=2$*. Is the intersection a line, a plane, or empty?*

Exercise 3.2.10 *Find the intersection of the line*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

and the plane $x+3y+z=6$*. Is the intersection a point, a line, or empty?*

Exercise 3.2.11 *Find the angle between the planes* $x+y=5$ *and* $2x+y-z=4$*.*

Exercise 3.2.12 *Find the angle between the line*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

and the plane $4x+7y+4z=6$*.*

Exercise 3.2.13 *In Example 3.25, we calculated the angle* θ *between a line and a plane by calculating the angle* ϕ *between the direction vector of the line and the normal vector of the plane according to the formula*

$$\cos\phi = \frac{\mathbf{n} \bullet \mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|}$$

and then taking $\theta = \frac{\pi}{2} - \phi$*.*

(a) *Explain what happens when the dot product is negative. How should we adjust the formula to ensure that* θ *is always between* 0 *and* $\frac{\pi}{2}$*?*

(b) *Show that one can get the answer in a single step with the formula*

$$\sin\theta = \frac{|\mathbf{n} \bullet \mathbf{d}|}{\|\mathbf{n}\|\,\|\mathbf{d}\|}.$$

Exercise 3.2.14 *Find the shortest distance from the point* $P=(1,1,-1)$ *to the plane given by* $x+2y+2z=6$*, and find the point Q on the plane that is closest to P.*

Exercise 3.2.15 *Use Exercise 2.7.15 to find an equation of a plane containing the two vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *and the point* 0*.* ***Hint:*** *If* (x,y,z) *is a point in this plane, the volume of the parallelepiped determined by* (x,y,z) *and the vectors* $\mathbf{p}$*,* $\mathbf{q}$ *equals 0.*

4. Matrices

4.1 Definition and equality

Outcomes

A. Identify the dimension and entries of a matrix.

B. Check equality of matrices.

We have solved systems of equations by writing them in terms of an augmented matrix and then doing row operations. It turns out that matrices are important not only for systems of equations but also for many other purposes.

Definition 4.1: Matrix

A ***matrix*** *is a rectangular array of numbers*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where the a_{ij} are scalars, called the ***entries*** *or* ***components*** *of A. The* ***size*** *or* ***dimension*** *of a matrix is defined as $m \times n$, where m is the number of rows and n is the number of columns.*

For example, here is a 3×4-matrix (pronounced "three-by-four matrix"):

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix}.$$

This is a 3×4-matrix because there are three rows and four columns. When specifying the size of a matrix, we always list the number of rows before the number of columns.

Entries of the matrix are identified according to their position. The (i, j)**-entry** of a matrix is the entry in the i^{th} row and j^{th} column, and is often denoted a_{ij}. For example, in the above matrix, the $(2,3)$-entry is the entry in the second row and the third column, and is equal to 8. We sometimes use $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ as a short-hand notation for the entire $m \times n$-matrix whose (i, j)-entry is equal to a_{ij} for all $i = 1, \ldots m$ and $j = 1, \ldots, n$.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added and subtracted, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

Definition 4.2: Equality of matrices

*Two matrices are **equal** if they have the same size and the same corresponding entries. More precisely, if $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are two $m \times n$-matrices, then $A = B$ means that $a_{ij} = b_{ij}$ for all $i = 1, \ldots m$ and $j = 1, \ldots, n$.*

For example,

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \neq \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

because they are different sizes. Also,

$$\left[\begin{array}{cc} 0 & 1 \\ 3 & 2 \end{array}\right] \neq \left[\begin{array}{cc} 1 & 0 \\ 2 & 3 \end{array}\right]$$

because, although they are the same size, their corresponding entries are not identical.

There are special names for matrices of certain dimensions: some matrices are called square matrices, columns vectors, or row vectors.

Definition 4.3: Square matrix

*A matrix of size $n \times n$ is called a **square matrix**. In other words, A is a square matrix if it has the same number of rows and columns.*

Definition 4.4: Column vectors and row vectors

*A matrix of size $n \times 1$ is called a **column vector**. A matrix of size $1 \times n$ is called a **row vector**. Here is an example of a column vector X and a row vector Y:*

$$X = \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right], \quad Y = \left[\begin{array}{ccc} y_1 & \cdots & y_n \end{array}\right].$$

We have already encountered column vectors in Chapter 2. When we use the term **vector** without further qualification, we always mean a column vector. Also recall from Definition 2.1 that the set of n-dimensional column vectors is called $\mathbb{R}^n$.

Exercises

Exercise 4.1.1 *Can a column vector ever be equal to a row vector?*

Exercise 4.1.2 *Find scalars x, y, z such that the following two matrices are equal.*

$$\begin{bmatrix} x & -1 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & y \\ z & 4 \end{bmatrix}.$$

Exercise 4.1.3 *What are the dimensions of the following matrices?*

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 3 & 1 \\ 6 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 4 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercise 4.1.4 *What is the $(2,3)$-entry of the matrix* $\begin{bmatrix} 1 & 2 & 1 \\ -4 & 4 & 7 \\ 6 & -5 & 3 \end{bmatrix}$*?*

4.2 Addition

Outcomes

A. *Perform the operations of matrix addition and subtraction.*

B. *Identify when these operations are not defined.*

C. *Apply the algebraic properties of matrix addition to manipulate an algebraic expression involving matrices.*

To add two matrices, the matrices have to be of the same size. The addition works by simply adding corresponding entries of the matrices.

Definition 4.5: Addition of matrices

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be two $m \times n$-matrices. Then $A + B = C$ where C is the $m \times n$-matrix $C = \left[c_{ij}\right]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

Example 4.6: Addition of matrices

Add the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution. Notice that both A and B are of size 2×3. Since A and B are of the same size, the addition is possible. Using Definition 4.5, the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}.$$

♠

On the other hand, the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, because one has size 3×2 while the other has size 2×3.

Definition 4.7: The zero matrix

*The $m \times n$ **zero matrix** is the $m \times n$-matrix in which all entries are equal to zero. It is denoted by 0.*

Note there is a zero matrix for every size. For example, there is a 2×3 zero matrix, a 3×4 zero matrix, and so on.

Example 4.8: The zero matrix

The 2×3 zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Definition 4.9: Negative of a matrix and subtraction

*The **negative** of a matrix $A = \left[a_{ij}\right]$ is defined to be $-A = \left[-a_{ij}\right]$. In other words, it is obtained by negating every entry of A. To **subtract** two matrices, we simply add the negative of the second matrix to the first one, i.e., $A - B = A + (-B)$. This is just the same as componentwise subtraction.*

Example 4.10: Subtraction

Subtract the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution.

$$A-B=\begin{bmatrix}1&2&3\\1&0&4\end{bmatrix}-\begin{bmatrix}5&2&3\\-6&2&1\end{bmatrix}=\begin{bmatrix}1-5&2-2&3-3\\1-(-6)&0-2&4-1\end{bmatrix}=\begin{bmatrix}-4&0&0\\7&-2&3\end{bmatrix}.$$

♠

Addition of matrices obeys the same properties as addition of vectors.

Proposition 4.11: Properties of matrix addition

Let A, B and C be matrices of the same size. Then, the following properties hold.

- *The commutative law of addition*
$$A+B=B+A.$$
- *The associative law of addition*
$$(A+B)+C=A+(B+C).$$
- *The existence of an additive unit*
$$A+0=A.$$
- *The existence of an additive inverse*
$$A+(-A)=0.$$

Proof. To prove the commutative law of addition, let A and B be matrices of the same size. We want to show that $A+B=B+A$. To do so, we use the definition of matrix addition given in Definition 4.5. We have
$$A+B=\left[a_{ij}+b_{ij}\right]=\left[b_{ij}+a_{ij}\right]=B+A.$$
The proof of the other properties are similar, and are left as an exercise. ♠

Exercises

Exercise 4.2.1 *For the following pairs of matrices, determine if the sum $A+B$ and the difference $A-B$ are defined. If so, calculate them.*

(a) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad B=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$

(b) $A=\begin{bmatrix}2&1&2\\1&1&0\end{bmatrix}, \quad B=\begin{bmatrix}-1&0&3\\0&1&4\end{bmatrix}.$

(c) $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 4 \end{bmatrix}.$

Exercise 4.2.2 *For each matrix A, find the matrix $-A$.*

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

Exercise 4.2.3 *Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. Find a matrix X such that $(A+X)-(B+0) = B+A$. Hint: first use the properties of matrix addition to simplify the equation and solve for X.*

Exercise 4.2.4 *Using only the properties given in Proposition 4.11, show that if $A+B=0$, then $B=-A$.*

Exercise 4.2.5 *Using only the properties given in Proposition 4.11, show $A+B=A$ implies $B=0$.*

4.3 Scalar multiplication

Outcomes

A. *Multiply a matrix by a scalar, and take linear combinations of matrices.*

B. *Identify when these operations are not defined.*

C. *Apply the algebraic properties of matrix addition and scalar multiplication to manipulate an algebraic expression involving matrices.*

The multiplication of a scalar by a matrix is called the **scalar multiplication** of matrices. The new matrix is obtained by multiplying every entry of the original matrix by the given scalar, as in the following example.

$$3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}.$$

The formal definition of scalar multiplication is as follows.

Definition 4.12: Scalar multiplication of a matrix

If k is a scalar and $A = \left[a_{ij}\right]$ is a matrix, then $kA = \left[ka_{ij}\right]$.

Example 4.13: Linear combination of matrices

Find $2A - 3B$, where

$$A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 1 & 0 & 4 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{rrr} 5 & 2 & 3 \\ -6 & 2 & 1 \end{array}\right]$$

Solution.

$$2A - 3B = 2\left[\begin{array}{rrr} 1 & 2 & 3 \\ 1 & 0 & 4 \end{array}\right] - 3\left[\begin{array}{rrr} 5 & 2 & 3 \\ -6 & 2 & 1 \end{array}\right] = \left[\begin{array}{rrr} 2 & 4 & 6 \\ 2 & 0 & 8 \end{array}\right] - \left[\begin{array}{rrr} 15 & 6 & 9 \\ -18 & 6 & 3 \end{array}\right] = \left[\begin{array}{rrr} -13 & -2 & -3 \\ 20 & -6 & 5 \end{array}\right].$$

♠

Scalar multiplication of matrices obeys the same properties as scalar multiplication of vectors.

Proposition 4.14: Properties of scalar multiplication

Let A and B be matrices of the same size, and let k, ℓ be scalars. Then the following properties hold.

- *The distributive law over matrix addition*

$$k(A + B) = kA + kB.$$

- *The distributive law over scalar addition*

$$(k + \ell)A = kA + \ell A.$$

- *The associative law for scalar multiplication*

$$k(\ell A) = (k\ell)A.$$

- *The rule for multiplication by 1*

$$1A = A.$$

Exercises

Exercise 4.3.1 *For each matrix A, find the products $(-2)A$, $0A$, and $3A$.*

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

Exercise 4.3.2 *Find scalars x,y,z,w such that*

$$x\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + z\begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} + w\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}.$$

Exercise 4.3.3 *Using only the properties given in Propositions 4.11 and 4.14, show that $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero matrix of appropriate size.*

4.4 Matrix multiplication

Outcomes

A. *Use two different methods to multiply a matrix and a vector.*

B. *Multiply two matrices using the componentwise method, the column method, or the row method.*

C. *Identify when these operations are not defined.*

D. *Write a system of linear equations in vector form and matrix form.*

E. *Demonstrate that matrix multiplication is not commutative.*

F. *Use algebraic properties of matrix multiplication to solve matrix equations.*

4.4.1. Multiplying a matrix and a vector

One of the most important uses of a matrix is to multiply a matrix by a vector. In fact, this is one of the reasons matrices were invented. Let us start by considering an alternative way of writing a system of linear equations.

Definition 4.15: The vector form of a system of linear equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

We can express this system in ***vector form****, which is as follows:*

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector. The left-hand side is a linear combination of column vectors. Linear combinations of column vectors are so important that we introduce a special notation for them.

Definition 4.16: The product of a matrix and a vector, by columns

The product of an $m \times n$-matrix A and an n-dimensional column vector $\mathbf{x}$ is an m-dimensional column vector, defined as a linear combination of the columns of A as follows:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

In other words, we can think of the vector $\mathbf{x}$ as encoding instructions for how to take a linear combination of the columns of A. The product $A\mathbf{x}$ is computed by taking x_1 times the first column of A, plus x_2 times the second column of A, and so on. For this to work, A must have the same number of columns as $\mathbf{x}$ has components.

Example 4.17: Multiplying a matrix and a vector, by columns

Compute the product

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Solution. We have

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}.$$

♠

There is another way of looking at the product of a matrix and a vector. Instead of looking at the columns of A, we can look at the rows.

Proposition 4.18: The product of a matrix and a vector, by rows

Let A be an $m \times n$-matrix and let $\mathbf{x}$ be an n-dimensional column vector. The product $A\mathbf{x}$ can also be written like this:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}.$$

Example 4.19: Multiplying a matrix and a vector, by rows

Compute the product

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

by rows.

Solution. We have

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \\ 4 \cdot 7 + 5 \cdot 8 + 6 \cdot 9 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}.$$

Note that the is exactly the same answer as before. ♠

When we use Definition 4.16, we calculate the product by looking at one column of A at a time. When we use Proposition 4.18, we calculate the product by looking at one row of A at a time. As the above examples show, both methods give exactly the same answer. Please convince yourself that this is true in general. This ability to switch back and forth between a column-based viewpoint and a row-based viewpoint is one of the central tools of linear algebra.

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector.

Definition 4.20: The matrix form of a system of linear equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

Then we can express this system in ***matrix form****, which is as follows:*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix form of a system of equations is therefore written as $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix of the system, $\mathbf{x}$ is an n-dimensional column vector constructed from the variables of the system, and $\mathbf{b}$ is an m-dimensional column vector constructed from the constant terms of the system. Any system of linear equations can be written in this form.

4.4.2. Matrix multiplication

The multiplication of a matrix and a vector from the previous section is a special case of the operation of multiplying two matrices, which we now define.

Definition 4.21: Matrix multiplication

Let $A = \left[a_{ij}\right]$ *be an* $m \times n$*-matrix, and let* $B = \left[b_{jk}\right]$ *be an* $n \times p$*-matrix. Then their product is the* $m \times p$*-matrix* $AB = [c_{ik}]$ *whose* (i,k)*-entry is defined by*

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

For matrices A and B, in order to form the product AB, the number of columns of A must equal the number of rows of B. Consider a product AB where A has dimensions $m \times n$ and B has dimensions $n \times p$. Then the dimensions of the product are given by

$$(m \times \overbrace{n)\ (n}^{\text{these must match!}} \times p) = m \times p.$$

Note that the two outside numbers give the dimensions of the product. If the two middle numbers do not match, we cannot multiply the matrices.

To better visualize the rule of matrix multiplication, suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}.$$

Then their product

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

is an $m \times p$-matrix whose (i,k)-entry is defined by

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

Note that we can also write this as

$$c_{ik} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}.$$

In other words, the (i,k)-entry of the matrix product AB is a kind of dot product of the i^{th} row of A with the k^{th} column of B.

Example 4.22: Matrix multiplication

Find AB, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Solution. First, let us note that since A has size 2×3 and B has size 3×3, the product AB is well-defined and has size 2×3. Let $C = AB$. We compute each of the six entries of C:

- The $(1,1)$-entry is the first row of A times the first column of B: $c_{11} = 1 \cdot 1 + 2 \cdot 0 + 1 \cdot (-2) = -1$.
- The $(1,2)$-entry is the first row of A times the second column of B: $c_{12} = 1 \cdot 2 + 2 \cdot 3 + 1 \cdot 1 = 9$.
- The $(1,3)$-entry is the first row of A times the third column of B: $c_{13} = 1 \cdot 0 + 2 \cdot 1 + 1 \cdot 1 = 3$.
- The $(2,1)$-entry is the second row of A times the first column of B: $c_{21} = 0 \cdot 1 + 2 \cdot 0 + 1 \cdot (-2) = -2$.
- The $(2,2)$-entry is the second row of A times the second column of B: $c_{22} = 0 \cdot 2 + 2 \cdot 3 + 1 \cdot 1 = 7$.
- The $(2,3)$-entry is the second row of A times the third column of B: $c_{23} = 0 \cdot 0 + 2 \cdot 1 + 1 \cdot 1 = 3$.

Therefore, we have

$$AB = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}.$$

♠

As this example shows, calculating matrix products one component at a time can be an extremely repetitive and tedious process. Fortunately, we can speed this up by considering whole columns at once.

Proposition 4.23: Matrix multiplication, column method

Let A be an $m\times n$-matrix, and let B be an $n\times p$-matrix. Suppose that the columns of B are $\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_p$. Then the columns of AB are

$$A\mathbf{b}_1,\ A\mathbf{b}_2,\ \ldots,\ A\mathbf{b}_p.$$

In other words, the k^{th} column of the matrix product AB is equal to A times the k^{th} column of B.

Example 4.24: Matrix multiplication by the column method

Find the matrix product AB by the column method, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Solution. We multiply A by each of the columns of B:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The resulting three column vectors form the columns of AB. Thus,

$$AB = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}.$$

♠

Of course, the answer in Example 4.24 is the same as that in Example 4.22. Please convince yourself that both methods of matrix multiplication give the same answer, since they each ultimately calculate the same thing. Nevertheless, with a bit of practice, the column method is much faster, and you can even learn to multiply matrices in your head! The key to understanding the column method is that each column of B provides instructions for taking a linear combination of the columns of A. The method works especially well if B contains many zeros and ones.

Since column vectors are simply $n \times 1$-matrices, and row vectors are $1 \times m$-matrices, we can also multiply a column vector by a row vector or vice versa.

Example 4.25: Column vector times row vector

Multiply $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$.

Solution. Here we are multiplying a 3×1-matrix by a 1×4-matrix, so the result will be a 3×4-matrix. Using the column method, we can compute this product as follows:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix} = \left[\overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{Third column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}^{\text{Fourth column}} \right] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

♠

Example 4.26: Row vector times column vector

Multiply $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution. Here we are multiplying a 1×3-matrix by a 3×1-matrix, so the result will be a 1×1-matrix, or in other words, a scalar. (We regard a scalar and a 1×1-matrix as the same thing). We have:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot (-1) = 2.$$

Therefore, multiplying a row vector by a column vector works very similarly to an ordinary dot product (except that the dot product is defined between two column vectors, not a row vector and a column vector).

♠

Example 4.27: A multiplication that is not defined

Find BA, where

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Solution. The product BA is not defined, since B is a 3×3-matrix and A is a 2×3-matrix. Since the number of columns of B does not match the number of rows of A, the product is not defined. ♠

Notice that the matrices in Example 4.27 are the same as those in Example 4.22. This demonstrates an important property of matrix multiplication: it is possible that AB is defined by BA is undefined. Even if AB and BA are both defined, they may not be equal, as the following example shows. Therefore, matrix multiplication is not commutative.

Example 4.28: Matrix multiplication is not commutative

Compute AB and BA, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Are they equal?

Solution. We have

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

Therefore, AB and BA are not equal. Matrix multiplication is not commutative. ♠

We have seen two methods for matrix multiplication: one component at a time, and by the column method. There is also a third method, called the row method. It is exactly symmetric to the column method.

Proposition 4.29: Matrix multiplication, row method

Let A be an $m\times n$-matrix, and let B be an $n\times p$-matrix. Suppose that the rows of A are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. Then the rows of AB are

$$\mathbf{a}_1 B,\ \mathbf{a}_2 B,\ \ldots,\ \mathbf{a}_m B.$$

In other words, the i^{th} column of the matrix product AB is equal to the i^{th} column of A times B.

Example 4.30: Matrix multiplication by the row method

Find the matrix product AB by the row method, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Solution. We multiply each of the rows of A by B:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 3 & 1 \end{bmatrix} + 1\begin{bmatrix} -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = 0\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 3 & 1 \end{bmatrix} + 1\begin{bmatrix} -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 & 3 \end{bmatrix}.$$

The resulting two row vectors form the rows of AB. Thus,

$$AB = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}.$$

Once again this is the same answer as in Examples 4.24 and 4.22. All three methods give the same result. But notice how in the row method, each row of A provides instructions for taking a linear combinations of the rows of B. ♠

We finish this section by introducing an important square matrix called the identity matrix.

Definition 4.31: Identity matrix

*The **identity matrix** of size $n \times n$ has ones along the diagonal, and zeros everywhere else. In other words, it is the matrix $\left[\delta_{ij}\right]$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The identity matrix is always a square matrix. Here are some identity matrices of various sizes.*

$$\begin{bmatrix} 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

Example 4.32: Multiplying by the identity matrix

Calculate AI, where I is the 2×2 identity matrix and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Solution. We need to calculate

$$AI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the column method, we find that the first column of AI is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + 0\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix},$$

which is exactly the same as the first column of A. Similarly, the second column of AI is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + 1 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix},$$

which is exactly the same as the second column of A. Therefore

$$AI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = A.$$

♠

The calculation of the last example generalizes to matrices of all sizes, and is summarized in the following proposition.

Proposition 4.33: Multiplying by the identity matrix

Let A be any $m \times n$-matrix. Then

$$I_m A = A = A I_n.$$

We can also raise a square matrix to a power. For example, A^5 means $A \cdot A \cdot A \cdot A \cdot A$.

Example 4.34: Raising a matrix to a power

Compute A^3, where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}.$$

Solution. We have

$$A^3 = A \cdot A \cdot A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -19 & 18 \\ -18 & -1 \end{bmatrix}.$$

♠

4.4.3. Properties of matrix multiplication

We have already seen that matrix multiplication is not in general commutative, i.e., AB and BA may be different, even if they are both defined. Sometimes it can happen that $AB = BA$ for specific matrices A and B. In this case, we say that A and B **commute**.

The following are some properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 4.35: Properties of matrix multiplication

The following properties hold for matrices A,B,C of appropriate dimensions and for scalars r.

- *The associative law of multiplication*

$$(AB)C \;=\; A(BC).$$

- *The existence of multiplicative units*

$$I_mA \;=\; A \;=\; AI_n,$$

where A is an $m\times n$-matrix.

- *Compatibility with scalar multiplication*

$$(rA)B \;=\; r(AB) \;=\; A(rB).$$

- *The distributive laws of multiplication over addition*

$$\begin{aligned} A(B+C) &= AB+AC, \\ (B+C)A &= BA+CA. \end{aligned}$$

Proof. First, we will prove the associative law. In the proof, it will be useful to use *summation notation*. We write

$$\sum_{i=1}^{n} x_i \;=\; x_1+x_2+\ldots+x_n$$

for the sum of the n numbers $x_1,\ldots,x_n$. Assume A is an $m\times n$-matrix, B is an $n\times p$-matrix, and C is a $p\times q$-matrix. Then both $(AB)C$ and $A(BC)$ are $m\times q$-matrices. We must show that they have the same entries. The (i,ℓ)-entry of the matrix $(AB)C$ is

$$((AB)C)_{i\ell} = \sum_{k=1}^{p}(AB)_{ik}c_{k\ell} = \sum_{k=1}^{p}(\sum_{j=1}^{n} a_{ij}b_{jk})c_{k\ell} = \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}c_{k\ell}.$$

On the other hand, the (i,ℓ)-entry of the matrix $A(BC)$ is

$$(A(BC))_{i\ell} = \sum_{j=1}^{n} a_{ij}(BC)_{j\ell} = \sum_{j=1}^{n} a_{ij}(\sum_{k=1}^{p} b_{jk}c_{k\ell}) = \sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}b_{jk}c_{k\ell}.$$

Both sums are equal, since they are both summing over all the terms where $j=1,\ldots,n$ and $k=1,\ldots,p$. Therefore, $(AB)C=A(BC)$. The fact that identity matrices act as multiplicative units was already mentioned in Proposition 4.33. We leave compatibility with scalar multiplication as an exercise. To prove the first distributive law, assume A is an $m\times n$-matrix, and B and C are $n\times p$-matrices. Then both $A(B+C)$ and $AB+AC$ are $m\times p$-matrices. We have

$$(A(B+C))_{ik} = \sum_{j=1}^{n} a_{ij}(B+C)_{jk} = \sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk}) = \sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk} = (AB+AC)_{ik}.$$

Thus $A(B+C) = AB + AC$ as claimed. The proof of the other distributive law is similar. ♠

Exercises

Exercise 4.4.1 *Let* $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. *Multiply A by each of the following vectors.*

$$(a) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (b) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad (c) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad (d) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (e) \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Exercise 4.4.2 *Write the following system of equations in vector form and matrix form.*

$$\begin{aligned} 2x + 3y + z &= 4, \\ x - 2z &= 3, \\ 2y + z &= 1. \end{aligned}$$

Exercise 4.4.3 *Compute the following by columns and by rows. Convince yourself that both method give the same result.*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Exercise 4.4.4 *Write the vector*

$$\begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{bmatrix}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ *where A is an appropriate matrix.*

Exercise 4.4.5 *Write the vector*

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1 \end{bmatrix}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ *where A is an appropriate matrix.*

Exercise 4.4.6 *Consider the matrices*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Find the following if possible. If it is not possible explain why.

(a) $-3A$

(b) $3B - A$

(c) AC

(d) CB

(e) AE

(f) EA

Exercise 4.4.7 *Consider the matrices*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find the following if possible. If it is not possible explain why.

(a) $-3A$

(b) $3B - A$

(c) AC

(d) CA

(e) AE

(f) EA

(g) BE

(h) DE

Exercise 4.4.8 *Let* $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. *Find all* 2×2*-matrices* B *such that* $AB = 0$.

Exercise 4.4.9 *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. *Is it possible to find* k *such that* $AB = BA$*? If so, what should* k *equal?*

Exercise 4.4.10 *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$. *Is it possible to choose* k *such that* $AB = BA$*? If so, what should* k *equal?*

Exercise 4.4.11 *For each pair of matrices, find the* $(1,2)$*-entry and* $(2,3)$*-entry of the product* AB.

(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 16 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

Exercise 4.4.12 *Compute A^4, where*

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}.$$

Hint: you can save some work by calculating A^2 times A^2.

Exercise 4.4.13 *Find 2×2-matrices A, B, and C such that $A \neq 0$, $C \neq B$, but $AC = AB$.*

Exercise 4.4.14 *Find 2×2-matrices A and B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.*

Exercise 4.4.15 *Find 3×3-matrices A and B such that $AB \neq BA$.*

Exercise 4.4.16 *Give an example of a matrix A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.*

Exercise 4.4.17 *A matrix A is called* ***idempotent*** *if $A^2 = A$. Let*

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}.$$

Show that A is idempotent.

Exercise 4.4.18 *Suppose A and B are square matrices of the same size. Which of the following are necessarily true?*

(a) $(A+B)^2 = A^2 + 2AB + B^2$.

(b) $(A-B)^2 = A^2 - 2AB + B^2$.

(c) $(AB)^2 = A^2B^2$.

(d) $(A+B)^2 = A^2 + AB + BA + B^2$.

(e) $A^2B^2 = A(AB)B$.

(f) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

(g) $(A+B)(A-B) = A^2 - B^2$.

4.5 Matrix inverses

Outcomes

A. *Determine whether a matrix is invertible, and compute the inverse if it exists.*

B. *Solve a system of linear equations using matrix algebra.*

C. *Prove algebraic properties of matrix inverses.*

D. *Determine whether a matrix is a left inverse, right inverse, or inverse of another matrix.*

4.5.1. Definition and uniqueness

We now define a matrix operation which in some ways plays the role of division. We cannot divide by a matrix, but we can multiply by the inverse of a matrix, which is almost as good.

Definition 4.36: The inverse of a matrix

Let A and B be $n \times n$-matrices. We say that B is an ***inverse*** *of A if*

$$BA = I \quad \text{and} \quad AB = I.$$

If this is the case, we also write $B = A^{-1}$. When a matrix has an inverse, it is called ***invertible****.*

Example 4.37: Verifying the inverse of a matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Check that $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is an inverse of A.

Solution. To check this, multiply

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

This shows that B is indeed an inverse of A. ♠

Unlike multiplication of scalars, it can happen that $A \neq 0$ but A does not have an inverse. This is illustrated in the following example.

Example 4.38: A non-zero matrix with no inverse

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. *Show that* A *is not invertible.*

Solution. One might think A has an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If an inverse A^{-1} existed, we would have the following:

$$\begin{aligned} \begin{bmatrix} -1 \\ 1 \end{bmatrix} &= I\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= (A^{-1}A)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= A^{-1}\left(A\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\ &= A^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

This says that

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is impossible! Therefore, A does not have an inverse. ♠

Can a matrix have more than one inverse? It turns out that this is not the case: the following theorem shows that if A has an inverse, then the inverse is unique. We can therefore speak of "the" inverse, rather than just "an" inverse, of A.

Theorem 4.39: Uniqueness of inverse

Suppose A *is an* $n \times n$*-matrix such that both* B *and* C *are inverses of* A*. Then* $B = C$*.*

Proof. By assumption, both B and C are inverses of A, so we have $AB = I$, $BA = I$, $AC = I$, and $CA = I$. Using the associative and unit properties of matrix multiplication, we have:

$$B = BI = B(AC) = (BA)C = IC = C.$$

Therefore, $B = C$, as desired. ♠

4.5.2. Computing inverses

In Example 4.37, we verified that a matrix A had an inverse. But we did not actually compute the inverse: the inverse B was already given, and we merely checked that $AB = I$ and $BA = I$. We now explore a method for finding the inverse when it is not already known what it is.

Example 4.40: Finding the inverse of a matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

Solution. To find A^{-1}, we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can multiply these two matrices, and see that in order for this equation to be true, we must solve the systems of equations

$$\begin{aligned} x - 2y &= 1, \\ 2x - 3y &= 0, \end{aligned}$$

and

$$\begin{aligned} z - 2w &= 0, \\ 2z - 3w &= 1. \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -3 & 0 \end{array}\right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -3 & 1 \end{array}\right]$$

for the second one. Note that both systems have A as their coefficient matrix. Since both systems have the same coefficient matrix, they both require exactly the same row operations, and we can use the method of Example 1.33 to solve both systems at the same time. To do so, we create a single augmented matrix containing both of the right-hand sides:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right].$$

Then we perform row operations until the coefficient matrix is in reduced echelon form:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\simeq} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \overset{R_1 \leftarrow R_1 + 2R_2}{\simeq} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]. \tag{4.1}$$

This corresponds to the following reduced echelon forms for the two original systems of equations:

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right].$$

The solution of the first system is $x = -3$ and $y = -2$. The solution for the second system is $z = 2$ and $w = 1$. If we take the values found for x, y, z, and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

Notice that this is exactly the right-hand side in the last augmented matrix of (4.1). In other words, all we really had to do to find the inverses were the row operations in (4.1). The inverse can be read off directly from the result. ♠

The example suggests a general method for finding the inverse of a matrix, which we summarize in the following algorithm.

Algorithm 4.41: Finding the inverse of a matrix

Suppose A is an $n \times n$-matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$-matrix

$$[A \mid I].$$

If possible, do row operations until you obtain an $n \times 2n$-matrix of the form

$$[I \mid B].$$

If this can be done, then A is invertible and $A^{-1} = B$. If it is not possible (i.e., if the reduced echelon form of A has less than n pivot entries), then A is not invertible.

This algorithm shows how to find the inverse if it exists. It also tells us if A does not have an inverse.

Example 4.42: Finding the inverse of a matrix

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. *Find A^{-1} if it exists.*

Solution. We set up the augmented matrix and reduce it to reduced echelon form.

$$\begin{aligned}
[A \mid I] \quad &= \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \\
\overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\simeq} &\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \\
\overset{\substack{R_1 \leftarrow 7R_1 \\ R_3 \leftarrow -2R_3}}{\simeq} &\left[\begin{array}{ccc|ccc} 7 & 14 & 14 & 7 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 10 & 14 & 6 & 0 & -2 \end{array}\right] \\
\overset{\substack{R_1 \leftarrow R_1 + 7R_2 \\ R_3 \leftarrow R_3 + 5R_2}}{\simeq} &\left[\begin{array}{ccc|ccc} 7 & 0 & 14 & 0 & 7 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right] \\
\overset{R_1 \leftarrow R_1 - R_3}{\simeq} &\left[\begin{array}{ccc|ccc} 7 & 0 & 0 & -1 & 2 & 2 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]
\end{aligned}$$

$$\overset{\substack{R_1 \leftarrow \frac{1}{7}R_1 \\ R_2 \leftarrow -\frac{1}{2}R_2 \\ R_3 \leftarrow \frac{1}{14}R_3}}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right].$$

Notice that the last augmented matrix is of the form $[I \mid B]$, where the left-hand side is the 3×3 identity matrix. Therefore, the inverse is the 3×3-matrix on the right-hand side, given by

$$A^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{bmatrix}.$$

♠

When looking for the inverse of a matrix, it can happen that the left-hand side cannot be row reduced to the identity matrix. The following is an example of this situation.

Example 4.43: A non-invertible matrix

Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \\ 2 & -2 & 4 \end{bmatrix}$. *Find* A^{-1} *if it exists.*

Solution. We write the augmented matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{array}\right]$$

and proceed to do row operations attempting to obtain $\left[I \mid A^{-1}\right]$. After a few row operations, we have

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & -2 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right].$$

At this point, we see that the coefficient matrix has rank 2, i.e., there are only two pivot entries. This means there is no way to obtain I on the left-hand side of this augmented matrix. Hence, there is no way to complete the algorithm, and the inverse of A does not exist. ♠

If the algorithm provides an inverse, it is always possible to double-check that your answer is correct. To do so, use the method demonstrated in Example 4.37. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always ensure that you have calculated A^{-1} properly.

4.5.3. Using the inverse to solve a system of equations

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 4.20 that we can write a system of equations in matrix form, which is in the form

$$A\mathbf{x} = \mathbf{b}.$$

Suppose we find the inverse A^{-1} of the matrix A. Then we can multiply both sides of this equation by A^{-1} on the left and simplify to obtain

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Therefore we can find $\mathbf{x}$, the solution to the system, by computing $\mathbf{x} = A^{-1}\mathbf{b}$. Note that once we have found A^{-1}, we can easily get the solution for different right-hand sides (different $\mathbf{b}$). It is always just $A^{-1}\mathbf{b}$.

Example 4.44: Using the inverse to solve a system of equations

Consider the following system of equations. Use the inverse of a suitable matrix to solve this system.

$$\begin{aligned} x+z &= 1 \\ x-y+z &= 3 \\ x+y-z &= 2 \end{aligned}$$

Solution. First, we can write the system in matrix form

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \mathbf{b}.$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

From here, the solution to the system $A\mathbf{x} = \mathbf{b}$ is found by $\mathbf{x} = A^{-1}\mathbf{b}$, i.e.,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}.$$

♠

What if the right-hand side had been $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In this case, the solution would be given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

This illustrates that for a system $A\mathbf{x} = \mathbf{b}$ where A^{-1} exists, it is easy to find the solution when the vector $\mathbf{b}$ is changed.

4.5.4. Properties of the inverse

The following are some algebraic properties of matrix inverses.

Proposition 4.45: Properties of the inverse

Let A and B be $n \times n$-matrices, I the $n \times n$-identity matrix. Then the following hold.

1. *I is invertible and $I^{-1} = I$.*
2. *If A and B are invertible then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$.*
4. *If A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.*
5. *If A is invertible and p is a non-zero scalar, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$.*

4.5.5. Right and left inverses

So far, we have only talked about the inverses of square matrices. But what about matrices that are not square? Can they be invertible? It turns out that non-square matrices can never be invertible. However, they can have left inverses or right inverses.

Definition 4.46: Left and right inverses

*Let A be an $m \times n$-matrix and B an $n \times m$-matrix. We say that B is a **left inverse** of A if*

$$BA = I.$$

*We say that B is a **right inverse** of A if*

$$AB = I.$$

*If A has a left inverse, we also say that A is **left invertible**. Similarly, if A has a right inverse, we say that A is **right invertible**.*

Example 4.47: Right inverse

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that B is a right inverse, but not a left inverse, of A.

Solution. We compute

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I.$$

Therefore, B is a right inverse, but not a left inverse, of A. ♠

Recall from Definition 4.36 that B is called an **inverse** of A if it is both a left inverse and a right inverse. A crucial fact is that invertible matrices are always square.

Theorem 4.48: Invertible matrices are square

Let A be an $m \times n$-matrix.

- *If A is left invertible, then $m \geq n$.*
- *If A is right invertible, then $m \leq n$.*
- *If A is invertible, then $m = n$.*

In particular, only square matrices can be invertible.

Proof. To prove the first claim, assume that A is left invertible, i.e., assume that $BA = I$ for some $n \times m$-matrix B. We must show that $m \geq n$. Assume, for the sake of obtaining a contradiction, that this is not the case, i.e., that $m < n$. Then the matrix A has more columns than rows. It follows that the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution; let $\mathbf{x}$ be such a solution. We obtain a contradiction by a similar method as in Example 4.38. Namely, we have

$$\mathbf{x} = I\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0},$$

contradicting the fact that $\mathbf{x}$ was non-trivial. Since we got a contradiction from the assumption that $m < n$, it follows that $m \geq n$.

The second claim is proved similarly, but exchanging the roles of A and B. The third claim follows directly from the first two claims, because every invertible matrix is both left and right invertible. ♠

Of course, not all square matrices are invertible. In particular, zero matrices are not invertible, along with many other square matrices.

Exercises

Exercise 4.5.1 *For each of the following pairs of matrices, determine whether B is an inverse of A.*

(a)

$$A=\left[\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}\right], \quad B=\frac{1}{2}\left[\begin{array}{rr} 3 & -4 \\ -1 & 2 \end{array}\right].$$

(b)

$$A=\left[\begin{array}{rr} 1 & -2 \\ 4 & -7 \end{array}\right], \quad B=\left[\begin{array}{rr} -1 & 2 \\ -4 & 7 \end{array}\right].$$

(c)

$$A=\left[\begin{array}{ccc} 4 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right], \quad B=\left[\begin{array}{rrr} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -1 & 1 & 2 \end{array}\right].$$

Exercise 4.5.2 *Suppose $AB = AC$ and A is an invertible $n\times n$-matrix. Does it follow that $B = C$? Explain why or why not.*

Exercise 4.5.3 *Suppose $AB = AC$ and A is a non-invertible $n \times n$-matrix. Does it follow that $B = C$? Explain why or why not.*

Exercise 4.5.4 *For each of the following matrices, find the inverse if possible. If the inverse does not exist, explain why.*

$$A=\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \end{array}\right], \quad B=\left[\begin{array}{cc} 0 & 1 \\ 5 & 3 \end{array}\right], \quad C=\left[\begin{array}{cc} 2 & 1 \\ 3 & 0 \end{array}\right], \quad D=\left[\begin{array}{cc} 2 & 1 \\ 4 & 2 \end{array}\right], \quad E=\left[\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 5 \end{array}\right].$$

Exercise 4.5.5 *For each of the following matrices, find the inverse if possible. If the inverse does not exist, explain why.*

$$A=\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{array}\right], \quad B=\left[\begin{array}{ccc} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{array}\right], \quad C=\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{array}\right], \quad D=\left[\begin{array}{cccc} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{array}\right].$$

Exercise 4.5.6 *Let A be a 2×2 invertible matrix, with $A=\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$. Find a formula for A^{-1} in terms of a,b,c,d.*

Exercise 4.5.7 *Using the inverse of the matrix, find the solution to the systems:*

(a)

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Now give the solution in terms of a and b to

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Exercise 4.5.8 *Using the inverse of the matrix, find the solution to the systems:*

(a)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Now give the solution in terms of a, b, and c to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Exercise 4.5.9 *Show that if A is an $n \times n$ invertible matrix and X and B are $n \times 1$-matrices such that $AX = B$, then $X = A^{-1}B$.*

Exercise 4.5.10 *Prove that if A^{-1} exists and $AX = 0$ then $X = 0$.*

Exercise 4.5.11 *Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that*

$$AB(B^{-1}A^{-1}) = I \quad \text{and} \quad B^{-1}A^{-1}(AB) = I.$$

Exercise 4.5.12 *Is it possible to have matrices A and B such that $AB = I$, while $BA = 0$? If it is possible, give an example of such matrices. If it is not possible, explain why.*

Exercise 4.5.13 *Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that*

$$(ABC)(C^{-1}B^{-1}A^{-1}) = I \quad \text{and} \quad (C^{-1}B^{-1}A^{-1})(ABC) = I.$$

Exercise 4.5.14 *If A is invertible, show that A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$.*

Exercise 4.5.15 *If A is invertible, show $(A^{-1})^{-1} = A$. Hint: Use the uniqueness of inverses.*

Exercise 4.5.16 *Determine whether B is a right inverse, left inverse, both, or neither of A.*

(a)

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

(d)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Exercise 4.5.17 *Show that right inverses are not unique by giving an example of matrices A, B, C such that both B and C are right inverses of A, but $B \neq C$.*

Exercise 4.5.18 *Solve the following system of equations by using the inverse of a suitable matrix.*

$$\begin{aligned} 8x + 2y + 3z &= -1 \\ y - 2z &= 2 \\ x + z &= 1 \end{aligned}$$

Exercise 4.5.19 *Suppose that A, B, C, D are $n \times n$-matrices, and that all relevant matrices are invertible. Further, suppose that $(A+B)^{-1} = CB^{-1}$. Solve this equation for A (in terms of B and C), B (in terms of A and C), and C (in terms of A and B).*

Exercise 4.5.20 *Which of the following matrices is right invertible? Find a right inverse if one exists. If possible, find two different right inverses.*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Exercise 4.5.21 *Which of the following matrices is left invertible? Find a left inverse if one exists. If possible, find two different left inverses.*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}$$

4.6 Elementary matrices

Outcomes

A. *Use multiplication by elementary matrices to apply row operations.*

B. *Find the elementary matrix corresponding to a particular row operation.*

C. *Write the reduced echelon form of a matrix A in the form $R = UA$, where U is invertible.*

D. *Write a matrix as a product of elementary matrices.*

4.6.1. Elementary matrices and row operations

Recall from Definition 1.14 that there are three kinds of elementary row operations on matrices:

1. Switch two rows.
2. Multiply a row by a non-zero number.
3. Add a multiple of one row to another row.

The purpose of this section is to show that each of these row operations corresponds to a special type of invertible matrix called an **elementary matrix**.

Example 4.49: Elementary matrix for switching two rows

Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

What is the effect of multiplying E by an arbitrary $3 \times n$-matrix A?

Solution. Consider an arbitrary $3 \times n$-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix}.$$

We compute the product EA by the row method:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}.$$

So the effect of multiplying A by E on the left is exactly the same as switching rows 2 and 3. We say that E is the **elementary matrix for switching rows 2 and 3**. ♠

Example 4.50: Elementary matrix for multiplying a row by a non-zero number

Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the effect of multiplying E by an arbitrary $3 \times n$-matrix A?

Solution. We compute the product EA by the row method:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix}.$$

So the effect of multiplying A by E on the left is exactly the same as multiplying row 2 by the scalar k. We say that E is the **elementary matrix for multiplying row 2 by** k. ♠

Example 4.51: Elementary matrix for adding a multiple of one row to another row

Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}.$$

What is the effect of multiplying E by an arbitrary $3 \times n$-matrix A?

Solution. Once again we compute the product EA:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & \cdots & a_{3n}+ka_{2n} \end{bmatrix}.$$

So the effect of multiplying A by E on the left is exactly the same as adding k times row 2 to row 3. We say that E is the **elementary matrix for adding k times row 2 to row 3**. ♠

As these examples show, performing each type of elementary row operation is the same as multiplying (on the left) by a certain invertible matrix. These matrices are called the **elementary matrices**. In the above examples, we have only considered 3×3-elementary matrices, but they exist for other sizes too. The following definition makes this precise. It also shows how to calculate the elementary matrix corresponding to any elementary row operation.

Definition 4.52: Elementary matrices and row operations

*Let E be an $n \times n$-matrix. Then E is an **elementary matrix** if it is the result of applying one elementary row operation to the $n \times n$ identity matrix.*

Example 4.53: Finding an elementary matrix

Consider the elementary row operation of adding 5 times row 3 to row 1 of a $4 \times n$-matrix. Find the elementary matrix E corresponding to this row operation.

Solution. Following Definition 4.52, all we have to do is apply the desired row operation to the 4×4-identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \overset{R_1 \leftarrow R_1 + 5R_3}{\simeq} \quad \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E.$$

♠

We can double-check that multiplying E by any $4 \times n$-matrix does indeed have the desired effect:

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix} = \begin{bmatrix} a_{11}+5a_{31} & a_{12}+5a_{32} & \cdots & a_{1n}+5a_{3n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}.$$

The fact that this always works is the content of the following theorem.

Theorem 4.54: Multiplication by an elementary matrix and row operations

Performing any of the three elementary row operations on a matrix A is the same as taking the product EA, where E is the elementary matrix obtained by applying the desired row operation to the identity matrix.

4.6.2. Inverses of elementary matrices

Suppose we have applied a row operation to a matrix A. Consider the row operation required to return A to its original form, i.e., to undo the row operation. It turns out that this action is described by the inverse of an elementary matrix. The following theorem ensures that the inverse of each elementary matrix is itself an elementary matrix.

Theorem 4.55: Inverses of elementary matrices

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I. E^{-1} is obtained by performing the row operation which would carry E back to I.

- If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j.
- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.

- If E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j.

Example 4.56: Inverse of an elementary matrix

Find E^{-1}, where E is the elementary matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Solution. E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence, E^{-1} is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

♠

4.6.3. Elementary matrices and reduced echelon forms

Suppose an $m \times n$-matrix A is row reduced to its reduced echelon form. By tracking each row operation completed, this row reduction can be performed through multiplication by elementary matrices. The following theorem uses this fact.

Theorem 4.57: The form $R = UA$

Let A be any $m \times n$-matrix and let R be its reduced echelon form. Then there exists an invertible $m \times m$-matrix U such that

$$R = UA.$$

Specifically, U can be computed as the product (from right to left) of the elementary matrices of all row operations used to convert A to reduced echelon form.

Example 4.58: The form $R = UA$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}.$$

Find the reduced echelon form of A and write it in the form $R = UA$, where U is invertible.

Solution. To find the reduced echelon form R, we row reduce A. For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\simeq} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

The corresponding elementary matrix is $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Next, subtract 2 times the first row from the third row.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \overset{R_3 \leftarrow R_3 - 2R_1}{\simeq} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding elementary matrix is $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Notice that the resulting matrix is R, the required reduced echelon form of A. We can then write

$$\begin{aligned} R &= E_2 E_1 A \\ &= UA. \end{aligned}$$

It remains to compute U:

$$U = E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

We can verify that $R = UA$ holds for this matrix U:

$$UA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R.$$

♠

While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to R. The following theorem provides an alternate way to find the matrix U.

Theorem 4.59: Finding the matrix U

Let A be an $m \times n$-matrix and let R be its reduced echelon form. Then $R = UA$, where U is an invertible $m \times m$-matrix found by forming the augmented matrix $[A \mid I]$ and row reducing to $[R \mid U]$.

Let's revisit the above example using the process outlined in Theorem 4.59.

Example 4.60: The form $R = UA$, revisited

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. *Use the process of Theorem 4.59 to find U such that $R = UA$.*

Solution. First, we set up the augmented matrix $[A \mid I]$:

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right].$$

Now, we row reduce until the left-hand side is in reduced echelon form:

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\simeq} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - 2R_1}{\simeq} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right].$$

The left-hand side of this augmented matrix is R, and the right-hand side is U. Comparing this to the matrices R and U we found in Example 4.58, we see that the same matrices are obtained regardless of which process is used. ♠

4.6.4. Writing an invertible matrix as a product of elementary matrices

Recall from Algorithm 4.41 that an $n \times n$-matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using elementary row operations. Combining this with our discussion of elementary matrices we see that A is invertible if and only if it can be written as a product of elementary matrices. This is the content of the following theorem.

Theorem 4.61: Product of elementary matrices

Let A be an $n \times n$-matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Proof. If A is an invertible $n \times n$-matrix, then its reduced echelon form is the $n \times n$ identity matrix I. By Theorem 4.57, we can write $I = UA$, where $U = E_k \cdots E_2 E_1$ is a product of elementary matrices. Then

$$A = U^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

By Theorem 4.55, if E_i is an elementary matrix, then so is E_i^{-1}. Therefore, A has been written as a product of elementary matrices. Conversely, if A can be written as a product of elementary matrices, then A is clearly invertible, because each elementary matrix is invertible. ♠

Example 4.62: Product of elementary matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. *Write A as a product of elementary matrices.*

Solution. Following the process of Theorem 4.61, we first row-reduce A to its reduced echelon form, recording each row operation as an elementary matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\simeq} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{with elementary matrix} \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \overset{R_1 \leftarrow R_1 - R_2}{\simeq} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{with elementary matrix} \quad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \overset{R_3 \leftarrow R_3 + 2R_2}{\simeq} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with elementary matrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Notice that the reduced echelon form of A is I. Hence $I = UA$ where $U = E_3E_2E_1$. It follows that $A = U^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$, and so we have succeeded in writing A as a product of elementary matrices

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

In particular, it follows that A is invertible. ♠

4.6.5. More properties of inverses

In this section, we will use elementary matrices to prove a useful theorem about the inverse of a square matrix. We start with an observation about the echelon form of a right invertible matrix.

Lemma 4.63: Echelon form of a right invertible matrix

Suppose that A is right invertible. Then the reduced echelon form of A does not have a row of zeros.

Proof. Let R be the reduced echelon form of A. Then by Theorem 4.57, we can write $R = UA$ for some invertible square matrix U. By assumption, we have $AB = I$, and therefore

$$R(BU^{-1}) = (UA)(BU^{-1}) = U(AB)U^{-1} = UIU^{-1} = UU^{-1} = I.$$

If R had a row of zeros, then so would the product $R(BU^{-1})$. But since the identity matrix I does not have a row of zeros, neither does R. ♠

Theorem 4.64: Right invertible square matrices are invertible

Suppose A and B are square matrices such that $AB = I$. Then it follows that $BA = I$, and therefore $B = A^{-1}$. In particular, a square matrix is right invertible if and only if it is left invertible if and only if it is invertible.

Proof. Assume A and B are square matrices such that $AB = I$. Let R be the reduced echelon form of A. Then by Theorem 4.57, we can write $R = UA$ where U is an invertible matrix. Since $AB = I$, we know by Lemma 4.63 that R does not have a row of zeros. Since R is a square reduced echelon form with no row of zeros, each column must be a pivot column, and it follows that $R = I$. Hence, $UA = I$, and therefore A is left invertible. Moreover, we have

$$B \;=\; IB \;=\; (UA)B \;=\; U(AB) \;=\; UI \;=\; U,$$

and therefore $B = U$. It follows that $BA = UA = I$, as claimed.

To prove the last claim, note that we just proved that for square matrices, $AB = I$ implies $BA = I$. Therefore, every right inverse of A is also a left inverse, and therefore an inverse. But of course, we also have that $BA = I$ implies $AB = I$. This is just the same theorem, with the roles of A and B interchanged. Therefore, every left inverse of A is also a right inverse. ♠

This theorem is very useful, because it allows us to test only one of the products AB or BA in order to check that B is the inverse of A, saving us half of the work. It is important to stress, however, that this only works for *square* matrices. As we saw in Example 4.47, non-square matrices can be right invertible without being left invertible, or vice versa.

Exercises

Exercise 4.6.1 *For each of the following pairs of matrices, suppose a row operation is applied to A and the result is B. Find the elementary matrix E that represents this row operation.*

(a)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}.$$

(c)

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}.$$

Exercise 4.6.2 *For each of the following pairs of matrices, suppose a row operation is applied to A and the result is B.*

- *Find the elementary matrix E such that $EA = B$.*
- *Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.*

(a)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 5 & 1 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}.$$

(c)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}.$$

(d)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}.$$

Exercise 4.6.3 *Find the reduced echelon form of each of the following matrices A, and write it in the form $R = UA$ where U is invertible.*

$$(a) \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 2 & 6 & -2 \end{bmatrix}, \quad (b) \begin{bmatrix} 0 & 1 \\ 3 & -1 \\ 2 & 6 \end{bmatrix}, \quad (c) \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 2 \end{bmatrix}, \quad (d) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Exercise 4.6.4 *Write each of the following matrices as a product of elementary matrices, if possible, or else say why it is not possible.*

$$(a) \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad (b) \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \quad (c) \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}, \quad (d) \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 7 \\ 1 & 1 & 3 \end{bmatrix}.$$

4.7 The transpose

Outcomes

A. *Calculate the transpose of a matrix.*

B. *Determine whether a matrix is symmetric, antisymmetric, or neither.*

C. *Manipulate algebraic expressions involving the transpose of matrices.*

Another important operation on matrices is that of taking the **transpose**. The transpose of a matrix is obtained by turning the rows into columns and vice versa.

Definition 4.65: The transpose of a matrix

Let A be an $m \times n$-matrix. Then the **transpose** *of A, denoted A^T, is the $n \times m$-matrix whose (i,j)-entry is the (j,i)-entry of A.*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Example 4.66: The transpose of a matrix

Find the transpose of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 5 & 4 \end{bmatrix}.$$

Solution. The transpose is

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 4 \end{bmatrix}.$$

Notice that A is a 2×3-matrix, while A^T is a 3×2-matrix. ♠

We have already used a special case of the transpose since Chapter 2, when we wrote $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ as a space-saving notation for the column vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The transpose of a matrix satisfies the following properties:

Proposition 4.67: Properties of the transpose

Let A and B be matrices of appropriate sizes, and r a scalar. Then the following hold.

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$.
3. $(rA)^T = rA^T$.
4. $(AB)^T = B^T A^T$.
5. $0^T = 0$.
6. $I^T = I$.
7. $(A^{-1})^T = (A^T)^{-1}$, *if A is invertible.*

Recall that a column vector is the same thing as a $n \times 1$-matrix. Using the transpose, we can make precise the connection between the dot product and the matrix product. Namely, let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

by column vectors. Then

$$\mathbf{v} \cdot \mathbf{w} \;=\; v_1 w_1 + \ldots + v_n w_n \;=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \;=\; \mathbf{v}^T \mathbf{w}.$$

In other words, the dot product of column vectors $\mathbf{v}$ and $\mathbf{w}$ is the same thing as the matrix product $\mathbf{v}^T \mathbf{w}$.

We can also use the notion of transpose to define what it means for a matrix to be **symmetric** and **antisymmetric**.

Definition 4.68: Symmetric and antisymmetric matrices

An $n \times n$-matrix A is said to be ***symmetric*** *if $A^T = A$. It is said to be* ***antisymmetric*** *(sometimes also called* ***skew symmetric****) if $A^T = -A$.*

Example 4.69: Symmetric and antisymmetric matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 5 & -3 \\ 1 & 3 & 0 \end{bmatrix}.$$

Then A is symmetric because $A^T = A$, B is antisymmetric because $B^T = -B$, and C is neither symmetric nor antisymmetric because C^T is equal to neither C nor $-C$.

Exercises

Exercise 4.7.1 *Let* $X = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ *and* $Y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. *Find* $X^T Y$ *and* XY^T *if possible.*

Exercise 4.7.2 *Consider the matrices*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Find the following if possible. If it is not possible explain why.

(a) $-3A^T$.

(b) $3B - A^T$.

(c) $E^T B$.

(d) EE^T.

(e) $B^T B$.

(f) CA^T.

(g) $D^T BE$.

Exercise 4.7.3 *Which of the following matrices are symmetric, antisymmetric, both, or neither?*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercise 4.7.4 *Suppose* A *is a matrix that is both symmetric and antisymmetric. Show that* $A = 0$.

Exercise 4.7.5 *Let* A *be an* $n \times n$*-matrix. Show* A *equals the sum of a symmetric and an antisymmetric matrix.* ***Hint:*** *Show that* $\frac{1}{2}(A^T + A)$ *is symmetric and then consider using this as one of the matrices.*

Exercise 4.7.6 *Show that the main diagonal of every antisymmetric matrix consists of only zeros. Recall that the main diagonal consists of every entry of the matrix which is of the form* a_{ii}.

Exercise 4.7.7 *Show that for* $m \times n$*-matrices* A, B *and scalars* r, s*, the following holds:*

$$(rA + sB)^T = rA^T + sB^T.$$

Exercise 4.7.8 *Let* A *be a real* $m \times n$*-matrix and let* $\mathbf{u} \in \mathbb{R}^n$ *and* $\mathbf{v} \in \mathbb{R}^m$. *Show* $(A\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (A^T \mathbf{v})$.

Exercise 4.7.9 *Show that if* A *is an invertible* $n \times n$*-matrix, then so is* A^T *and* $(A^T)^{-1} = (A^{-1})^T$.

Exercise 4.7.10 *Suppose* A *is invertible and symmetric. Show that* A^{-1} *is symmetric.*

4.8 Matrix arithmetic modulo p

Outcomes

A. *Perform matrix operations over the field $\mathbb{Z}_p$.*

In Section 1.8, you learned that most of linear algebra can be done over scalars from any field K, and not just the real numbers. You also learned that $\mathbb{Z}_p$, the set of integers modulo p, is a field whenever p is a prime number.

Indeed, all of the operations on matrices that we covered in this chapter make sense over any field: addition, scalar multiplication, matrix multiplication, inverses, elementary matrices, and the transpose.

Example 4.70: A matrix product over $\mathbb{Z}_5$

Compute the matrix product AB over the field $\mathbb{Z}_5$, where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 2 & 2 \end{bmatrix}.$$

Solution. For example, the $(1,1)$-entry of AB is calculated by multiplying the first row of A by the first column of B, i.e.,

$$c_{11} \;=\; 1\cdot 3+0\cdot 4+4\cdot 2 \;=\; 3+0+3 \;=\; 1,$$

keeping in mind that all arithmetic operations are done in $\mathbb{Z}_5$. We repeat the same for the other entries and obtain

$$AB \;=\; \begin{bmatrix} 1 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 2 & 2 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}.$$

♠

Example 4.71: An inverse over $\mathbb{Z}_7$

Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 6 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

with scalars in the field $\mathbb{Z}_7$.

Solution. We use exactly the method of Algorithm 4.41, i.e., we set up the augmented matrix $[A \mid I]$ and reduce it to reduced echelon form. The only thing we have to keep in mind is that all operations are done modulo 7. Also, as usual, instead of dividing by a scalar, we must multiply by its inverse.

$$
\begin{aligned}
[A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 5 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \\
&\overset{R_3 \leftarrow R_3+2R_1}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right] \\
&\overset{R_2 \leftrightarrow R_3}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 6 & 2 & 0 & 1 & 0 \end{array}\right] \\
&\overset{R_3 \leftarrow R_3+R_2}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 \end{array}\right] \\
&\overset{R_3 \leftarrow 2^{-1}R_3 = 4R_3}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right] \\
&\overset{R_1 \leftarrow R_1-4R_2}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right] \\
&\overset{R_1 \leftarrow R_1-2R_3}{\simeq} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 2 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right].
\end{aligned}
$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 5 & 6 & 2 \\ 2 & 0 & 1 \\ 1 & 4 & 4 \end{bmatrix}.$$

As usual, we can double-check that we didn't make any mistakes by calculating

$$AA^{-1} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 6 & 2 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 2 \\ 2 & 0 & 1 \\ 1 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

So indeed, we have calculated the inverse correctly. ♠

Exercises

Exercise 4.8.1 *Let*

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

and compute the following over $\mathbb{Z}_{11}$*:*

(a) $3A$,

(b) A^2,

(c) AB,

(d) BC,

(e) C^{-1}.

4.9 Application: Cryptography

Cryptography is about encoding a message so that it is hard for a third party to read. The original message is called the **plaintext** and the encrypted message is called the **ciphertext**. The process of turning a plaintext into the corresponding ciphertext is called **encryption**, and the process of turning a ciphertext into the corresponding plaintext is called **decryption**. An encryption and decryption method is also called a **cipher**. Modern ciphers are designed in such a way that the cipher itself is not secret, but the encryption depends on a secret **key**. A cipher should be designed so that decryption is easy for a person who knows the key, but difficult for everybody else. The art of designing ciphers is called **cryptography**, and the art of breaking ciphers is called **cryptanalysis**.

In order to be able to define ciphers using algebraic operations, we start by encoding strings as sequences of numbers. To that end, we assign a number to each letter of the alphabet, as well as the special symbols "space", "comma", and "period", according to the following scheme.

Space	`A`	`B`	`C`	`D`	...	`Z`	Comma	Period
0	1	2	3	4	...	26	27	28

In practical applications, one would probably use a larger set of symbols and a standard encoding such as ASCII or UTF-8. But the above 29 symbols will be sufficient for our purposes. It will also come in handy that 29 is prime.

Example 4.72: Representing strings as sequences of numbers

Convert the string "Attack at dawn" to a sequence of numbers. Convert the sequence of numbers $9,0,12,9,11,5,0,3,15,4,5,19,28$ *to a string.*

Solution. We have `A` $=1$, `T` $=20$, `T` $=20$, `A` $=1$, `C` $=3$, `K` $=11$, Space $=0$, and so on. Continuing in this way, the encoding of "Attack at dawn" is $1,20,20,1,3,11,0,1,20,0,4,1,23,14$. Conversely, we have $9=$ `I`, $0=$ Space, $12=$ `L`, $9=$ `I`, $11=$ `K`, $5=$ `E`, and so on. We find that the decoded string is "I like codes."

♠

There are many different ways to define ciphers. Some of the oldest known ciphers date back thousands of years. An example of such a "classic" cipher is a **substitution cipher**, where each letter of the alphabet is replaced by a different letter, for example `A` $\mapsto$ `D`, `B` $\mapsto$ `E`, and so on. Substitution ciphers have the property that changing one letter of the plaintext always changes exactly one letter of the ciphertext. This is not a desirable property, because it makes the cipher easy to break. Therefore, modern ciphers are designed to satisfy a property called **diffusion**: changing one letter of the plaintext should change many letters of the ciphertext.

In a **block cipher**, the plaintext is first divided into blocks of equal size, and then each block is encrypted separately. The **block size** is the number of plaintext symbols in each block. If the length of the plaintext is not divisible by the block size, we pad the final block with additional spaces. In the context of a block cipher, the diffusion property means that changing one symbol of a plaintext block potentially affects every symbol of the ciphertext block. The following is an example of a block cipher.

Definition 4.73: Hill cipher

*The **Hill cipher** of block size n has as its key an invertible $n \times n$-matrix A with scalars from $\mathbb{Z}_{29}$. Each ciphertext block $c_1, \ldots, c_n$ is computed from the corresponding plaintext block $p_1, \ldots, p_n$ by matrix multiplication modulo $\mathbb{Z}_{29}$:*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}.$$

*The matrix A is called the **encryption matrix** of the cipher. It inverse A^{-1} is called the **decryption matrix**.*

Example 4.74: Hill cipher: encryption

Encrypt the message "Meet me tomorrow" using the Hill cipher with block size 3 and encryption matrix

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}.$$

Solution. We start by converting the message "Meet me tomorrow" to a sequence of scalars. We have $\texttt{M} = 13$, $\texttt{E} = 5$, and so on. The encoded plaintext is $13,5,5,20,0,13,5,0,20,15,13,15,18,18,15,23$. Next, we divide the plaintext into blocks of length 3. Since the length of the plaintext is not a multiple of three, we pad the final block with spaces, i.e., with zeros.

$$\text{Plaintext blocks:} \quad (13,5,5),\ (20,0,13),\ (5,0,20),\ (15,13,15),\ (18,18,15),\ (23,0,0).$$

To compute the ciphertext, we regard each plaintext block as a 3-dimensional column vector and multiply by the encryption matrix A. All calculations are done modulo 29. For example, for the first block, we have

$$A \begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 11 \\ 9 \end{bmatrix},$$

so the first ciphertext block is $(22,11,9)$. We repeat the same with the remaining plaintext blocks.

$$A\begin{bmatrix}20\\0\\13\end{bmatrix}=\begin{bmatrix}24\\9\\17\end{bmatrix},\quad A\begin{bmatrix}5\\0\\20\end{bmatrix}=\begin{bmatrix}1\\28\\16\end{bmatrix},\quad A\begin{bmatrix}15\\13\\15\end{bmatrix}=\begin{bmatrix}10\\17\\26\end{bmatrix},$$

$$A\begin{bmatrix}18\\18\\15\end{bmatrix}=\begin{bmatrix}7\\2\\15\end{bmatrix},\quad A\begin{bmatrix}23\\0\\0\end{bmatrix}=\begin{bmatrix}17\\11\\23\end{bmatrix}.$$

Therefore, we have found the following ciphertext blocks:

$$\text{Ciphertext blocks:}\quad (22,11,9),\ (24,9,17),\ (1,28,16),\ (10,17,26),\ (7,2,15),\ (17,11,23).$$

Finally, we can convert the ciphertext to a list of symbols: "`VKIXIQA.PJQZGBOQKW`". ♠

Example 4.75: Hill cipher: decryption

Decrypt the message "`RNOLFPHHCIGH DE`" *using the Hill cipher with block size* 3 *and encryption matrix*

$$A=\begin{bmatrix}2&4&1\\3&1&5\\1&3&2\end{bmatrix}.$$

Solution. The process is analogous to encryption, except that we need to use the decryption matrix A^{-1} instead of A. We first calculate A^{-1}, keeping in mind that scalars are from the field $\mathbb{Z}_{29}$. The method is the same as in Example 4.71; we skip the individual steps in the interest of brevity.

$$[A\mid I]=\left[\begin{array}{ccc|ccc}2&4&1&1&0&0\\3&1&5&0&1&0\\1&3&2&0&0&1\end{array}\right]\simeq\dots\simeq\left[\begin{array}{ccc|ccc}1&0&0&23&20&11\\0&1&0&4&17&28\\0&0&1&26&8&11\end{array}\right]=\left[I\mid A^{-1}\right].$$

Next, we convert the 15 ciphertext symbols "`RNOLFPHHCIGH DE`" to scalars and divide them into blocks of length 3:

$$\text{Ciphertext blocks:}\quad (18,14,15),\ (12,6,16),\ (8,8,3),\ (9,7,8),\ (0,4,5).$$

Now we decrypt each ciphertext block by a matrix multiplication with A^{-1}.

$$A^{-1}\begin{bmatrix}18\\14\\15\end{bmatrix}=\begin{bmatrix}18\\5\\20\end{bmatrix},\quad A^{-1}\begin{bmatrix}12\\6\\16\end{bmatrix}=\begin{bmatrix}21\\18\\14\end{bmatrix},\quad A^{-1}\begin{bmatrix}8\\8\\3\end{bmatrix}=\begin{bmatrix}0\\20\\15\end{bmatrix},$$

$$A^{-1}\begin{bmatrix}9\\7\\8\end{bmatrix}=\begin{bmatrix}0\\2\\1\end{bmatrix},\quad A^{-1}\begin{bmatrix}0\\4\\5\end{bmatrix}=\begin{bmatrix}19\\5\\0\end{bmatrix}.$$

This yields the following plaintext blocks:

$$\text{Plaintext blocks:}\quad (18,5,20),\ (21,18,14),\ (0,20,15),\ (0,2,1),\ (19,5,0).$$

Converting these back to letters, and omitting the trailing space, we find that the plaintext is "return to base". ♠

It is important to note that, despite its good diffusion properties, the Hill cipher is not secure. The cipher has many weaknesses. For one, because $A\mathbf{0}=\mathbf{0}$, a block of spaces in the plaintext will always be encrypted as a block of spaces in the ciphertext, regardless of the encryption matrix A. More importantly, the cipher is subject to a so-called **known plaintext attack**. If an eavesdropper intercepts some ciphertext for which a small amount of the corresponding plaintext happens to be known, it is immediately possible to recover the key and therefore decrypt the rest of the ciphertext. Carrying out this attack only requires some basic knowledge of linear algebra. The following example illustrates how this is done.

Example 4.76: Cryptanalysis of the Hill cipher: known plaintext attack

Eve intercepts the following encrypted message sent by Alice:

"`EFNOR.AHIFNEPL.TSZS,RSKT.ZBBRFVUPFVZLFHNTV`".

Eve knows that Alice uses a Hill cipher with block length 3, but she does not know the secret encryption matrix. Eve also knows that Alice begins all of her correspondence with "My dear love". Decrypt the message.

Solution. The first three blocks of the ciphertext are "`EFNOR.AHI`", i.e.,

$$\text{Ciphertext blocks:}\quad (5,6,14),\ (15,18,28),\ (1,8,9).$$

Eve also knows that the first three blocks of the plaintext are "`MY DEAR L`", i.e.,

$$\text{Plaintext blocks:}\quad (13,25,0),\ (4,5,1),\ (18,0,12).$$

These facts allow Eve to deduce the following information about the unknown decryption matrix A^{-1}:

$$A^{-1}\begin{bmatrix}5\\6\\14\end{bmatrix}=\begin{bmatrix}13\\25\\0\end{bmatrix},\quad A^{-1}\begin{bmatrix}15\\18\\28\end{bmatrix}=\begin{bmatrix}4\\5\\1\end{bmatrix},\quad A^{-1}\begin{bmatrix}1\\8\\9\end{bmatrix}=\begin{bmatrix}18\\0\\12\end{bmatrix}.$$

Since Eve remembers the column method of matrix multiplication, she knows that these three equations can be written as a single equation in matrix form:

$$A^{-1}\begin{bmatrix}5&15&1\\6&18&8\\14&28&9\end{bmatrix}=\begin{bmatrix}13&4&18\\25&5&0\\0&1&12\end{bmatrix}.$$

Note that this equation is of the form $A^{-1}C=P$. (Here, C stands for "ciphertext" and P for "plaintext"). Multiplying both sides of the equation by C^{-1} on the right, we get $A^{-1}=PC^{-1}$. Thus, assuming that C is invertible, Eve can easily compute the decryption matrix A^{-1}. Eve computes:

$$C^{-1}=\begin{bmatrix}5&15&1\\6&18&8\\14&28&9\end{bmatrix}^{-1}=\begin{bmatrix}19&8&23\\0&5&2\\22&1&0\end{bmatrix}.$$

This allows Eve to compute the decryption matrix:

$$A^{-1} = PC^{-1} = \begin{bmatrix} 13 & 4 & 18 \\ 25 & 5 & 0 \\ 0 & 1 & 12 \end{bmatrix} \begin{bmatrix} 19 & 8 & 23 \\ 0 & 5 & 2 \\ 22 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 26 & 17 \\ 11 & 22 & 5 \\ 3 & 17 & 2 \end{bmatrix}.$$

Armed with the decryption matrix A^{-1}, Eve can now decrypt Alice's entire message, using the same method as in Example 4.75. The plaintext is "My dear love, run away with me at midnight". ♠

As the example shows, the Hill cipher is not secure at all. The main problem is that the cipher is *linear*, i.e., each component of a ciphertext block is a simple linear combination of the components of the plaintext block. This linearity property enables Eve to break the cipher by solving a system of linear equations.

For this reason, all modern block ciphers have a non-linear component. Often this takes the form of so-called **S-boxes**. An S-box is an operation that scrambles the symbols of the alphabet in a non-linear way. For example, consider the following S-box, which is an operation from $\mathbb{Z}_{29}$ to $\mathbb{Z}_{29}$:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$S(x)$	17	9	27	2	20	12	21	26	16	18	4	24	23	7	19	14	28	29	1	15	10	22	6	5	25	11	13	3	8

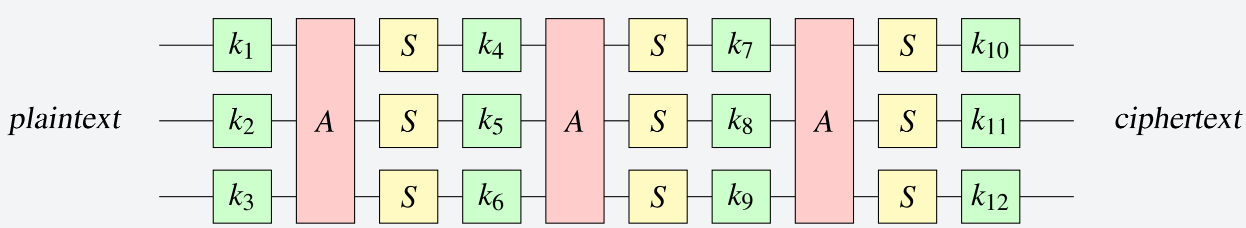
The inputs of the S-box are shown in the top row, and the corresponding outputs in the bottom row. For example, this S-box maps the input 7 to the output 26. We write $S(7) = 26$.

Definition 4.77: A toy block cipher

Consider the following block cipher on the alphabet $\mathbb{Z}_{29}$ with block size 3. The key consists of 12 elements $k_1, \ldots, k_{12}$ of $\mathbb{Z}_{29}$. To encrypt a plaintext block, regard the block as a 3-dimensional column vector. Then repeat the following steps 3 times. All operations are carried out modulo 29.

- *Key mixing: add the next three components of the key to the components of the vector.*
- *Diffusion: multiply the vector by the fixed 3×3-matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.
- *S-box application: apply the S-box to each component of the vector.*

Finally, apply one more key mixing step at the end. The resulting vector is the ciphertext block. The cipher can be visualized as follows:



Note that the three basic steps (key mixing, diffusion, and S-box application) are repeated several times; each such repetition is called a **round** of the block cipher. The more rounds a block cipher has, the better its diffusion and non-linearity properties. The final round is short: it only consists of a key mixing step, with no final diffusion or S-box application. The reason is that performing a final diffusion and S-box application would not add anything to the security of the cipher. An attacker could simply undo these last two steps, since they do not depend on the key.

The matrix A is called the **diffusion matrix** of the cipher. Note that, unlike for the Hill cipher, the matrix A is fixed once and for all and is not part of the key. Instead, the key consists of scalars that are added to the current block at the beginning of each round.

Example 4.78: Toy block cipher: encryption

Encrypt the message "I like math" using the block cipher of Definition 4.77 and the key $1,1,3,3,5,5,7,7,9,9,11,11$.

Solution. We first represent the plaintext as a sequence of blocks, padding the final block with zeros:

$$\text{Plaintext blocks:} \quad (9,0,12),\ (9,11,5),\ (0,13,1),\ (20,8,0).$$

To encrypt the first block, we start with the vector $[9,0,12]^T$ and apply the following steps:

Round 1:

- Key mixing: the first three components of the key are $1,1,3$. We add them to the plaintext.

$$\begin{bmatrix} 9 \\ 0 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 15 \end{bmatrix}.$$

- Diffusion: multiply by the matrix A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 9 \end{bmatrix}.$$

- S-box application: apply the S-box to each component of the vector.

$$\begin{bmatrix} S(28) \\ S(3) \\ S(9) \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 18 \end{bmatrix}.$$

Round 2:

- Key mixing: the next three components of the key are $3,5,5$.

$$\begin{bmatrix} 8 \\ 2 \\ 18 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 23 \end{bmatrix}.$$

- Diffusion:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 7 \\ 23 \end{bmatrix} = \begin{bmatrix} 7 \\ 28 \\ 8 \end{bmatrix}.$$

- S-box application:

$$\begin{bmatrix} S(7) \\ S(28) \\ S(8) \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ 16 \end{bmatrix}.$$

Round 3:

- Key mixing: the next three components of the key are $7, 7, 9$.

$$\begin{bmatrix} 26 \\ 8 \\ 16 \end{bmatrix} + \begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \\ 25 \end{bmatrix}.$$

- Diffusion:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 15 \\ 25 \end{bmatrix} = \begin{bmatrix} 22 \\ 19 \\ 20 \end{bmatrix}.$$

- S-box application:

$$\begin{bmatrix} S(22) \\ S(19) \\ S(20) \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 10 \end{bmatrix}.$$

Round 4 (the final round is abbreviated):

- Key mixing: the next three components of the key are $9, 11, 11$.

$$\begin{bmatrix} 6 \\ 15 \\ 10 \end{bmatrix} + \begin{bmatrix} 9 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 15 \\ 26 \\ 21 \end{bmatrix}.$$

Therefore, the first ciphertext block is $(15, 26, 21)$. We repeat the same procedure with the remaining plaintext blocks, and obtain the following ciphertext blocks:

$$\text{Ciphertext blocks:} \quad (15,26,21),\ (7,24,1),\ (2,16,23),\ (7,20,22).$$

The corresponding ciphertext is "`OZUGXABPWGTV`". ♠

Are ciphers like this actually used in the real world? The answer is yes. While the cipher of Definition 4.77 is greatly simplified, it has the same basic structure as modern real-world block ciphers (such as AES, the Advanced Encryption Standard). Naturally, these real-world ciphers differ in some details, such as the alphabet size, the block size, the number of rounds, the design of the S-boxes, the way the key is computed, and the precise order in which the operations are applied. However, their basic structure is very similar to our toy cipher, and indeed, all such ciphers rely on key mixing, diffusion, and non-linear S-boxes as their key components.

For example, AES uses an alphabet size of 256 instead of 29 (i.e., it operates on bytes, rather than elements of $\mathbb{Z}_{29}$). Although $\mathbb{Z}_{256}$ is not a field (because 256 is not prime), it nevertheless turns out that there exists a field with 256 elements, and AES uses it for its algebraic operations. Our toy cipher's block size of 3 is much too small to achieve effective diffusion; modern real-world ciphers use block sizes between 16 and 32 bytes (128 to 256 bits). The design of the S-boxes is a bit of a black art; at minimum, they must be designed to withstand two common types of cryptanalysis known as **linear cryptanalysis** and **differential cryptanalysis**. Among other things, this means that the S-box should be "as far from linear" as possible.

A detailed discussion of the design and cryptanalysis of modern block ciphers is far beyond the scope of this book, but we hope that you have gotten a taste of this fascinating subject, and the role that linear algebra over finite fields plays in it.

Exercises

Exercise 4.9.1 *Encrypt the message "Rendezvous at dawn" using the Hill cipher with block size 3 and encryption matrix*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Exercise 4.9.2 *Decrypt the message* "`ERM DXYBJUWW.JWQLD,HL`" *using the Hill cipher with block size* 3 *and encryption matrix*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Exercise 4.9.3 *Eve intercepts the following encrypted message sent by Bob:*

"`TGVXKHGSW,JU,JHYJSCDSBQIRPEV`"

Eve knows that Alice uses a Hill cipher with block length 2, but she does not know the secret encryption matrix. Eve also knows that Bob begins all of his letters with "Hello". Decrypt the message.

Exercise 4.9.4 *Encrypt the message "Lost contact" using the block cipher of Definition 4.77 and the key* $2,3,4,1,1,1,5,5,5,4,3,2$.

Exercise 4.9.5 *Decrypt the message* "`NRQEUAPOM GLFN`", *using the block cipher of Definition 4.77 and the key* $1,1,1,2,2,2,3,3,3,4,4,4$. ***Hint:*** *To decrypt, we must perform all the encryption steps in reverse. To undo a key mixing step, we subtract the relevant key components. To undo an S-box application, we apply the S-box in reverse. To undo a diffusion step, multiply by the inverse of the diffusion matrix.*

5. Spans, linear independence, and bases in $\mathbb{R}^n$

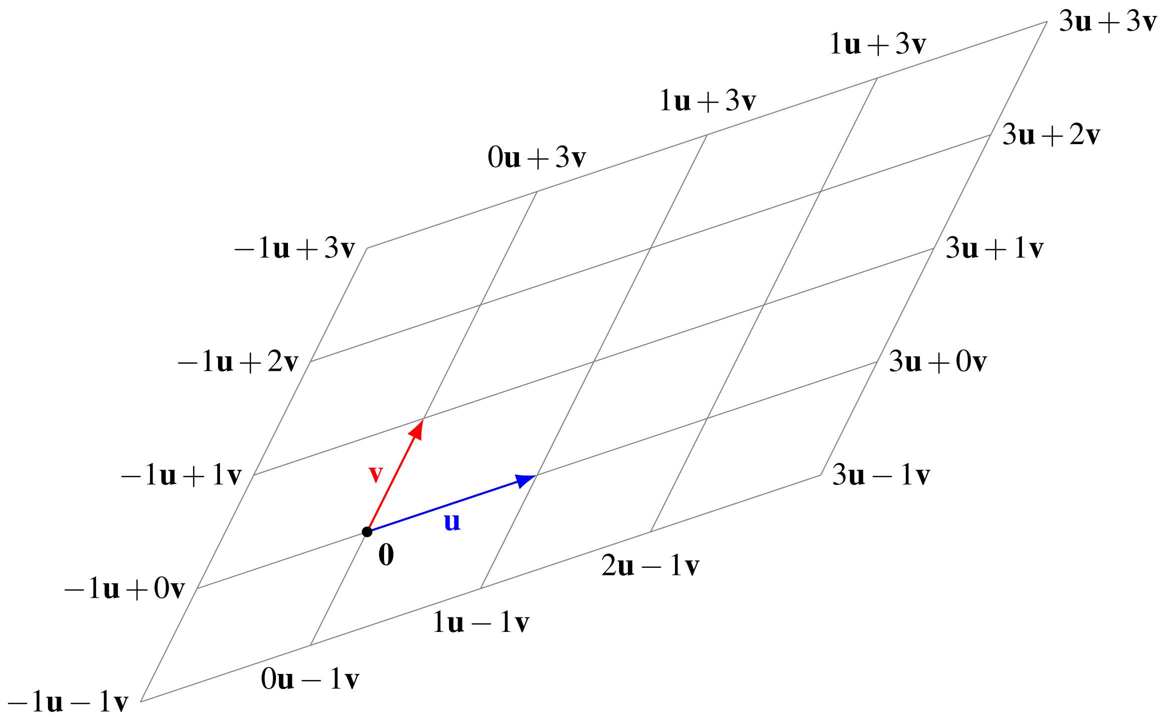
5.1 Spans

Outcomes

A. *Determine the span of a set of vectors.*

B. *Determine if a vector is contained in a specified span.*

Let $\mathbf{u}$ and $\mathbf{v}$ be two non-parallel vectors in $\mathbb{R}^n$. We can picture the set of their linear combinations as follows:



As the picture shows, the linear combinations of $\mathbf{u}$ and $\mathbf{v}$ form a 2-dimensional plane through the origin. We say that this plane is **spanned** by the vectors $\mathbf{u}$ and $\mathbf{v}$. This concept generalizes to more than two vectors. For example, three vectors may span a 3-dimensional space (although sometimes, they span only a 2-dimensional space, or even a line). This motivates the following definition.

Definition 5.1: Span of a set of vectors

The set of all linear combinations of the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ *in* $\mathbb{R}^n$ *is known as the* ***span*** *of these vectors and is written as* $\operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$. *Using set notation, we can write*

$$\operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\} = \{a_1\mathbf{u}_1+\ldots+a_k\mathbf{u}_k \mid a_1,\ldots,a_k \in \mathbb{R}\}.$$

Example 5.2: Vectors in a span

Let $\mathbf{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}3\\2\\1\end{bmatrix}$. *Which of the following vectors are elements of* $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$?

$$(a)\quad \mathbf{w} = \begin{bmatrix}2\\3\\4\end{bmatrix}, \qquad (b)\quad \mathbf{z} = \begin{bmatrix}1\\1\\2\end{bmatrix}.$$

Solution. (a) For a vector to be in $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$, it must be a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Therefore, $\mathbf{w} \in \operatorname{span}\{\mathbf{u},\mathbf{v}\}$ if and only if we can find find scalars a,b such that $a\mathbf{u}+b\mathbf{v} = \mathbf{w}$. We must therefore solve the equation

$$a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}.$$

We write this as an augmented matrix and solve.

$$\left[\begin{array}{cc|c}1&3&2\\1&2&3\\1&1&4\end{array}\right] \simeq \ldots \simeq \left[\begin{array}{cc|c}1&0&5\\0&1&-1\\0&0&0\end{array}\right].$$

The solution is $a = 5$ and $b = -1$. This means that $\mathbf{w} = 5\mathbf{u}+(-1)\mathbf{v}$. Therefore, $\mathbf{w}$ is an element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$.

(b) We repeat the same method with the vector $\mathbf{z}$. This time, we have to find a,b such that $a\mathbf{u}+b\mathbf{v} = \mathbf{z}$. The system of equations is

$$\left[\begin{array}{cc|c}1&3&1\\1&2&1\\1&1&2\end{array}\right] \simeq \ldots \simeq \left[\begin{array}{cc|c}1&3&1\\0&-1&0\\0&0&1\end{array}\right],$$

which is inconsistent. Therefore, there is no solution. We conclude that $\mathbf{z}$ is not an element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$.

♠

Example 5.3: Describing the span

Describe the span of the vectors $\mathbf{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ *in* $\mathbb{R}^3$.

Solution. Let $\mathbf{w} = [x,y,z]^T$ be any vector. Proceeding as in the previous example, we know that $\mathbf{w}$ is an element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ if and only if the equation

$$a\begin{bmatrix}1\\1\\1\end{bmatrix}+b\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

is consistent. Note that the variables of this equation are a,b; we regard x,y,z as constants for the moment. We write the augmented matrix of this system and reduce to echelon form:

$$\left[\begin{array}{cc|c}1&3&x\\1&2&y\\1&1&z\end{array}\right]\overset{\substack{R_2\leftarrow R_2-R_1\\R_2\leftarrow R_3-R_1}}{\simeq}\left[\begin{array}{cc|c}1&3&x\\0&-1&y-x\\0&-2&z-x\end{array}\right]\overset{R_3\leftarrow R_3-2R_2}{\simeq}\left[\begin{array}{cc|c}1&3&x\\0&-1&y-x\\0&0&(z-x)-2(y-x)\end{array}\right],$$

From the echelon form, we see that the system is consistent if and only if $(z-x)-2(y-x)=0$, or equivalently $x-2y+z=0$. Therefore, the vector $\mathbf{w}$ is in $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ if and only if $x-2y+z=0$. In other words, the span of $\mathbf{u}$ and $\mathbf{v}$ is the plane $x-2y+z=0$. ♠

Example 5.4: Span of redundant vectors

Let $\mathbf{u}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}3\\2\\1\end{bmatrix}$, *and* $\mathbf{w}=\begin{bmatrix}11\\8\\5\end{bmatrix}$. *Show that* $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}=\operatorname{span}\{\mathbf{u},\mathbf{v}\}$.

Solution. Observe that $\mathbf{w}=2\mathbf{u}+3\mathbf{v}$. Therefore, $\mathbf{w}$ is already in the span of $\mathbf{u}$ and $\mathbf{v}$. Two sets are equal if they have the same elements, i.e., each element of the first set is an element of the second set and vice versa. Therefore, to show $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}=\operatorname{span}\{\mathbf{u},\mathbf{v}\}$, we must show (a) that every element of $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ and (b) vice versa.

(a) Let $\mathbf{z}$ be an arbitrary element of $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$. Then, by definition of span, there exist scalars a,b,c such that

$$\mathbf{z} = a\mathbf{u}+b\mathbf{v}+c\mathbf{w}.$$

But as observed above, we have $\mathbf{w}=2\mathbf{u}+3\mathbf{v}$, and therefore we can also write

$$\begin{aligned}\mathbf{z} &= a\mathbf{u}+b\mathbf{v}+c(2\mathbf{u}+3\mathbf{v})\\ &= (a+2c)\mathbf{u}+(b+3c)\mathbf{v}.\end{aligned}$$

It follows that $\mathbf{z}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and therefore, $\mathbf{z}\in\operatorname{span}\{\mathbf{u},\mathbf{v}\}$.

(b) Clearly every linear combination of $\mathbf{u}$ and $\mathbf{v}$ is also a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, namely, taking the coefficient of $\mathbf{w}$ to be 0. Therefore, every element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ is an element of $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$.

Because we have shown that every element of $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an element of $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ and vice versa, it follows that $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ and $\operatorname{span}\{\mathbf{u},\mathbf{v}\}$ are the same set of vectors. ♠

In the situation of the last example, we say that the vector $\mathbf{w}$ is **redundant**; it does not contribute anything to $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$. Geometrically, the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in a plane. Since the two vectors $\mathbf{u}$ and $\mathbf{v}$

are sufficient to span this plane, the third vector $\mathbf{w}$ is not really needed. We will study this situation more systematically in the next section.

Example 5.5: Span of the empty set

We talked about the span of k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. What if $k = 0$? What is the span of an empty set of vectors?

Solution. Consider what happens when we compute the sum of three numbers. We usually write this as $b_1 + b_2 + b_3$. We can also compute the sum of three numbers by starting from 0 and then adding each of the three numbers to it. I.e., the sum can be computed as $0 + b_1 + b_2 + b_3$. Similarly, we can write the sum of two numbers as $0 + b_1 + b_2$, and the sum of just one number as $0 + b_1$. Continuing the pattern, it follows that the sum of zero numbers should be 0:

$$\begin{array}{ll} \text{Sum of 3 numbers:} & 0 + b_1 + b_2 + b_3. \\ \text{Sum of 2 numbers:} & 0 + b_1 + b_2. \\ \text{Sum of 1 numbers:} & 0 + b_1. \\ \text{Sum of 0 numbers:} & 0. \end{array}$$

The sum of zero numbers is also called the **empty sum**. It is equal to the unit of addition, i.e., 0. By an analogous argument, the empty sum of vectors is equal to the unit of vector addition, i.e., to the zero vector $\mathbf{0}$. In general, if we have k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, the span consists of all vectors of the form $a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k$, which is a sum of k vectors. In case $k = 0$, the span consists only of the empty sum, i.e., the zero vector $\mathbf{0}$, which we also call the **empty linear combination**. Therefore, the span of the empty set of vectors is $\{\mathbf{0}\}$.

♠

Exercises

Exercise 5.1.1 *Consider the vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Which of the following vectors are in $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_4\}$*? For each vector that is in the span, exhibit a linear combination of* $\mathbf{u}_1, \ldots, \mathbf{u}_4$ *that equals this vector.*

$$(a) \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad (b) \quad \mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}, \qquad (c) \quad \mathbf{z} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}.$$

Exercise 5.1.2 *Describe the span of the vectors* $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ *and* $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ *in* $\mathbb{R}^3$.

Exercise 5.1.3 *Describe the span of the following vectors in* $\mathbb{R}^4$*:*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 5.1.4 *Let* $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. *Show that* $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\} = \operatorname{span}\{\mathbf{u},\mathbf{v}\}$.

Exercise 5.1.5 *Suppose* $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ *is a set of vectors from* $\mathbb{R}^n$. *Show that* $\mathbf{0} \in \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$.

Exercise 5.1.6 *In this exercise, we use scalars from the field* $\mathbb{Z}_5$ *of integers modulo 5 instead of real numbers (see Section 1.8, "Fields"). Consider the vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

Which of the following vectors are in $\operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_4\}$*? For each vector that is in the span, exhibit a linear combination of* $\mathbf{u}_1,\ldots,\mathbf{u}_4$ *that equals this vector.*

$$(a) \quad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \qquad (b) \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad (c) \quad \mathbf{z} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

5.2 Linear independence

Outcomes

A. *Find the redundant vectors in a set of vectors.*

B. *Determine whether a set of vectors is linearly independent.*

C. *Find a linearly independent subset of a set of spanning vectors.*

D. *Write a vector as a unique linear combination of a set of linearly independent vectors.*

5.2.1. Redundant vectors and linear independence

In Example 5.4, we encountered three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\} = \operatorname{span}\{\mathbf{u},\mathbf{v}\}$. If this happens, then the vector $\mathbf{w}$ does not contribute anything to the span of $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$, and we say that $\mathbf{w}$ is **redundant**. The following definition generalizes this notion.

Definition 5.6: Redundant vectors, linear dependence, and linear independence

*Consider a sequence of k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. We say that the vector $\mathbf{u}_j$ is **redundant** if it can be written as a linear combination of earlier vectors in the sequence, i.e., if*

$$\mathbf{u}_j = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \ldots + a_{j-1} \mathbf{u}_{j-1}$$

*for some scalars $a_1, \ldots, a_{j-1}$. We say that the sequence of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is **linearly dependent** if it contains one or more redundant vectors. Otherwise, we say that the vectors are **linearly independent**.*

Example 5.7: Redundant vectors

Find the redundant vectors in the following sequence of vectors. Are the vectors linearly independent?

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_6 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \end{bmatrix}.$$

Solution.

- The vector $\mathbf{u}_1$ is redundant, because it is a linear combination of earlier vectors. (Although there are no earlier vectors, recall from Example 5.5 that the empty sum of vectors is equal to the zero vector $\mathbf{0}$. Therefore, $\mathbf{u}_1$ is indeed an (empty) linear combination of earlier vectors.)

- The vector $\mathbf{u}_2$ is not redundant, because it cannot be written as a linear combination of $\mathbf{u}_1$. This is because the system of equations

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 2 \\ 0 & 2 \\ 0 & 3 \end{array}\right]$$

  has no solution.

- The vector $\mathbf{u}_3$ is not redundant, because it cannot be written as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. This is because the system of equations

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \end{array}\right]$$

  has no solution.

- The vector $\mathbf{u}_4$ is redundant, because $\mathbf{u}_4 = \mathbf{u}_2 + \mathbf{u}_3$.

- The vector $\mathbf{u}_5$ is not redundant, because This is because the system of equations

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 3 & 2 \\ 0 & 3 & 1 & 4 & 3 \end{array}\right]$$

has no solution.

- The vector $\mathbf{u}_6$ is redundant, because $\mathbf{u}_6 = \mathbf{u}_2 + 2\mathbf{u}_3 - \mathbf{u}_5$.

In summary, the vectors $\mathbf{u}_1$, $\mathbf{u}_4$, and $\mathbf{u}_6$ are redundant, and the vectors $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_5$ are not. It follows that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_6$ are linearly dependent. ♠

5.2.2. The casting-out algorithm

The last example shows that it can be a lot of work to find the redundant vectors in a sequence of k vectors. Doing so in the naive way require us to solve up to k systems of linear equations! Fortunately, there is a much faster and easier method, the so-called *casting-out algorithm*.

Algorithm 5.8: Casting-out algorithm

Input: *a list of* k *vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$.

Output: *the set of indices* j *such that* $\mathbf{u}_j$ *is redundant.*

Algorithm: *Write the vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_k$ *as the columns of an* $n \times k$*-matrix, and reduce to echelon form. Every* non-pivot *column, if any, corresponds to a redundant vector.*

Example 5.9: Casting-out algorithm

Use the casting-out algorithm to find the redundant vectors among the vectors from Example 5.7.

Solution. Following the casting-out algorithm, we write the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_6$ as the columns of a matrix and reduce to echelon form.

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & 3 & 2 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 \end{bmatrix} \simeq \ldots \simeq \begin{bmatrix} 0 & \textcircled{1} & 1 & 2 & 0 & 3 \\ 0 & 0 & \textcircled{1} & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns are columns 2, 3, and 5. The non-pivot columns are columns 1, 4, and 6. Therefore, the vectors $\mathbf{u}_1$, $\mathbf{u}_4$, and $\mathbf{u}_6$ are redundant. Note that this is the same answer we got in Example 5.7. ♠

The above version of the casting-out algorithm only tells us which of the vectors (if any) are redundant, but it does not give us a specific way to write the redundant vectors as linear combinations of previous vectors. However, we can easily get this additional information if we reduce the matrix all the way to reduced echelon form. We call this version of the algorithm the *extended casting-out algorithm*.

Algorithm 5.10: Extended casting-out algorithm

Input: *a list of k vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k \in \mathbb{R}^n$.*

Output: *the set of indices j such that $\mathbf{u}_j$ is redundant, and a set of coefficients for writing each redundant vector as a linear combination of previous vectors.*

Algorithm: *Write the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ as the columns of an $n\times k$-matrix, and reduce to reduced echelon form. Every* non-pivot *column, if any, corresponds to a redundant vector. If $\mathbf{u}_j$ is a redundant vector, then the entries in the j^{th} column of the reduced echelon form are coefficients for writing $\mathbf{u}_j$ as a linear combination of previous non-redundant vectors.*

Example 5.11: Extended casting-out algorithm

Use the casting-out algorithm to find the redundant vectors among the vectors from Example 5.7, and write each redundant vector as a linear combination of previous non-redundant vectors.

Solution. Once again, we write the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_6$ as the columns of a matrix. This time we use the extended casting-out algorithm, which means we reduce the matrix to reduced echelon form instead of echelon form.

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & 3 & 2 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 \end{bmatrix} \simeq \cdots \simeq \begin{bmatrix} 0 & \textcircled{1} & 0 & 1 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As before, the non-pivot columns are columns 1, 4, and 6, and therefore, the vectors $\mathbf{u}_1$, $\mathbf{u}_4$, and $\mathbf{u}_6$ are redundant. The non-redundant vectors are $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_5$. Moreover, the entries in the sixth column are 1, 2, and -1. Note that this means that the sixth column can be written as 1 times the second column plus 2 times the third column plus (-1) times the fourth column. The same coefficients can be used to write $\mathbf{u}_6$ as a linear combination of previous *non-redundant* columns, namely:

$$\mathbf{u}_6 = 1\,\mathbf{u}_2 + 2\,\mathbf{u}_3 - 1\,\mathbf{u}_5.$$

Also, the entries in the fourth column are 1 and 1, which are the coefficients for writing $\mathbf{u}_4$ as a linear combination of previous non-redundant columns, namely:

$$\mathbf{u}_4 = 1\,\mathbf{u}_2 + 1\,\mathbf{u}_3.$$

Finally, there are no non-zero entries in the first column. This means that $\mathbf{u}_1$ is the empty linear combination

$$\mathbf{u}_1 = \mathbf{0}.$$

♠

5.2.3. Alternative characterization of linear independence

Our definition of redundant vectors depends on the order in which the vectors are written. This is because each redundant vector must be a linear combination of *earlier* vectors in the sequence. For example, in the sequence of vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 11 \\ 8 \\ 5 \end{bmatrix},$$

the vector $\mathbf{w}$ is redundant, because it is a linear combination of earlier vectors $\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$. Neither $\mathbf{u}$ nor $\mathbf{v}$ are redundant. On the other hand, in the sequence of vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 11 \\ 8 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

$\mathbf{v}$ is redundant because $\mathbf{v} = \frac{1}{3}\mathbf{w} - \frac{2}{3}\mathbf{u}$, but neither $\mathbf{u}$ nor $\mathbf{w}$ are redundant. Note that none of the vectors have changed; only the order in which they are written is different. Yet $\mathbf{w}$ is the redundant vector in the first sequence, and $\mathbf{v}$ is the redundant vector in the second sequence.

Because we defined linear independence in terms of the absence of redundant vectors, you may suspect that the concept of linear independence also depends on the order in which the vectors are written. However, this is not the case. The following theorem gives an alternative characterization of linear independence that is more symmetric (it does not depend on the order of the vectors).

Theorem 5.12: Characterization of linear independence

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors. Then $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent if and only if the homogeneous equation

$$a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k = \mathbf{0}$$

has only the trivial solution.

Proof. Let A be the $n \times k$-matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_k$. We know from the theory of homogeneous systems that the system $a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k = \mathbf{0}$ has no non-trivial solution if and only if every column of the echelon form of A is a pivot column. By the casting-out algorithm, this is the case if and only if none of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are redundant, i.e., if and only if the vectors are linearly independent. ♠

Example 5.13: Characterization of linear independence

Use the method of Theorem 5.12 to determine whether the following vectors are linearly independent in $\mathbb{R}^4$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 1 \end{bmatrix}.$$

Solution. We must check whether the equation

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} + a_4 \begin{bmatrix} 2 \\ 3 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution. If it does, the vectors are linearly dependent. On the other hand, if there is only the trivial solution, the vectors are linearly independent. We write the augmented matrix and solve:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 7 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{cccc|c} \textcircled{1} & 0 & 1 & 2 & 0 \\ 0 & \textcircled{1} & 1 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{2} & 0 \end{array}\right].$$

Since every column is a pivot column, there are no free variables; the system of equations has a unique solution, which is $a_1 = a_2 = a_3 = a_4 = 0$, i.e., the trivial solution. Therefore, the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_4$ are linearly independent. ♠

Example 5.14: Characterization of linear independence

Use the method of Theorem 5.12 to determine whether the following vectors are linearly independent in $\mathbb{R}^3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},$$

Solution. As in the previous example, we must check whether the equation $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{0}$ has a non-trivial solution. Once again, we write the augmented matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 0 & 0 \\ 0 & \textcircled{2} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since column 3 is not a pivot column, a_3 is a free variable. Therefore, the system has a non-trivial solution, and the vectors are linearly dependent.

With a small amount of extra work, we can find an actual non-trivial solution of $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{0}$. All we have to do is set $a_3 = 1$ and do a back substitution. We find that $(a_1, a_2, a_3) = (2, -2, 1)$ is a solution. In other words,

$$2\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}.$$

We can also use this information to write $\mathbf{u}_3$ as a linear combination of previous vectors, namely, $\mathbf{u}_3 = -2\mathbf{u}_1 + 2\mathbf{u}_2$. ♠

The characterization of linear independence in Theorem 5.12 is mostly useful for theoretical reasons. However, it can also help in solving problems such as the following.

Example 5.15: Related sets of vectors

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in $\mathbb{R}^n$. Are the vectors $\mathbf{u}+\mathbf{v}$, $2\mathbf{u}+\mathbf{w}$, and $\mathbf{v}-5\mathbf{w}$ linearly independent?

Solution. By Theorem 5.12, to check whether the vectors are linearly independent, we must check whether the equation

$$a(\mathbf{u}+\mathbf{v}) + b(2\mathbf{u}+\mathbf{w}) + c(\mathbf{v}-5\mathbf{w}) = \mathbf{0} \tag{5.1}$$

has non-trivial solutions. If it does, the vectors are linearly dependent, if it does not, they are linearly independent. We can simplify the equation as follows:

$$(a+2b)\mathbf{u}+(a+c)\mathbf{v}+(b-5c)\mathbf{w}=\mathbf{0}. \tag{5.2}$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, we know, again by Theorem 5.12, that equation (5.2) only has the trivial solution. Therefore,

$$\begin{aligned} a+2b &= 0, \\ a+c &= 0, \\ b-5c &= 0. \end{aligned}$$

We can solve this system of three equations in three variables, and we find that it has the unique solution $a=b=c=0$. Therefore, $a=b=c=0$ is the only solution to equation (5.1), which means that the vectors $\mathbf{u}+\mathbf{v}$, $2\mathbf{u}+\mathbf{w}$, and $\mathbf{v}-5\mathbf{w}$ are linearly independent. ♠

5.2.4. Properties of linear independence

The following are some properties of linearly independent sets.

Proposition 5.16: Properties of linear independence

1. ***Linear independence and reordering.*** *If a sequence $\mathbf{u}_1,\ldots,\mathbf{u}_k$ of k vectors is linearly independent, then so is any reordering of the sequence (i.e., whether or not the vectors are linearly independent does not depend on the order in which the vectors are written down).*
2. ***Linear independence of a subset.*** *If $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are linearly independent, then so are $\mathbf{u}_1,\ldots,\mathbf{u}_j$ for any $j<k$.*
3. ***Linear independence and dimension.*** *Let $\mathbf{u}_1,\ldots,\mathbf{u}_k$ be a sequence of k vectors in $\mathbb{R}^n$. If $k>n$, then the vectors are linearly dependent (i.e., not linearly independent).*

Proof.

1. This follows from Theorem 5.12, because whether or not the equation $a_1\mathbf{u}_1+\ldots+a_k\mathbf{u}_k=\mathbf{0}$ has a non-trivial solution does not depend on the order in which the vectors are written.
2. If one of the vectors in the sequence $\mathbf{u}_1,\ldots,\mathbf{u}_j$ were redundant, then it would be redundant in the longer sequence $\mathbf{u}_1,\ldots,\mathbf{u}_k$ as well.
3. Let A be the $n\times k$-matrix that has the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ as its columns and suppose that $k>n$. Then the rank of A is at most n, so the echelon form of A has some non-pivot columns. Therefore, the system $a_1\mathbf{u}_1+\ldots+a_k\mathbf{u}_k=\mathbf{0}$ has non-trivial solutions, and the vectors are linearly dependent by Theorem 5.12.

♠

Example 5.17: Linear dependence

Are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution. Since these are 3 vectors in $\mathbb{R}^2$, they are linearly dependent by Proposition 5.16. No calculation is necessary. ♠

5.2.5. Linear independence and linear combinations

In general, there is more than one way of writing a given vector as a linear combination of some spanning vectors. For example, consider

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

We can write $\mathbf{v}$ in many different ways as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_3$, for example

$$\begin{aligned} \mathbf{v} &= \quad -\mathbf{u}_1 + 2\mathbf{u}_3, \\ \mathbf{v} &= \quad \mathbf{u}_2 + \mathbf{u}_3, \\ \mathbf{v} &= \quad \mathbf{u}_1 + 2\mathbf{u}_2, \\ \mathbf{v} &= 2\mathbf{u}_1 + 3\mathbf{u}_2 - \mathbf{u}_3. \end{aligned}$$

However, when the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, this does not happen. In this case, the linear combination is always unique, as the following theorem shows.

Theorem 5.18: Unique linear combination

Assume $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent. Then every vector $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ can be written as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in a unique way.

Proof. We already know that every vector $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ can be written as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$, because that is the definition of span. So what must be proved is the uniqueness. Suppose, therefore, that there are two ways of writing $\mathbf{v}$ as such a linear combination, i.e., that

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_k\mathbf{u}_k \quad \text{and} \\ \mathbf{v} &= b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_k\mathbf{u}_k. \end{aligned}$$

Subtracting one equation from the other, we get

$$\mathbf{0} = (a_1 - b_1)\mathbf{u}_1 + (a_2 - b_2)\mathbf{u}_2 + \ldots + (a_k - b_k)\mathbf{u}_k.$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, we know by Theorem 5.12 that the last equation only has the trivial solution, i.e., $a_1 - b_1 = 0$, $a_2 - b_2 = 0$, ..., $a_k - b_k = 0$. It follows that $a_1 = b_1$, $a_2 = b_2$, ..., $a_k = b_k$.

We have shown that any two ways of writing $\mathbf{v}$ as a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are equal. Therefore, there is only one way of doing so. ♠

5.2.6. Removing redundant vectors

Consider the span of some vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$. As we just saw in the previous subsection, the span is especially nice when the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are linearly independent, because in that case, every element $\mathbf{v}$ of the span can be *uniquely* written in the form $\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k$.

But what if we have a span of some vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ that are not linearly independent? It turns out that we can always find some linearly independent vectors that span the same set. In fact, this can be done by simply removing the redundant vectors from $\mathbf{u}_1,\ldots,\mathbf{u}_k$. This is the subject of the following theorem.

Theorem 5.19: Removing redundant vectors

Let $\mathbf{u}_1,\ldots,\mathbf{u}_k$ be a sequence of vectors, and suppose that $\mathbf{u}_{j_1},\ldots,\mathbf{u}_{j_\ell}$ is the subsequence of vectors that is obtained by removing all of the redundant vectors. Then $\mathbf{u}_{j_1},\ldots,\mathbf{u}_{j_\ell}$ are linearly independent and

$$\operatorname{span}\left\{\mathbf{u}_{j_1},\ldots,\mathbf{u}_{j_\ell}\right\} = \operatorname{span}\left\{\mathbf{u}_1,\ldots,\mathbf{u}_k\right\}.$$

Proof. Remove the redundant vectors one by one, from right to left. Each time a redundant vector is removed, the span does not change; the proof of this is similar to Example 5.4. Moreover, the resulting sequence of vectors span$\left\{\mathbf{u}_{j_1},\ldots,\mathbf{u}_{j_\ell}\right\}$ is linearly independent, because if any of these vectors were a linear combination of earlier ones, then it would have been redundant in the original sequence of vectors, and would have therefore been removed. ♠

Example 5.20: Finding a linearly independent set of spanning vectors

Find a subset of $\{\mathbf{u}_1,\ldots,\mathbf{u}_4\}$ that is linearly independent and has the same span as $\{\mathbf{u}_1,\ldots,\mathbf{u}_4\}$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -6 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}.$$

Solution. We use the casting-out algorithm to find the redundant vectors:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ -2 & 4 & 2 & 2 \\ 3 & -6 & 1 & 5 \end{bmatrix} \simeq \ldots \simeq \begin{bmatrix} \textcircled{1} & -2 & 1 & 3 \\ 0 & 0 & \textcircled{2} & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the redundant vectors are $\mathbf{u}_2$ and $\mathbf{u}_4$. We remove them ("cast them out") and are left with $\mathbf{u}_1$ and $\mathbf{u}_3$. Therefore, by Theorem 5.19, $\{\mathbf{u}_1,\mathbf{u}_3\}$ is linearly independent and $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_3\} = \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_4\}$. ♠

Exercises

Exercise 5.2.1 *Which of the following vectors are redundant? If there are redundant vectors, write each of them as a linear combination of previous vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}.$$

Exercise 5.2.2 *Which of the following vectors are redundant? If there are redundant vectors, write each of them as a linear combination of previous vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ -4 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 2 \\ -3 \\ -6 \end{bmatrix}.$$

Exercise 5.2.3 *Use the method of Theorem 5.12 to determine whether the following vectors are linearly independent. If they are linearly dependent, find a non-trivial linear combination of the vectors that is equal to* **0**.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -3 \\ -4 \\ -2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}.$$

Exercise 5.2.4 *Use the method of Theorem 5.12 to determine whether the following vectors are linearly independent. If they are linearly dependent, find a non-trivial linear combination of the vectors that is equal to* **0**.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 6 \\ 7 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 5.2.5 *Are the following vectors linearly independent? If not, write one of them as a linear combination of the others.*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Exercise 5.2.6 *Find a linearly independent set of vectors that has the same span as the given vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

Exercise 5.2.7 *Find a linearly independent set of vectors that has the same span as the given vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 5.2.8 *Here are some vectors in* $\mathbb{R}^4$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Explain why these vectors cannot possibly be linearly independent. Then obtain a linearly independent subset of these vectors that has the same span as these vectors.

Exercise 5.2.9 *Here are some vectors in* $\mathbb{R}^4$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 3 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 2 \\ -9 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Explain why these vectors cannot possibly be linearly independent. Then find a non-trivial linear combination of these vectors that equals $\mathbf{0}$.

Exercise 5.2.10 *Here are some vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 5 \\ 7 \\ -10 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 12 \\ 17 \\ -24 \end{bmatrix}.$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 5.2.11 *Here are some vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 5.2.12 *In this exercise, we use scalars from the field* $\mathbb{Z}_3$ *of integers modulo* 3 *instead of real numbers (see Section 1.8, "Fields"). Use the extended casting-out algorithm to determine which of the following vectors are redundant. If there are redundant vectors, write each of them as a linear combination of previous vectors.*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 5.2.13 *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be linearly independent vectors in* $\mathbb{R}^n$. *Are the vectors* $\mathbf{u}+\mathbf{v}$, $2\mathbf{u}+\mathbf{w}$, *and* $\mathbf{w}-2\mathbf{v}$ *linearly independent?*

Exercise 5.2.14 *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be linearly independent vectors in* $\mathbb{R}^n$. *Are the vectors* $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{w}$, *and* $\mathbf{w}+\mathbf{v}$ *linearly independent?*

Exercise 5.2.15 *Suppose* A *is an* $m \times n$*-matrix and* $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *is a linearly independent set of vectors in* $\mathbb{R}^m$. *Now suppose* $A\mathbf{z}_i = \mathbf{w}_i$. *Show* $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ *is also linearly independent.*

5.3 Subspaces of $\mathbb{R}^n$

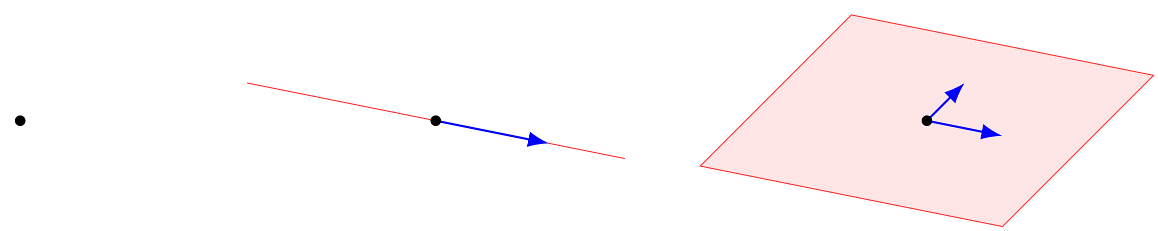
Outcomes

A. *Determine whether a subset of* $\mathbb{R}^n$ *is a subspace.*

B. *Recognize that spans are subspaces of* $\mathbb{R}^n$.

C. *Recognize that solution sets of homogeneous systems of equations are subspaces of* $\mathbb{R}^n$.

As we saw earlier, the span of 0 vectors in $\mathbb{R}^n$ is a point, namely the set $\{\mathbf{0}\}$. The span of one non-zero vector is a line through the origin, and the span of two linearly independent vectors is a plane through the origin.



Span of 0 vectors: a point — Span of one vector: a line — Span of two vectors: a plane

We also call these sets, respectively, a 0-*dimensional subspace*, a 1-*dimensional subspace*, and a 2-*dimensional subspace* of $\mathbb{R}^n$. The purpose of this section is to generalize this concept of subspace to arbitrary dimensions.

Definition 5.21: Subspace

A subset V *of* $\mathbb{R}^n$ *is called a* ***subspace*** *of* $\mathbb{R}^n$ *if*

1. V *contains the zero vector of* $\mathbb{R}^n$, *i.e.,* $\mathbf{0} \in V$;

2. V *is closed under addition, i.e., for all* $\mathbf{u}, \mathbf{w} \in V$, *we have* $\mathbf{u}+\mathbf{w} \in V$;

3. V *is closed under scalar multiplication, i.e., for all* $\mathbf{u} \in V$ *and scalars* k, *we have* $k\mathbf{u} \in V$.

Notice that the subset $V = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$ (called the **zero subspace**). Every line or plane through the origin is a subspace. Moreover, the entire space $\mathbb{R}^n$ is a subspace of itself. A subspace that is not the entire space $\mathbb{R}^n$ is referred to as a **proper subspace** of $\mathbb{R}^n$.

Proposition 5.22: Spans are subspaces

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$. Then $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a subspace of $\mathbb{R}^n$.

Proof. Let $S = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. To verify that S is a subspace of $\mathbb{R}^n$, we must check that the three conditions of Definition 5.21 hold.

- We have $\mathbf{0} \in S$ because $\mathbf{0} = 0\mathbf{u}_1 + \ldots + 0\mathbf{u}_k$.
- Suppose $\mathbf{u}, \mathbf{w} \in S$. By definition of span, there exist scalars $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ such that $\mathbf{u} = a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k$ and $\mathbf{w} = b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k$. Therefore,
$$\mathbf{u} + \mathbf{w} = (a_1 + b_1)\mathbf{u}_1 + \ldots + (a_k + b_k)\mathbf{u}_k.$$
It follows that $\mathbf{u} + \mathbf{w} \in S = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, so that S is closed under addition.
- Suppose $\mathbf{u} \in S$ and t is a scalar. Then by definition of span, there exists scalars $a_1, \ldots, a_k$ such that $\mathbf{u} = a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k$. Then
$$t\mathbf{u} = (ta_1)\mathbf{u}_1 + \ldots + (ta_k)\mathbf{u}_k,$$
and thus $t\mathbf{u} \in S$. It follows that S is closed under scalar multiplication.

Since $S = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ satisfies all three conditions, it follows that it is a subspace of $\mathbb{R}^n$. ♠

Example 5.23: A line in $\mathbb{R}^3$

In $\mathbb{R}^3$, let L be the line through the origin that is parallel to the vector

$$\mathbf{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}.$$

Show that L is a subspace of $\mathbb{R}^3$.

Solution. The line L is simply the span of the vector $\mathbf{d}$, i.e., $L = \operatorname{span}\{\mathbf{d}\}$. Therefore, it is a subspace by Proposition 5.22. ♠

Proposition 5.24: Solution space of a homogeneous system of equations

Consider a homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$-matrix. Then the set of solutions,

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\},$$

*is a subspace of $\mathbb{R}^n$. It is called the **solution space** of the system.*

Proof. To show that V is a subspace of $\mathbb{R}^n$, we check the three conditions of Definition 5.21.

- We have $\mathbf{0} \in V$ because $A\mathbf{0} = \mathbf{0}$.
- To show that V is closed under addition, suppose $\mathbf{u}, \mathbf{w} \in V$. Then by definition of V, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. Therefore,
$$A(\mathbf{u}+\mathbf{w}) = A\mathbf{u}+A\mathbf{w} = \mathbf{0}+\mathbf{0} = \mathbf{0}.$$
It follows that $\mathbf{u}+\mathbf{w} \in V$.
- To show that V is closed under scalar multiplication, suppose $\mathbf{u} \in V$ and t is a scalar. Then by definition of V, we have $A\mathbf{u} = \mathbf{0}$. It follows that
$$A(t\mathbf{u}) = t(A\mathbf{u}) = t\mathbf{0} = \mathbf{0}.$$
Therefore, $t\mathbf{u} \in V$.

Since the solution space $V = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$ satisfies all three conditions, it is a subspace of $\mathbb{R}^n$. ♠

Example 5.25: A plane in $\mathbb{R}^3$

Show that the plane $2x+3y-z=0$ is a subspace of $\mathbb{R}^3$.

Solution. Since $2x+3y-z=0$ is a homogeneous equation, its solution space
$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; 2x+3y-z=0 \right\}$$
is a subspace of $\mathbb{R}^3$ by Proposition 5.24. ♠

Example 5.26: Non-examples

Which of the following are subspaces of $\mathbb{R}^3$?

(a) The line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(b) The plane $2x+3y-z=5$.

(c) The set of vectors
$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x,y,z \geq 0 \right\}.$$

Solution. None of them are subspaces. Neither the line (a) nor the plane (b) contains the origin $\mathbf{0}$, so they fail to satisfy the first condition of subspaces. The set of vectors in (c) contains $\mathbf{0}$. It is also closed under addition. However, it fails to be closed under scalar multiplication. For example, let $\mathbf{u} = [1,1,1]^T$. Then $\mathbf{u} \in W$, but $(-1)\mathbf{u} \notin W$. ♠

Exercises

Exercise 5.3.1 *Which of the following sets are subspaces of* $\mathbb{R}^3$*? Explain.*

(a) $V_1 = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;\middle|\; |u_1| \leq 4 \right\}$.

(b) $V_2 = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;\middle|\; u_i \geq 0 \text{ for each } i = 1,2,3 \right\}$.

(c) $V_3 = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;\middle|\; u_3 + u_1 = 2u_2 \right\}$.

(d) $V_4 = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;\middle|\; u_3 \geq u_1 \right\}$.

(e) $V_5 = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;\middle|\; u_3 = u_1 = 0 \right\}$.

Exercise 5.3.2 *Let* $\mathbf{w} \in \mathbb{R}^4$ *be a given fixed vector. Let*

$$M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \mathbf{w} \bullet \mathbf{u} = 0 \right\}.$$

Is M a subspace of $\mathbb{R}^4$*? Explain.*

Exercise 5.3.3 *Let* $\mathbf{w}, \mathbf{v}$ *be given vectors in* $\mathbb{R}^4$ *and define*

$$M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \mathbf{w} \bullet \mathbf{u} = 0 \text{ and } \mathbf{v} \bullet \mathbf{u} = 0 \right\}.$$

Is M a subspace of $\mathbb{R}^4$*? Explain.*

Exercise 5.3.4 *In this exercise, we use scalars from the field* $\mathbb{Z}_2$ *of integers modulo 2 instead of real numbers (see Section 1.8, "Fields"). Which of the following sets are subspaces of* $(\mathbb{Z}_2)^3$*?*

(a) $V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$

(b) $V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$

(c) $V_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$

(d) $V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

Exercise 5.3.5 *Suppose V, W are subspaces of $\mathbb{R}^n$. Let $V \cap W$ be the set of all vectors that are in both V and W. Show that $V \cap W$ is also a subspace.*

Exercise 5.3.6 *Let V be a subset of $\mathbb{R}^n$. Show that V is a subspace if and only if it is non-empty and the following condition holds: for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $a, b \in \mathbb{R}$,*

$$a\mathbf{u} + b\mathbf{v} \in V.$$

Exercise 5.3.7 *Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$, and let $S = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Show that S is the* smallest *subspace of $\mathbb{R}^n$ that contains $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Specifically, this means you have to show: if V is any other subspace of $\mathbb{R}^n$ such that $\mathbf{u}_1, \ldots, \mathbf{u}_k \in V$, then $S \subseteq V$.*

5.4 Basis and dimension

Outcomes

A. *Find a basis for a subspace of $\mathbb{R}^n$.*

B. *Use the casting-out algorithm to find a basis for a subspace given as a span.*

C. *Use basic solutions to find a basis for a subspace given as the solution space of a homogeneous system of equations.*

D. *Find the coordinates of a vector with respect to a basis.*

E. *Find the dimension of a subspace of $\mathbb{R}^n$.*

F. *Extend a set of linearly independent vectors to a basis.*

G. *Shrink a spanning set to a basis by removing redundant vectors.*

H. *Determine whether k vectors form a basis of a k-dimensional space.*

5.4.1. Definition of basis

We saw in Proposition 5.22 that spans are subspaces of $\mathbb{R}^n$. Interestingly, the converse is also true: every subspace of $\mathbb{R}^n$ is the span of some finite set of vectors.

Theorem 5.27: Subspaces are spans

Let V be a subspace of $\mathbb{R}^n$. Then there exist linearly independent vectors $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ in V such that

$$V = \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}.$$

Proof. We proceed as follows.

0. If $V = \{\mathbf{0}\}$, then V is the empty span, and we are done.
1. Otherwise, V contains some non-zero vector. Pick a non-zero vector $\mathbf{u}_1$ in V. If $V = \operatorname{span}\{\mathbf{u}_1\}$, we are done.
2. Otherwise, pick a vector $\mathbf{u}_2$ in V that is not in $\operatorname{span}\{\mathbf{u}_1\}$. If $V = \operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2\}$, we are done.
3. Otherwise, pick a vector $\mathbf{u}_3$ in V that is not in $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2\}$. If $V = \operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, we are done.
4. Otherwise, pick a vector $\mathbf{u}_4$ in V that is not in $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_4\}$, and so on.

Continue in this way. Note that after the j^{th} step of this process, the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_j$ are linearly independent. This is because, by construction, no vector is in the span of the previous vectors, and therefore no vector is redundant. By Proposition 5.16(3), there can be at most n linearly independent vectors in $\mathbb{R}^n$. Therefore the process must stop after k steps for some $k \leq n$. But then $V = \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$, as desired. ♠

In summary, every subspace of $\mathbb{R}^n$ is spanned by a finite, linearly independent collection of vectors. Such a collection of vectors is called a **basis** of the subspace.

Definition 5.28: Basis of a subspace

Let V be a subspace of $\mathbb{R}^n$. Then $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is a ***basis*** *for V if the following two conditions hold:*

1. $\operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\} = V$, *and*
2. $\mathbf{u}_1,\ldots,\mathbf{u}_k$ *are linearly independent.*

Note that the plural of basis is **bases**.

5.4.2. Examples of bases

Proposition 5.29: Standard basis of $\mathbb{R}^n$

Let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ whose i^{th} component is 1 and all of whose other components are 0. In other words, $\mathbf{e}_i$ is the i^{th} column of the identity matrix.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

*Then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. It is called the **standard basis** of $\mathbb{R}^n$.*

Proof. To see that it is a basis of $\mathbb{R}^n$, first notice that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ span $\mathbb{R}^n$. Indeed, every vector $\mathbf{v} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ can be written as $\mathbf{v} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$. Second, the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are evidently linearly independent, because none of these vectors can be written as a linear combination of previous vectors. Since the vectors span $\mathbb{R}^n$ and are linearly independent, they form a basis of $\mathbb{R}^n$. ♠

Example 5.30: A non-standard basis of $\mathbb{R}^3$

Check that the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution. We must check that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent and span $\mathbb{R}^3$. To check linear independence, we use the casting-out algorithm.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & \textcircled{2} \end{bmatrix}.$$

Since all columns are pivot columns, there are no redundant vectors, so $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent. To check that they span all of $\mathbb{R}^3$, let $\mathbf{w} = [x, y, z]^T$ be an arbitrary element of $\mathbb{R}^3$. We must show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. This amounts to solving the system of equations

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{w},$$

or in augmented matrix form,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 2 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 1 & 2 & y-2x \\ 0 & 0 & 2 & z-x \end{array}\right].$$

The system is clearly consistent, so it has a solution, and therefore $\mathbf{w}$ is indeed a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Since $\mathbf{w}$ was an arbitrary vector of $\mathbb{R}^3$, it follows that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span $\mathbb{R}^3$. ♠

Generalizing the last example, we find that a set of n vectors forms a basis of $\mathbb{R}^n$ if and only if the matrix having those vectors as its columns is invertible. This is the content of the following proposition.

Proposition 5.31: Invertible matrices and bases of $\mathbb{R}^n$

Let A be an $n \times n$-matrix. Then the columns of A form a basis of $\mathbb{R}^n$ if and only if A is invertible.

We turn to the question of finding bases for subspaces of $\mathbb{R}^n$.

Example 5.32: Basis of a span

Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Find a basis of $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$.

Solution. Let $S = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$. By Theorem 5.19, we know that if we remove the redundant vectors from $\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$, then the remaining vectors will be linearly independent and will still span S. In other words, the remaining vectors will be a basis for S. We use the casting-out algorithm to identity the redundant vectors:

$$\begin{bmatrix} 2 & -1 & 1 & 3 & -1 \\ 0 & 0 & 3 & 5 & 1 \\ -2 & 1 & 5 & 7 & 3 \end{bmatrix} \simeq \begin{bmatrix} 2 & -1 & 1 & 3 & -1 \\ 0 & 0 & 3 & 5 & 1 \\ 0 & 0 & 6 & 10 & 2 \end{bmatrix} \simeq \begin{bmatrix} \textcircled{2} & -1 & 1 & 3 & -1 \\ 0 & 0 & \textcircled{3} & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since columns 2, 4, and 5 are the non-pivot columns, it follows that the vectors $\mathbf{u}_2$, $\mathbf{u}_4$, and $\mathbf{u}_5$ are redundant. Therefore, the desired basis is $\{\mathbf{u}_1, \mathbf{u}_3\}$. ♠

Example 5.33: Basis of the solution space of a homogeneous system of equations

Find a basis for the solution space of the system of equations

$$\begin{aligned} x + y - z + 3w - 2v &= 0, \\ x + y + z - 11w + 8v &= 0, \\ 4x + 4y - 3z + 5w - 3v &= 0. \end{aligned}$$

Solution. We solve the system of equations in the usual way:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 3 & -2 & 0 \\ 1 & 1 & 1 & -11 & 8 & 0 \\ 4 & 4 & -3 & 5 & -3 & 0 \end{array}\right] \simeq \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 2 & -14 & 10 & 0 \\ 0 & 0 & 1 & -7 & 5 & 0 \end{array}\right] \simeq \left[\begin{array}{ccccc|c} \textcircled{1} & 1 & 0 & -4 & 3 & 0 \\ 0 & 0 & \textcircled{1} & -7 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From the reduced echelon form, we see that y, w, and v are free variables. The general solution is:

$$\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = t\begin{bmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 4 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the solution space is spanned by the vectors

$$\left\{ \begin{bmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Moreover, these vectors are evidently linearly independent, because each vector contains a 1 in a position where all the previous vectors have 0 (and therefore, none of the vectors can be written as a linear combination of previous vectors). It follows that the above three vectors form a basis of the solution space. ♠

Note that the basis vectors of the solution space are exactly what we called the **basic solutions** in Section 1.6.

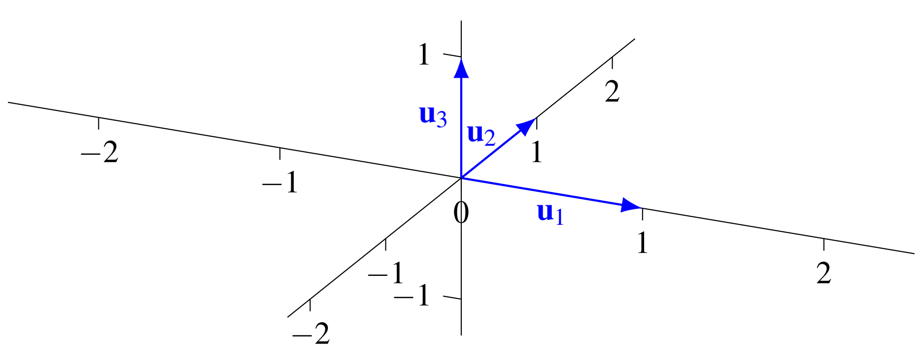
5.4.3. Bases and coordinate systems

Let V be a subspace of $\mathbb{R}^n$. A basis of V is essentially the same thing as a coordinate system for V. To see why, let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be some basis of V. This means that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent and span V. Because the basis vectors are spanning, every vector $\mathbf{v} \in V$ can be written as a linear combination of basis vectors

$$\mathbf{v} = a_1\,\mathbf{u}_1 + \ldots + a_k\,\mathbf{u}_k.$$

Moreover, because the basis vectors are linearly independent, it follows by Theorem 5.18 that the coefficients $a_1, \ldots, a_k$ are unique. We say that $a_1, \ldots, a_k$ are the **coordinates of v with respect to the basis** B, and we write

$$[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}.$$



Basis as coordinate system

Example 5.34: Find a vector from its coordinates in a basis

Find the vector $\mathbf{v}$ *that has coordinates*

$$[\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

with respect to the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *of* $\mathbb{R}^3$*, where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution. This simply means that $\mathbf{v} = 1\mathbf{u}_1 - 1\mathbf{u}_2 + 2\mathbf{u}_3$. We calculate

$$\mathbf{v} = 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

♠

In case the basis is the standard basis, the coordinates are just the usual ones, as the following example illustrates:

Example 5.35: Find a vector from its coordinates in the standard basis

Find the vector $\mathbf{v}$ *that has coordinates*

$$[\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

where B *is the standard basis of* $\mathbb{R}^3$.

Solution. The standard basis is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We have to calculate

$$\mathbf{v} = 1\mathbf{e}_1 - 1\mathbf{e}_2 + 2\mathbf{e}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

We see that the coordinates of any vector with respect to the standard basis are just the usual components of the vector. ♠

We can also ask to find the coordinates of a given vector in a given basis.

Example 5.36: Find the coordinates of a vector with respect to a basis

Find the coordinates of the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

with respect to the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *of* $\mathbb{R}^3$, *where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \textit{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution. To find the coordinates, we must solve the system of equations $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$. We solve:

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \simeq \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right] \simeq \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Therefore, the unique solution is $(a_1, a_2, a_3) = (2, -2, 1)$. The coordinates of $\mathbf{v}$ with respect to the basis B are

$$[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

♠

5.4.4. Dimension

One of the most important properties of bases is that any two bases for the same space must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Lemma. This lemma states that spanning sets have at least as many vectors as linearly independent sets.

Lemma 5.37: Exchange Lemma

Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ *are linearly independent elements of* $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$. *Then* $r \leq s$.

Proof. Since each $\mathbf{u}_j$ is an element of $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$, there exist scalars a_{ij} such that

$$\mathbf{u}_j = a_{1j}\mathbf{v}_1 + \ldots + a_{sj}\mathbf{v}_s.$$

Let $A = \left[a_{ij}\right]$. Note that this matrix has s rows and r columns, i.e., it is an $s \times r$-matrix. Now suppose, for the sake of obtaining a contradiction, that $r > s$. Then by Theorem 1.35, the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x}$, i.e., there exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$. In other words, for all $i = 1, \ldots, s$,

$$a_{i1}x_1 + \ldots + a_{ir}x_r = 0.$$

Therefore,

$$x_1\mathbf{u}_1 + \ldots + x_r\mathbf{u}_r \quad = \quad x_1(a_{11}\mathbf{v}_1 + \ldots + a_{s1}\mathbf{v}_s) + \ldots + x_r(a_{1r}\mathbf{v}_1 + \ldots + a_{sr}\mathbf{v}_s)$$

$$\begin{aligned} &= (a_{11}x_1+\ldots+a_{1r}x_r)\mathbf{v}_1+\ldots+(a_{s1}x_1+\ldots+a_{sr}x_r)\mathbf{v}_s \\ &= 0\,\mathbf{v}_1+\ldots+0\,\mathbf{v}_s \\ &= 0. \end{aligned}$$

This contradicts the assumption that $\mathbf{u}_1,\ldots,\mathbf{u}_r$ are linearly independent. Since we assumed $r > s$ and obtained a contradiction, it follows that $r \leq s$, as desired. ♠

Armed with the Exchange Lemma, we are now ready to show that any two bases of a space are of the same size.

Theorem 5.38: Bases are of the same size

Let V be a subspace of $\mathbb{R}^n$, and let B_1 and B_2 be bases of V. Suppose B_1 contains s vectors and B_2 contains r vectors. Then $s = r$.

Proof. This follows right away from the Exchange Lemma. Indeed, observe that $B_1 = \{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ is a spanning set for V while $B_2 = \{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ is linearly independent, so $s \geq r$. Similarly $B_2 = \{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ is a spanning set for V while $B_1 = \{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ is linearly independent, so $r \geq s$. ♠

Because every basis of V has the same number of vectors, we give this number a special name. It is called the **dimension** of V.

Definition 5.39: Dimension of a subspace

*Let V be a subspace of $\mathbb{R}^n$. Then the **dimension** of V, written $\dim(V)$, is defined to be the number of vectors in a basis.*

Example 5.40: Dimension of $\mathbb{R}^n$

What is the dimension of $\mathbb{R}^n$?

Solution. The standard basis of $\mathbb{R}^n$ is $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. Since it has n vectors, so $\dim(\mathbb{R}^n) = n$. ♠

Example 5.41: Dimension of a subspace

Let

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x-y+2z=0 \right\}.$$

What is the dimension of V?

Solution. We know that V is a subspace of $\mathbb{R}^3$, because it is the solution space of a system of a homogeneous system of equations (in this case, one equation in three variables). We can take $y = t$ and $z = s$ as the free variables and solve for $x = y - 2z = t - 2s$. Therefore, a general element of V is of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t-2s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since the two spanning vectors are linearly independent, they form a basis of V, and thus $\dim(V) = 2$. ♠

Note that the dimension of the solution space of a system of equations is equal to the number of parameters in the general solution, which is equal to the number of free variables. For this reason, the dimension is also sometimes called the number of **degrees of freedom**.

Example 5.42: Dimension of a span

Let

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

What is the dimension of W?

Solution. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix},$$

so that $W = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_6\}$. We use the casting-out algorithm to remove any redundant vectors from $\mathbf{u}_1, \ldots, \mathbf{u}_6$. The remaining vectors will be linearly independent, and therefore a basis of the span.

$$\begin{bmatrix} 1 & 1 & 8 & -6 & 1 & 1 \\ 2 & 3 & 19 & -15 & 3 & 5 \\ -1 & -1 & -8 & 6 & 0 & 0 \\ 1 & 1 & 8 & -6 & 1 & 1 \end{bmatrix} \simeq \begin{bmatrix} \textcircled{1} & 0 & 5 & -3 & 0 & -2 \\ 0 & \textcircled{1} & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the vectors $\mathbf{u}_3$, $\mathbf{u}_4$, and $\mathbf{u}_6$ are redundant, and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_5\}$ is a basis of W. It follows that $\dim(W) = 3$. ♠

5.4.5. More properties of bases and dimension

We begin by noting that every subspace V of $\mathbb{R}^n$ has a basis.

Theorem 5.43: Existence of bases

Every subspace of $\mathbb{R}^n$ has a basis.

Proof. This is just a restatement of Theorem 5.27. ♠

Of course, the theorem does not mean that the basis is unique. Usually, a subspace of $\mathbb{R}^n$ will have many different bases. The theorem just states that there exists at least one.

Sometimes, when we are looking for a basis of a space, we may already have a number of linearly independent vectors. We would like to obtain a basis by adding some *additional* linearly independent vectors to the ones we already have. The following lemma guarantees that this can always be done.

Lemma 5.44: Linearly independent set can be extended to a basis

Let V be a subspace of $\mathbb{R}^n$, and let $\mathbf{u}_1,\ldots,\mathbf{u}_\ell$ be linearly independent elements of V. Then it is possible to extend $\{\mathbf{u}_1,\ldots,\mathbf{u}_\ell\}$ to a basis of V. In other words, there exist zero or more vectors $\mathbf{w}_1,\ldots,\mathbf{w}_s$ such that

$$\{\mathbf{u}_1,\ldots,\mathbf{u}_\ell,\mathbf{w}_1,\ldots,\mathbf{w}_s\}$$

is a basis of V.

Proof. By Theorem 5.43, we know that V has some basis, say $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$. However, this may not be the basis we are looking for, because maybe it does not contain the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_\ell$. Consider the sequence of $\ell+k$ vectors

$$\mathbf{u}_1,\ldots,\mathbf{u}_\ell,\mathbf{v}_1,\ldots,\mathbf{v}_k.$$

Since V is spanned by the vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$, it is certainly also spanned by the larger set of vectors $\mathbf{u}_1,\ldots,\mathbf{u}_\ell,\mathbf{v}_1,\ldots,\mathbf{v}_k$. From Theorem 5.19, we know that we can obtain a basis of V by removing the redundant vectors from $\mathbf{u}_1,\ldots,\mathbf{u}_\ell,\mathbf{v}_1,\ldots,\mathbf{v}_k$. On the other hand, $\mathbf{u}_1,\ldots,\mathbf{u}_\ell$ are linearly independent, so none of them can be redundant. It follows that the resulting basis of V contains all of the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_\ell$. In other words, we have found a basis of V that is an extension of $\mathbf{u}_1,\ldots,\mathbf{u}_\ell$, which is what had to be shown. ♠

Example 5.45: Extending a linearly independent set to a basis

Extend $\{\mathbf{u}_1,\mathbf{u}_2\}$ to a basis of $\mathbb{R}^4$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}.$$

Solution. Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. We obtain the desired basis by applying the casting-out algorithm to $\mathbf{u}_1,\mathbf{u}_2,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} \circled{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \circled{1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \circled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \circled{2} & 1 \end{bmatrix}.$$

Therefore, we cast out the vectors $\mathbf{e}_1$ and $\mathbf{e}_4$ and keep the rest. The resulting basis is

$$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{e}_2,\mathbf{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

♠

Example 5.46: Extending a linearly independent set to a basis

Let

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \; x+2y+z-w=0 \right\}.$$

Note that $\mathbf{u}_1,\mathbf{u}_2 \in V$*, where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Extend $\{\mathbf{u}_1,\mathbf{u}_2\}$ *to a basis of* V*.*

Solution. We first find a basis of V by solving the linear equation $x+2y+z-w=0$. Taking $y=r$, $z=s$, and $w=t$ as the free variables, we get $x=-2y-z+w=-2r-s+t$, and therefore the general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2r-s+t \\ r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the vectors $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ form a basis of V, where

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

However, this is not the basis we are looking for, because it does not extend $\{\mathbf{u}_1,\mathbf{u}_2\}$. To get a basis of V that extends $\{\mathbf{u}_1,\mathbf{u}_2\}$, we perform the casting-out algorithm on the vectors $\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$:

$$\begin{bmatrix} 1 & -2 & -2 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \simeq \cdots \simeq \begin{bmatrix} \textcircled{1} & -2 & -2 & -1 & 1 \\ 0 & \textcircled{1} & 1 & 1 & -1 \\ 0 & 0 & \textcircled{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Casting out $\mathbf{v}_2$ and $\mathbf{v}_3$, we find that the desired basis is $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1\}$. ♠

We also have a kind of opposite of Lemma 5.44: every spanning set can be shrunk to a basis.

Lemma 5.47: Spanning set can be shrunk to a basis

Let V be a subspace of $\mathbb{R}^n$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ be a set of vectors spanning V. Then it is possible to obtain a basis of V by "shrinking" the set, i.e., by removing zero or more vectors from it.

Proof. This is merely a restatement of Theorem 5.19. We obtain the linearly independent subset by removing the redundant vectors, which can be achieved by the casting-out algorithm. See also Example 5.20. ♠

The following proposition tells us something about the size of a linearly independent set of vectors or the size of a spanning set of vectors.

Proposition 5.48: Size of a linearly independent or spanning set of vectors

Let V be a k-dimensional subspace of $\mathbb{R}^n$. Then

(a) Every linearly independent set of vectors in V has at most k vectors.

(b) Every spanning set of vectors in V has at least k vectors.

Proof. Both properties follow from the Exchange Lemma (Lemma 5.37). Since V is k-dimensional, it has some basis consisting of k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

(a) Suppose $\mathbf{u}_1, \ldots \mathbf{u}_r$ are linearly independent vectors in V. Since $\mathbf{u}_1, \ldots \mathbf{u}_r$ are linearly independent and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are spanning, the Exchange Lemma implies that $r \leq k$.

(b) Suppose the vectors $\mathbf{u}_1, \ldots \mathbf{u}_s$ span V. Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent and $\mathbf{u}_1, \ldots \mathbf{u}_s$ are spanning, the Exchange Lemma implies that $k \leq s$.

♠

The next proposition often comes in handy when we need to check that some set of vectors is a basis for a subspace V, where the dimension of V is already known. If $\dim(V) = k$, we know that any basis has to have size k. Interestingly, to check that a set of k vectors is a basis of V, it is sufficient to check *either* that it is linearly independent *or* that it is spanning. This can save half the work in checking that some set of vectors is a basis (but it only works if the number of vectors is exactly k, the dimension of V).

Proposition 5.49: Basis test for k vectors in k-dimensional space

Let V be a k-dimensional subspace of $\mathbb{R}^n$, and consider k vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in V.

- *If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, then they form a basis for V.*
- *If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ span V, then they form a basis for V.*

Proof. The first claim is an easy consequence of Lemma 5.44. Assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent. By Lemma 5.44, we can add zero or more vectors to $\mathbf{u}_1, \ldots, \mathbf{u}_k$ to obtain a basis of V. On the other hand, since V is k-dimensional, every basis must have exactly k elements, so that the only possibility is that we have added zero vectors. Therefore, $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is already a basis of V, as claimed.

To prove the second claim, assume that $V = \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$. By Theorem 5.27, there exists a linearly independent subset of $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ that spans V, i.e., that is a basis for V. But since $\dim(V) = k$, every basis must have exactly k elements, so that the only possible such subset is $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ itself. Therefore, $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is a basis of V, as claimed. ♠

It is important to note that Proposition 5.49 does *not* say that every linearly independent set of vectors in V is a basis. For example, a set of $k-1$ or fewer linearly independent vectors will not be spanning. Also, the proposition does *not* say that every spanning set of vectors in V is a basis. For example, a set of $k+1$ or more spanning vectors will not be linearly independent. Rather, what the proposition is saying is that if we have exactly k vectors in a k-dimensional space, then linear independence implies spanning and vice versa.

Example 5.50: Basis test for 3 vectors in 3-dimensional space

Do the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$?

Solution. This is similar to Example 5.30. But because we know that $\mathbb{R}^3$ is a 3-dimensional space, and because we have exactly 3 vectors, by Proposition 5.49, we only need to check *either* whether $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ are linearly independent *or* whether they are spanning. We check whether they are linearly independent by using the casting-out algorithm.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since the matrix has rank 3, the vectors $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ are linearly independent. Therefore, by Proposition 5.49, they form a basis of $\mathbb{R}^3$. ♠

The following proposition is also a consequence of Lemma 5.44. It says that smaller subspaces have smaller dimension.

Proposition 5.51: Subspace of a subspace

Let V and W be subspaces of $\mathbb{R}^n$, and suppose that $V \subseteq W$. Then $\dim(V) \leq \dim(W)$, with equality only when $V = W$.

Proof. Consider any basis $\mathbf{u}_1,\ldots,\mathbf{u}_k$ of V. Because $V \subseteq W$, the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are linearly independent elements of W, and therefore can be extended to a basis of W by Lemma 5.44. The resulting basis of W has at least k elements, i.e., $\dim(V) \leq \dim(W)$. To prove the last claim, assume moreover that $\dim(V) = \dim(W)$. In that case, $\dim(W) = k$, so that the k linearly independent vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ form a basis of W by Proposition 5.49. Since both V and W are spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_k$, we must have $V = W$, as claimed. ♠

Exercises

Exercise 5.4.1 *For each of the following subspaces of $\mathbb{R}^4$, find a basis and determine the dimension.*

(a) $V_1 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \right\}.$

(b) $V_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$

(c) $V_3 = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -9 \\ 4 \\ 3 \\ -9 \end{bmatrix}, \begin{bmatrix} -33 \\ 15 \\ 12 \\ -36 \end{bmatrix}, \begin{bmatrix} -22 \\ 10 \\ 8 \\ -24 \end{bmatrix} \right\}.$

(d) $V_4 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ -3 \\ -6 \end{bmatrix} \right\}.$

Exercise 5.4.2 *Find a basis and the dimension of each of the following subspaces of $\mathbb{R}^n$.*

(a) $S_1 = \left\{ \begin{bmatrix} 4u+v-5w \\ 12u+6v-6w \\ 4u+4v+4w \end{bmatrix} \;\middle|\; u,v,w \in \mathbb{R} \right\}.$

(b) $S_2 = \left\{ \begin{bmatrix} 2u+6v+7w \\ -3u-9v-12w \\ 2u+6v+6w \\ u+3v+3w \end{bmatrix} \;\middle|\; u,v,w \in \mathbb{R} \right\}.$

(c) $S_3 = \left\{ \begin{bmatrix} 2u+v \\ 6v-3u+3w \\ 3v-6u+3w \end{bmatrix} \;\middle|\; u,v,w \in \mathbb{R} \right\}.$

Exercise 5.4.3 *Find a basis and the dimension of each of the following subspaces of $\mathbb{R}^n$.*

(a) $W_1 = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \middle| \; u+v=0 \text{ and } u-2w=0 \right\}.$

(b) $W_2 = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \middle| \; u+v+w=0 \right\}.$

(c) $S = \left\{ \begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} \middle| \; u+v=w+x \text{ and } u+w=v+x \right\}.$

Exercise 5.4.4 *Find the vector* $\mathbf{v}$ *that has coordinates*

$$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

with respect to the basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *of* $\mathbb{R}^3$*, where*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

Exercise 5.4.5 *Find the coordinates of each of* $\mathbf{v}$*,* $\mathbf{w}$ *with respect to the basis* $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$*, where*

$$\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 5.4.6 *Extend* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *to a basis of* $\mathbb{R}^3$*, where*

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

Exercise 5.4.7 *Let*

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \; x+y+z+2w=0 \right\}.$$

Note that $\mathbf{u}_1, \mathbf{u}_2 \in V$*, where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$

Is $\{\mathbf{u}_1, \mathbf{u}_2\}$ *a basis of* V*? If not, extend it to a basis of* V *by adding additional basis vectors.*

Exercise 5.4.8 *Shrink* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ *to a basis of* $\mathbb{R}^3$ *by removing redundant vectors, where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 5.4.9 *Use one of the basis tests of Proposition 5.49 to determine whether the vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \quad \textit{and} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Exercise 5.4.10 *Use one of the basis tests of Proposition 5.49 to determine whether the vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \textit{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Exercise 5.4.11 *In this exercise, we use scalars from the field* $\mathbb{Z}_5$ *of integers modulo 5 instead of real numbers (see Section 1.8, "Fields"). Find a basis and the dimension of each of the following subspaces of* $(\mathbb{Z}_5)^n$.

(a) $V_1 = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$

(b) $V_2 = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \;\middle|\; 2u + v = 0 \textit{ and } u + 4w = 0 \right\}.$

(c) $V_3 = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \;\middle|\; u + 2v + 3w = 0 \right\}.$

Exercise 5.4.12 *In this exercise, we use scalars from the field* $\mathbb{Z}_7$ *of integers modulo 7 instead of real numbers. Find the coordinates of* $\mathbf{v}$ *with respect to the basis* $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *in* $(\mathbb{Z}_7)^3$*, where*

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Exercise 5.4.13 *In this exercise, we use scalars from the field $\mathbb{Z}_2$ of integers modulo 2 instead of real numbers. Extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis of $(\mathbb{Z}_2)^4$, where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 5.4.14 *True or false? Explain.*

(a) *Every set of 5 vectors in $\mathbb{R}^5$ is linearly independent.*

(b) *Every set of 4 vectors in $\mathbb{R}^5$ is linearly independent.*

(c) *Every set of 6 vectors in $\mathbb{R}^5$ is linearly dependent.*

(d) *No set of 4 vectors spans $\mathbb{R}^5$.*

(e) *Every linearly independent set of 5 vectors in $\mathbb{R}^5$ is a basis of $\mathbb{R}^5$.*

(f) *Every linearly independent set of 4 vectors in $\mathbb{R}^5$ is a basis of $\mathbb{R}^5$.*

(g) *Some linearly independent set of 4 vectors in $\mathbb{R}^5$ is a basis of $\mathbb{R}^5$.*

(h) *Every spanning set of 6 vectors in $\mathbb{R}^5$ is a basis of $\mathbb{R}^5$.*

(i) *Every linearly independent set of 4 vectors in $\mathbb{R}^5$ spans a 4-dimensional subspace of $\mathbb{R}^5$.*

Exercise 5.4.15 *If you have 6 vectors in $\mathbb{R}^5$, is it possible they are linearly independent? Explain.*

Exercise 5.4.16 *Suppose V and W both have dimension equal to 7 and they are subspaces of $\mathbb{R}^{10}$. What are the possibilities for the dimension of $V \cap W$?* ***Hint:*** *Remember that a linear independent set can be extended to form a basis.*

5.5 Column space, row space, and null space of a matrix

Outcomes

A. *Find a basis for the column space, row space, and null space of a matrix.*

B. *Find the rank and nullity of a matrix.*

There are three important spaces we can associate to a matrix. They are called the column space, row space, and null space, and are defined as follows.

Definition 5.52: Column space, row space, null space

Let A be an $m \times n$-matrix. The **column space** *of A, written $\operatorname{col}(A)$, is the span of the columns. The* **row space** *of A, written $\operatorname{row}(A)$, is the span of the rows. The* **null space** *of A, written $\operatorname{null}(A)$, is the set*

$$\operatorname{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

Note that the column space is a subspace of $\mathbb{R}^m$ and the null space is a subspace of $\mathbb{R}^n$. The row space, on the other hand, is a set of row vectors. It can be regarded as a subspace of $\mathbb{R}^n$, but only if we regard $\mathbb{R}^n$ as the set of n-dimensional row vectors (and not column vectors, as usual).

Before we give an example, recall that two matrices are called **row equivalent** if one can be obtained from the other by performing a sequence of elementary row operations. The point of elementary row operations is that they do not affect the row space or the null space of the matrix. (They do, however, affect the column space). The following proposition makes this more precise.

Proposition 5.53: Effect of row operations

Let A and B be row equivalent matrices. Then $\operatorname{row}(A) = \operatorname{row}(B)$ and $\operatorname{null}(A) = \operatorname{null}(B)$.

Proof. The fact that elementary row operations do not change the null space is a special case of Theorem 1.11, applied to a homogeneous system. To prove that they do not change the row space is also easy; we just need to look at each kind of elementary row operation. For example, adding a multiple of one row to another clearly does not change the span of the rows. ♠

Example 5.54: Basis of column space, row space, and null space

Find a basis for the column space, row space, and null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}.$$

Solution. The column space of A is the span of the columns of A, i.e.,

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\}.$$

To find a basis for the column space, we use the casting-out algorithm. The reduced echelon form of A is

$$\begin{bmatrix} \textcircled{1} & 0 & -9 & 9 & 2 \\ 0 & \textcircled{1} & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.3}$$

Note that the first two columns of the reduced echelon form are pivot columns. Therefore, by the casting-out algorithm, the first two columns of A form a basis for the column space. Thus, the following is a basis for the column space:

$$\text{Basis of col}(A): \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

The rows space of A is the span of the rows of A, i.e.,

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 6 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 7 & 8 & 6 & 6 \end{bmatrix} \right\}$$

We could find a basis of the row space by writing all three rows as column vectors and using the casting-out algorithm. However, there is an easier way. By Proposition 5.53, the row space of A is equal to the row space of the reduced echelon form (5.3). Moreover, the non-zero rows of the reduced echelon form are clearly linearly independent (no non-zero row can be a linear combination of other rows below it, because each non-zero row has a pivot entry). Therefore, the non-zero rows of the reduced echelon form form a basis of the row space.

$$\text{Basis of row}(A): \quad \left\{ \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix} \right\}.$$

Finally, the null space of A is just the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Thus, finding a basis of the null space is the same as finding a set of basic solutions. From the reduced echelon form, we can easily find the general solution of $A\mathbf{x} = \mathbf{0}$, using three parameters r, s, t corresponding to the three non-pivot columns of (5.3). The general solution is:

$$\mathbf{x} = r \begin{bmatrix} 9 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the following is a basis of the null space:

$$\text{Basis of null}(A): \quad \left\{ \begin{bmatrix} 9 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

♠

As the example shows, all three bases, for the column space, the row space, and the null space of A, can be easily determined from the reduced echelon form. In the next proposition, we use this information to determine the dimensions of these three spaces. Recall from Definition 1.24 that the **rank** of a matrix is equal to the number of pivot entries of its reduced echelon form.

Proposition 5.55: Dimension of column space, row space, and null space

Let A be an $m \times n$-matrix. Then the dimensions of the column space, row space, and null space of A are as follows:

$$\begin{aligned} \dim(\text{col}(A)) &= \text{rank}(A), \\ \dim(\text{row}(A)) &= \text{rank}(A), \\ \dim(\text{null}(A)) &= n - \text{rank}(A). \end{aligned}$$

Proof. Let $r = \text{rank}(A)$. Following the same method as in Example 5.54, we can use the casting-out algorithm to find a basis for the column space. Since the reduced echelon form of A has r pivot columns, the basis has r elements, and therefore $\dim(\text{col}(A)) = r$. Also, the reduced echelon form has r non-zero rows (since each non-zero row contains exactly one pivot entry). These form a basis of the row space, and therefore $\dim(\text{row}(A)) = r$. Finally, the dimension of the null space is equal to the number of parameters in the general solution of the system of equations $A\mathbf{x} = \mathbf{0}$. There is one parameter for each non-pivot column, and since A has n columns and r pivot columns, it follows that $\dim(\text{null}(A)) = n - r$. ♠

Among other things, the proposition states that the "row rank" of a matrix (the dimension of its row space) is always equal to the "column rank" (the dimension of the column space). This fact is not at all obvious when one first considers the definition of a matrix. It is often called the **rank theorem** and is one of the deep and mysterious facts of linear algebra. It means, for example, that if we do elementary column operations instead of elementary row operations, we end up with exactly the same number of pivots. Since the "row rank" and "column rank" are always equal, we are justified in simply calling this quantity the "rank" of the matrix.

There is also a name for the dimension of the null space. It is called the **nullity** of the matrix, and is written $\text{nullity}(A)$. The last part of Proposition 5.55 is also called the **rank-nullity theorem**, and is often written in the form

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Example 5.56: Rank and nullity

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}.$$

Solution. The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\text{rank}(A) = 3$ and $\text{nullity}(A) = 5 - 3 = 2$. ♠

We conclude this section with two useful theorems about matrices.

Theorem 5.57:

The following are equivalent for an $m \times n$-matrix A.

1. $\operatorname{rank}(A) = n$.
2. $\operatorname{row}(A) = \mathbb{R}^n$, *i.e., the rows of A span $\mathbb{R}^n$.*
3. *The columns of A are linearly independent in $\mathbb{R}^m$.*
4. *The $n \times n$-matrix $A^T A$ is invertible.*
5. *A is left invertible, i.e., there exists an $n \times m$-matrix B such that $BA = I$.*
6. *The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Theorem 5.58:

The following are equivalent for an $m \times n$-matrix A.

1. $\operatorname{rank}(A) = m$.
2. $\operatorname{col}(A) = \mathbb{R}^m$, *i.e., the columns of A span $\mathbb{R}^m$.*
3. *The rows of A are linearly independent in $\mathbb{R}^n$.*
4. *The $m \times m$-matrix AA^T is invertible.*
5. *A is right invertible, i.e., there exists an $n \times m$-matrix B such that $AB = I$.*
6. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$.*

Exercises

Exercise 5.5.1 *Determine the rank and nullity and find a basis of the column space, row space, and null space of each of the following matrices.*

(a)

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 3 & 9 & 1 & 7 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

(d)

$$D = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

(e)

$$E = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

Exercise 5.5.2 *Find* $\operatorname{null}(A)$ *for the following matrices.*

(a)

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -2 \\ 1 & 2 & -2 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$$

Exercise 5.5.3 *In this exercise, we use scalars from the field* $\mathbb{Z}_5$ *of integers modulo* 5 *instead of real numbers (see Section 1.8, “Fields”). Determine the rank and nullity and find a basis of the column space, row space, and null space of the following matrix over* $\mathbb{Z}_5$.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Exercise 5.5.4 *Show that if A is an $m \times n$-matrix, then* $\text{null}(A)$ *is a subspace of* $\mathbb{R}^n$.

Exercise 5.5.5 *Let A be an $m \times n$-matrix. Show that* $\text{col}(A) = \{A\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n\}$.

Exercise 5.5.6 *Show that* $\text{rank}(A) = \text{rank}(A^T)$.

Exercise 5.5.7 *For invertible matrices B and C of appropriate size, show that* $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AC)$.

Exercise 5.5.8 *Suppose A is an $m \times n$-matrix and B is an $n \times p$-matrix. Show that*

$$\text{nullity}(AB) \leq \text{nullity}(A) + \text{nullity}(B).$$

Hint: *Consider the subspace* $\text{col}(B) \cap \text{null}(A)$ *and suppose a basis for this subspace is* $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. *Let* $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ *be such that* $B\mathbf{z}_i = \mathbf{w}_i$. *Now suppose* $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ *is a basis for* $\text{null}(B)$, *and argue that* $\text{null}(AB) \subseteq \text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{z}_1, \ldots, \mathbf{z}_k\}$.

6. Linear transformations in $\mathbb{R}^n$

6.1 Linear transformations

Outcomes

A. *Determine whether a vector function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.*

In calculus, a **function** (or **map**) $f : \mathbb{R} \to \mathbb{R}$ is a rule that maps a real number $x \in \mathbb{R}$ to a real number $f(x) \in \mathbb{R}$. In linear algebra, we can generalize this concept to vectors. A **vector function** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that inputs an n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ and outputs an m-dimensional vector $T(\mathbf{v}) \in \mathbb{R}^m$. The following are some examples of vector functions:

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x+y \\ y^2 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y+z \\ 0 \end{bmatrix}, \quad T_3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} e^{x+z} \\ \sqrt{y} \end{bmatrix}. \tag{6.1}$$

Of these, the first is a function $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$, the second is a function $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$, and the third is a function $T_3 : \mathbb{R}^3 \to \mathbb{R}^2$. We can evaluate a vector function by applying it to a vector, for example,

$$T_1\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \\ 1+2 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad T_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0^2 \\ 1+1 \\ 1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and so on. The study of arbitrary vector functions and their derivatives and integrals is the subject of *multivariable calculus*. In linear algebra, we will only be concerned with **linear vector functions**, which are also called **linear transformations** or **linear maps**. They are defined as follows.

Definition 6.1: Linear transformation

*A vector function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation**, or simply **linear**, if it satisfies the following two conditions:*

1. *T preserves addition, i.e., for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$;*
2. *T preserves scalar multiplication, i.e, for all $\mathbf{v} \in \mathbb{R}^n$ and scalars k, we have $T(k\mathbf{v}) = kT(\mathbf{v})$.*

Example 6.2: Linear and non-linear transformations

Which of the vector functions in (6.1) are linear transformations?

Solution.

(a) The function T_1 is not a linear transformation. For example, let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$T_1(\mathbf{v}) = T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T_1(2\mathbf{v}) = T_1\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $T_1(2\mathbf{v}) \neq 2T_1(\mathbf{v})$, the vector function T_1 does not preserve scalar multiplication, and therefore it is not a linear transformation.

(b) The function T_2 is a linear transformation. For example, to prove that T_2 preserves addition, consider two arbitrary vectors

$$\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

We have

$$T_2(\mathbf{v}+\mathbf{w}) = T_2\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ (x_1+x_2)+(y_1+y_2)+(z_1+z_2) \\ 0 \end{bmatrix}$$

and

$$T_2(\mathbf{v})+T_2(\mathbf{w}) = \begin{bmatrix} x_1+y_1 \\ x_1+y_1+z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ x_2+y_2+z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (x_1+y_1)+(x_2+y_2) \\ (x_1+y_1+z_1)+(x_2+y_2+z_2) \\ 0 \end{bmatrix}.$$

Since the two sides are evidently equal, T_2 preserves addition. The fact that it preserves scalar multiplication can be shown by a similar calculation.

(c) The function T_3 is not a linear transformation. For example, consider $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Then

$$T_3(\mathbf{v}+\mathbf{w}) = T_3\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e \\ \sqrt{2} \end{bmatrix},$$

and

$$T_3(\mathbf{v})+T_3(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} e \\ 1 \end{bmatrix} = \begin{bmatrix} e+1 \\ 2 \end{bmatrix}.$$

Since $T_3(\mathbf{v}+\mathbf{w}) \neq T_3(\mathbf{v})+T_3(\mathbf{w})$, the vector function T_3 does not preserve addition, and therefore it is not linear.

♠

An easy fact about linear transformation is that they preserve the origin, i.e., they satisfy $T(\mathbf{0}) = \mathbf{0}$. This can be seen, for example, by considering $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0})+T(\mathbf{0})$ and then subtracting $T(\mathbf{0})$ from both sides of the equation. This gives an easier way to see that T_3 in the above example is not a linear transformation, since $T_3(\mathbf{0}) \neq \mathbf{0}$. On the other hand, of course not every function that preserves the origin is linear. For example, T_1 is not linear although it satisfies $T_1(\mathbf{0}) = \mathbf{0}$.

The following characterization of linearity is often useful, as it permits us to check just one property instead of two.

Proposition 6.3: Alternative characterization of linear transformations

A vector function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it satisfies the following condition, for all $\mathbf{v},\mathbf{w} \in \mathbb{R}^n$ and scalars a,b:

$$T(a\mathbf{v}+b\mathbf{w}) = aT(\mathbf{v})+bT(\mathbf{w}).$$

Proof. First, assume that T is linear. Then from preservation of addition and scalar multiplication, we have $T(a\mathbf{v}+b\mathbf{w}) = T(a\mathbf{v})+T(b\mathbf{w}) = aT(\mathbf{v})+bT(\mathbf{w})$. Conversely, assume that T satisfies $T(a\mathbf{v}+b\mathbf{w}) = aT(\mathbf{v})+bT(\mathbf{w})$ for all vectors $\mathbf{v},\mathbf{w}$ and scalars a,b. Then we get preservation of addition by setting $a = b = 1$, and preservation of scalar multiplication by setting $b = 0$. ♠

Exercises

Exercise 6.1.1 *Which of the following vector functions are linear transformations?*

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x-2y \\ -x-y \end{bmatrix}, \quad T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y^2 \\ (x+y)z \\ 0 \end{bmatrix}, \quad T_3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Exercise 6.1.2 *Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Explain why each of these functions T is not linear.*

(a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z+1 \\ 2y-3x+z \end{bmatrix}$

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y^2+3z \\ 2y+3x+z \end{bmatrix}$

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin x+2y+3z \\ 2y+3x+z \end{bmatrix}$

(d) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y+3x-\ln z \end{bmatrix}$

Exercise 6.1.3 *Let A be an $m \times n$-matrix. Show the vector function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation.*

Exercise 6.1.4 *Let $\mathbf{u} \in \mathbb{R}^n$ be a fixed vector. Show that the function T defined by $T(\mathbf{v}) = \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ is a linear transformation.*

Exercise 6.1.5 *Let $\mathbf{u} \in \mathbb{R}^n$ be a fixed non-zero vector. The function T defined by $T(\mathbf{v}) = \mathbf{u} + \mathbf{v}$ has the effect of translating all vectors by adding $\mathbf{u}$. Show this is not a linear transformation.*

6.2 The matrix of a linear transformation

Outcomes

A. *Find the matrix corresponding to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.*

An important example of linear transformations are the so-called **matrix transformations**.

Proposition 6.4: Matrix transformations are linear transformations

Let A be an $m \times n$-matrix, and consider the vector function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$. Then T is a linear transformation.

Proof. This follows from the laws of matrix multiplication. Namely, by the distributive law, we have $A(\mathbf{v}+\mathbf{w}) = A\mathbf{v}+A\mathbf{w}$, showing that T preserves addition. And by the compatibility of matrix multiplication and scalar multiplication, we have $A(k\mathbf{v}) = k(A\mathbf{v})$, showing that T preserves scalar multiplication. ♠

In fact, matrix transformations are not just an example of linear transformations, but they are essentially the *only* example. One of the central theorems in linear algebra is that all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ are in fact matrix transformations. Therefore, a matrix can be regarded as a notation for a linear transformation, and vice versa. This is the subject of the following theorem.

Theorem 6.5: Linear transformations are matrix transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Then there exists an $m \times n$-matrix A such that for all $\mathbf{v} \in \mathbb{R}^n$,

$$T(\mathbf{v}) = A\mathbf{v}.$$

In other words, T is a matrix transformation.

Proof. Suppose $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation and consider the standard basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ of $\mathbb{R}^n$. For all i, define $\mathbf{u}_i=T(\mathbf{e}_i)$, and let A be the matrix that has $\mathbf{u}_1,\ldots,\mathbf{u}_n$ as its columns. We claim that A is the desired matrix, i.e., that $T(\mathbf{v})=A\mathbf{v}$ holds for all $\mathbf{v}\in\mathbb{R}^n$.

To see this, let

$$\mathbf{v}=\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix}$$

be some arbitrary element of $\mathbb{R}^n$. Then $\mathbf{v}=x_1\mathbf{e}_1+\ldots+x_n\mathbf{e}_n$, and we have:

$$\begin{array}{rcll} T(\mathbf{v}) & = & T(x_1\mathbf{e}_1+\ldots+x_n\mathbf{e}_n) & \\ & = & T(x_1\mathbf{e}_1)+\ldots+T(x_n\mathbf{e}_n) & \text{by linearity} \\ & = & x_1T(\mathbf{e}_1)+\ldots+x_nT(\mathbf{e}_n) & \text{by linearity} \\ & = & x_1\mathbf{u}_1+\ldots+x_n\mathbf{u}_n & \text{by definition of } \mathbf{u}_i \\ & = & A\mathbf{v} & \text{by the column method of matrix multiplication.} \end{array}$$

♠

In summary, the matrix corresponding to the linear transformation T has as its columns the vectors $T(\mathbf{e}_1),\ldots,T(\mathbf{e}_n)$, i.e., the images of the standard basis vectors. We can visualize this matrix as follows:

$$A=\begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix}.$$

Example 6.6: The matrix of a linear transformation

Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear transformation where

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}9\\-3\end{bmatrix},\quad \text{and}\quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix}.$$

Find the matrix A such that $T(\mathbf{v})=A\mathbf{v}$ for all $\mathbf{v}$.

Solution. By Theorem 6.5, the columns of A are $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$. Therefore,

$$A=\begin{bmatrix}1 & 9 & 1\\ 2 & -3 & 1\end{bmatrix}.$$

♠

Example 6.7: The matrix of a linear transformation

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+2y-z \end{bmatrix},$$

for all $x, y, z \in \mathbb{R}$. Find the matrix of this linear transformation.

Solution. We compute the images of the standard basis vectors:

$$\begin{aligned} T(\mathbf{e}_1) &= T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ T(\mathbf{e}_2) &= T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ T(\mathbf{e}_3) &= T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned}$$

The matrix A has $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ as its columns. Therefore,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}.$$

♠

Example 6.8: Matrix of a projection map

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map defined by*

$$T(\mathbf{v}) = \text{proj}_{\mathbf{u}}(\mathbf{v})$$

for all $\mathbf{v} \in \mathbb{R}^3$.

(a) Is T a linear transformation?

(b) If yes, find the matrix of T.

Solution.

(a) Recall the formula for the projection of $\mathbf{v}$ onto $\mathbf{u}$:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}.$$

In the situation we are interested in, $\mathbf{u}$ is a fixed vector, and $\mathbf{v}$ is the input to the function T. Given any two vectors $\mathbf{v},\mathbf{w}$, and using the distributive laws of the dot product and scalar multiplication, we have:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}+\mathbf{w}) = \frac{\mathbf{u}\bullet(\mathbf{v}+\mathbf{w})}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} = \left(\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}+\frac{\mathbf{u}\bullet\mathbf{w}}{\mathbf{u}\bullet\mathbf{u}}\right)\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u}+\frac{\mathbf{u}\bullet\mathbf{w}}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} = \text{proj}_{\mathbf{u}}(\mathbf{v})+\text{proj}_{\mathbf{u}}(\mathbf{w}).$$

Therefore, the function $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}(\mathbf{v})$ preserves addition. Also, given any scalar k, we have

$$\text{proj}_{\mathbf{u}}(k\mathbf{v}) = \frac{\mathbf{u}\bullet(k\mathbf{v})}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} = \left(k\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\right)\mathbf{u} = k\left(\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u}\right) = k\,\text{proj}_{\mathbf{u}}(\mathbf{v}).$$

Therefore, the function T preserves scalar multiplication. It follows that T is a linear transformation.

(b) To find the matrix of T, we must compute the images of the standard basis vectors $T(\mathbf{e}_1),\ldots,T(\mathbf{e}_3)$. We compute

$$\begin{array}{rcccccc} T(\mathbf{e}_1) & = & \text{proj}_{\mathbf{u}}(\mathbf{e}_1) & = & \dfrac{\mathbf{u}\bullet\mathbf{e}_1}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} & = & \dfrac{1}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}, \\ T(\mathbf{e}_2) & = & \text{proj}_{\mathbf{u}}(\mathbf{e}_2) & = & \dfrac{\mathbf{u}\bullet\mathbf{e}_2}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} & = & \dfrac{2}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}, \\ T(\mathbf{e}_3) & = & \text{proj}_{\mathbf{u}}(\mathbf{e}_3) & = & \dfrac{\mathbf{u}\bullet\mathbf{e}_3}{\mathbf{u}\bullet\mathbf{u}}\mathbf{u} & = & \dfrac{3}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}. \end{array}$$

Hence the matrix of T is

$$A = \frac{1}{14}\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}.$$

♠

Exercises

Exercise 6.2.1 *For each of the following vector functions $T:\mathbb{R}^n\to\mathbb{R}^n$, show that T is a linear transformation and find the corresponding matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.*

(a) T multiplies the j^{th} component of $\mathbf{v}$ by a non-zero number b.

(b) T adds b times the j^{th} component of $\mathbf{v}$ to the i^{th} component.

(c) T switches the i^{th} and j^{th} components.

Exercise 6.2.2 *Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and that $\mathbf{u}_1,\ldots,\mathbf{u}_n$ is a basis of $\mathbb{R}^n$. For all $i = 1,\ldots,n$, let $\mathbf{v}_i = T(\mathbf{u}_i)$. Let A be the matrix that has $\mathbf{u}_1,\ldots,\mathbf{u}_n$ as its columns, and let B be the matrix that has $\mathbf{v}_1,\ldots,\mathbf{v}_n$ as its columns. Show that A is invertible and the matrix of T is BA^{-1}.*

Exercise 6.2.3 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}, \quad T\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}.$$

Find the matrix of T. Hint: use Exercise 6.2.2.

Exercise 6.2.4 *Suppose T is a linear transformation such that*

$$T\begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}.$$

Find the matrix of T. Hint: use Exercise 6.2.2.

Exercise 6.2.5 *Consider the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$. For each, determine the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.*

(a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y-3x+z \end{bmatrix}$

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x+2y+z \\ 3x-11y+2z \end{bmatrix}$

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y+z \\ x+2y+6z \end{bmatrix}$

(d) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y-5x+z \\ x+y+z \end{bmatrix}$

Exercise 6.2.6 *Find the matrix for $T(\mathbf{w}) = \mathrm{proj}_{\mathbf{v}}(\mathbf{w})$, where $\mathbf{v} = [1,-2,3]^T$.*

Exercise 6.2.7 *Find the matrix for $T(\mathbf{w}) = \mathrm{proj}_{\mathbf{v}}(\mathbf{w})$, where $\mathbf{v} = [1,5,3]^T$.*

6.3 Geometric interpretation of linear transformations

Outcomes

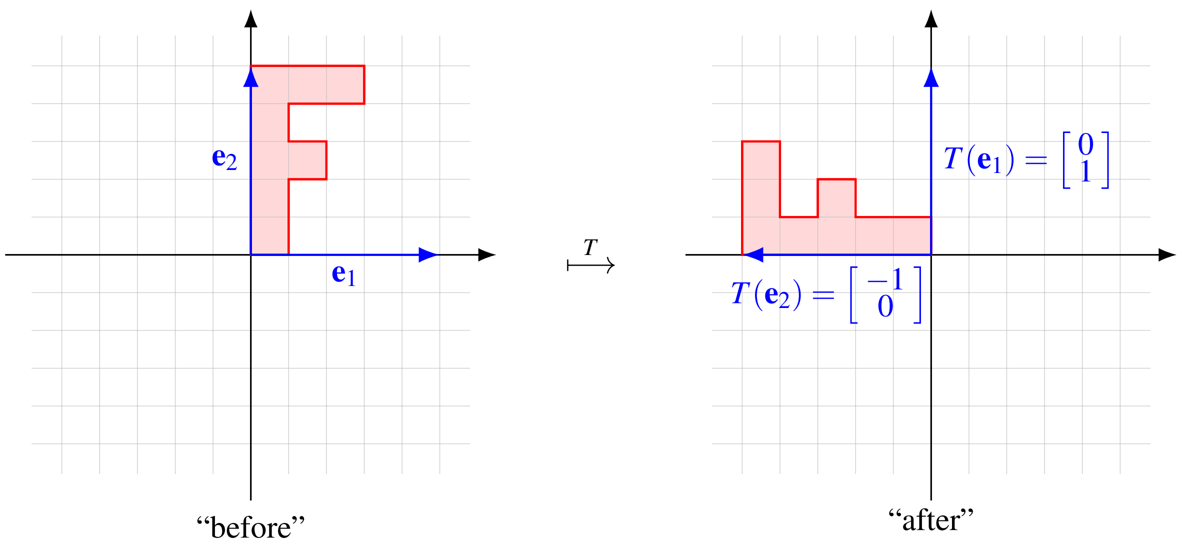
A. *Find the matrix of rotations, reflections, scalings, and shearings in $\mathbb{R}^2$ and $\mathbb{R}^3$.*

In this section, we will examine some special examples of linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ including rotations and reflections.

Example 6.9: Rotation by 90° in $\mathbb{R}^2$

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is given by a counterclockwise rotation by 90 degrees. Find the matrix A corresponding to this linear transformation. Find a formula for T.

Solution. To visualize a vector function on $\mathbb{R}^2$, it is often useful to consider a pair of before-and-after pictures such as the following:



The picture illustrates how the function T rotates the entire plane (including the pink letter "F") by 90 degrees counterclockwise. The picture also illustrates that when we apply the rotation T to the first and second standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, we obtain the vectors

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The matrix of T has these vectors as its columns. Therefore, the matrix of T is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Finally, we can use this to find a formula for the counterclockwise 90 degree rotation T:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

To illustrate how this works, consider the top right corner of the letter “F”. It has the coordinates $(0.6, 1)$. Applying the function T to the coordinate vector, we get

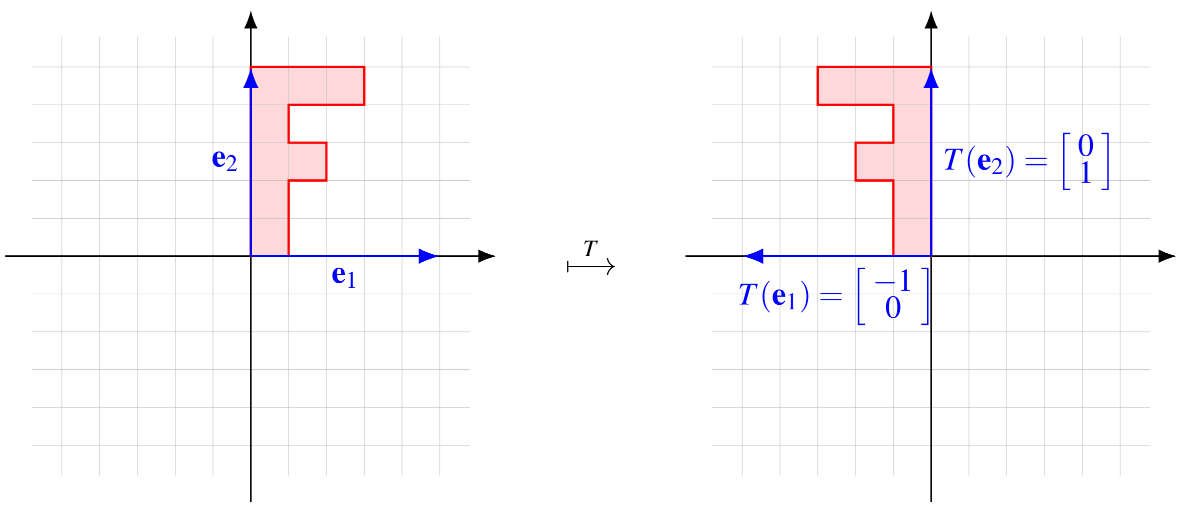
$$T\left(\begin{bmatrix} 0.6 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0.6 \end{bmatrix}.$$

These are precisely the coordinates of the corresponding point on the letter “F” after the rotation. ♠

Example 6.10: Reflection about the y-axis in $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection about the y-axis. Find the matrix A corresponding to this linear transformation, and a formula for T.

Solution. The before-and-after picture for a reflection about the y-axis looks like this:



We see that

$$T(\mathbf{e}_1) = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, the matrix of T is

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The formula for a reflection about the y-axis is:

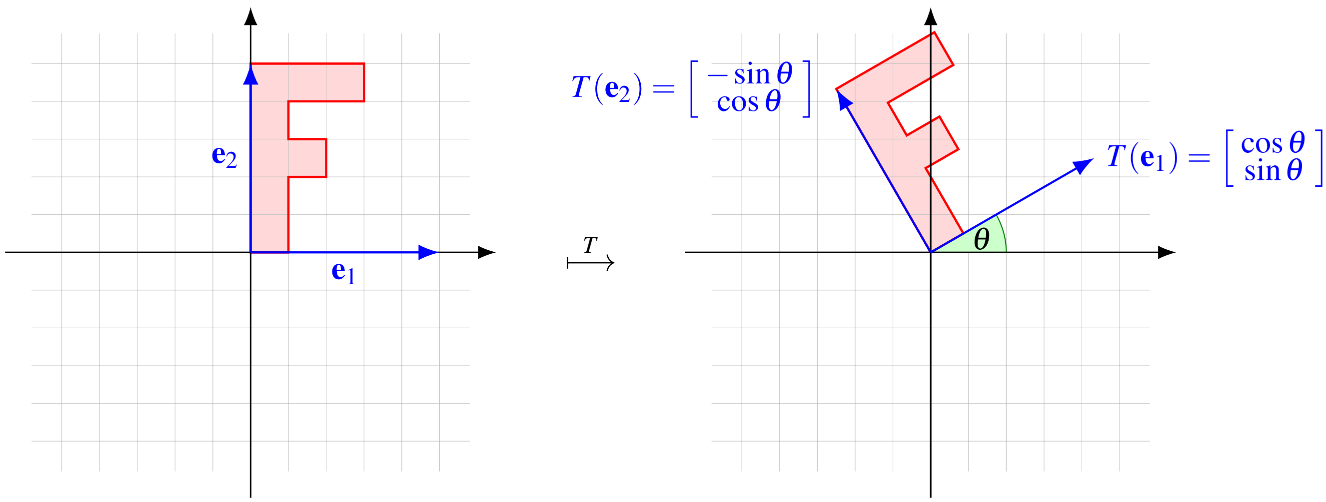
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

♠

Example 6.11: Rotation by an arbitrary angle in $\mathbb{R}^2$

Find the matrix A for a counterclockwise rotation by angle θ in $\mathbb{R}^2$.

Solution. The before-and-after picture is as follows:



Thus the matrix of T is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

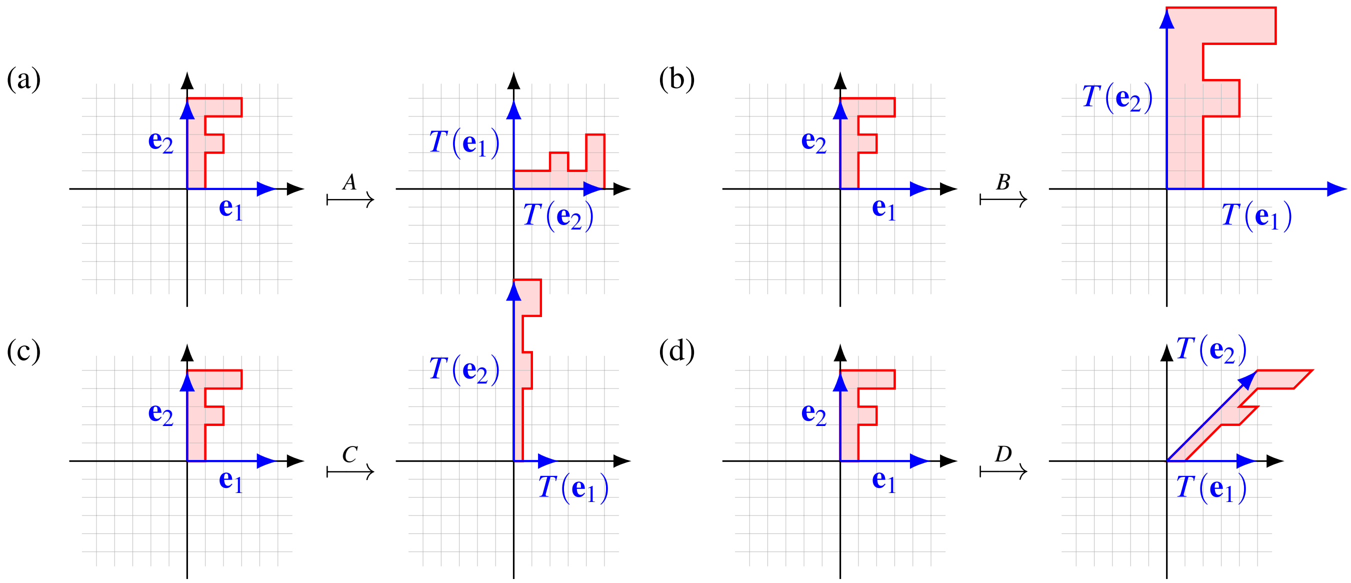
♠

Example 6.12: More linear transformations of the plane

Describe the linear transformation that is given by each of the following matrices. Draw a before-and-after picture for each.

$$(a)\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (b)\quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad (c)\quad C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \quad (d)\quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution. To draw each before-and-after picture, we can start by drawing the images of the two standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, which are the columns of the transformation matrix. We have also drawn the image of the letter "F", to better illustrate the effect of each transformation.

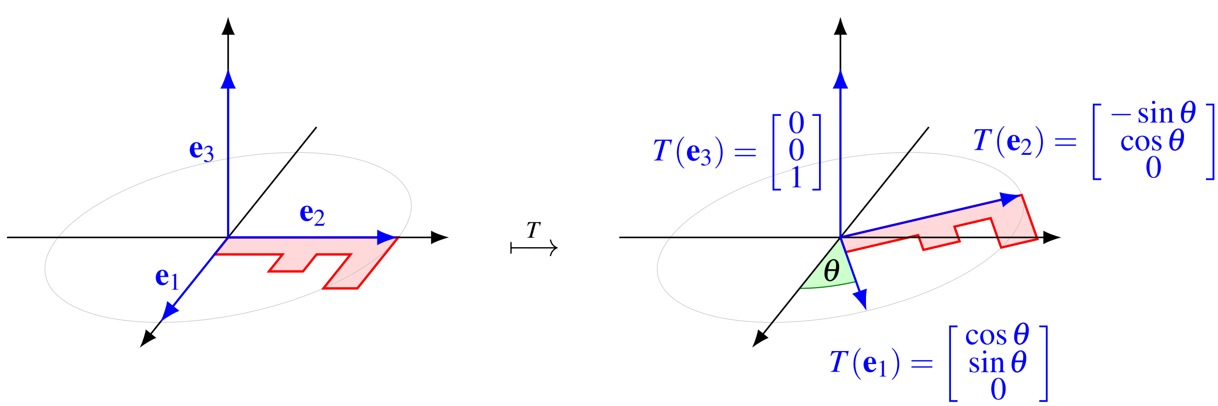


The transformation A is a reflection about the line $x = y$. The transformation B is a scaling by a factor of 2. The transformation C is also a scaling, but by a different factor in the x- and y-directions. It scales the x-direction by a factor of $\frac{1}{2}$ (or equivalently, shrinks it by a factor of 2), and scales the y-direction by a factor of 2. The transformation D is called a **shearing**. It keeps one line (the x-axis) fixed, while shifting all other points by varying distances along lines that are parallel to the x-axis. ♠

Example 6.13: Rotation in $\mathbb{R}^3$

Find the matrix of a rotation by angle θ about the z-axis in 3-dimensional space, counterclockwise when viewed from above.

Solution. Here is the before-and-after picture. A rotation in 3-dimensional space is usually harder to visualize than in the plane, but fortunately, the rotation is about the z-axis, so all the "action" is taking place in the xy-plane.



Therefore, the matrix of the rotation is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

♠

Exercises

Exercise 6.3.1 *Find the matrix for the linear transformation that rotates every vector in $\mathbb{R}^2$ by an angle of $\pi/3$.*

Exercise 6.3.2 *Find the matrix for the linear transformation that reflects every vector in $\mathbb{R}^2$ about the x-axis.*

Exercise 6.3.3 *Find the matrix for the linear transformation that reflects every vector in $\mathbb{R}^2$ about the line $y = -x$.*

Exercise 6.3.4 *Find the matrix for the linear transformation that stretches $\mathbb{R}^2$ by a factor of 3 in the vertical direction.*

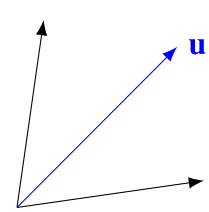
Exercise 6.3.5 *Find the matrix of the linear transformation that reflects every vector in $\mathbb{R}^3$ about the xy-plane.*

Exercise 6.3.6 *Find the matrix of the linear transformation that reflects every vector in $\mathbb{R}^3$ about plane $x = z$.*

Exercise 6.3.7 *Describe the linear transformation that is given by each of the following matrices. Draw a before-and-after picture for each.*

$$(a)\quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad (b)\quad B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad (c)\quad C = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad (d)\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercise 6.3.8 *Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector in $\mathbb{R}^2$. Find the matrix that reflects all vectors about this vector, as shown in the following picture.*



6.4 Properties of linear transformations

Outcomes

A. *Use properties of linear transformations to solve problems.*

B. *Find the composite of transformations and the inverse of a transformation.*

We begin by noting that linear transformations preserve the zero vector, negation, and linear combinations.

Proposition 6.14: Properties of linear transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

- *T preserves the zero vector: $T(\mathbf{0}) = \mathbf{0}$.*
- *T preserves negation: $T(-\mathbf{v}) = -T(\mathbf{v})$.*
- *T preserves linear combinations:*

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k) \;=\; a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_kT(\mathbf{v}_k).$$

Example 6.15: Linear combination

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}\right)$.

Solution. Using the third property in Proposition 6.14, we can find $T\left(\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}\right)$ by writing $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$. By solving the appropriate system of equations, we find that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} T\left(\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) \\ &= T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}. \end{aligned}$$

♠

Suppose that we first apply a linear transformation T to a vector, and then the linear transformation S to the result. The resulting two-step transformation is also a linear transformation, called the **composition** of T and S.

Definition 6.16: Composition of linear transformations

*Let $S : \mathbb{R}^k \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Then the **composition** of S and T (also called the **composite transformation** of S and T) is the linear transformation*

$$T \circ S : \mathbb{R}^k \to \mathbb{R}^m$$

that is defined by

$$(T \circ S)(\mathbf{v}) = T(S(\mathbf{v})),$$

for all $\mathbf{v} \in \mathbb{R}^k$.

Notice that the resulting vector will be in $\mathbb{R}^m$. Be careful to observe the order of transformations. The composite transformation $T \circ S$ means that we are *first* applying S, and *then* T. Composition of linear transformations is written from right to left. The composition $T \circ S$ is sometimes pronounced "T *after* S".

Theorem 6.17: Matrix of a composite transformation

Let $S : \mathbb{R}^k \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Let A be the matrix corresponding to S, and let B be the matrix corresponding to T. Then the matrix corresponding to the composite linear transformation $T \circ S$ is BA.

Proof. For all $\mathbf{v} \in \mathbb{R}^k$, we have

$$(T \circ S)(\mathbf{v}) = T(S(\mathbf{v})) = B(A\mathbf{v}) = (BA)\mathbf{v}.$$

Therefore, BA is the matrix corresponding to $T \circ S$. ♠

Example 6.18: Two rotations

Find the matrix for a counterclockwise rotation by angle $\theta + \phi$ in two different ways, and compare.

Solution. Let A_θ be the matrix of a rotation by θ, and let A_ϕ be the matrix of a rotation by angle ϕ. We calculated these matrices in Example 6.11. Then a rotation by the angle $\theta+\phi$ is given by the product of these two matrices:

$$\begin{aligned} A_\theta A_\phi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}. \end{aligned}$$

On the other hand, we can compute the matrix for a rotation by angle $\theta+\phi$ directly:

$$A_{\theta+\phi} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

The fact that these matrices are equal amounts to the well-known trigonometric identities for the sum of two angles, which we have here derived using linear algebra concepts:

$$\begin{aligned} \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi, \\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi. \end{aligned}$$

♠

Example 6.19: Multiple rotations in $\mathbb{R}^3$

Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is given as follows: a rotation by 30 degrees about the z-axis, followed by a rotation by 45 degrees about the x-axis.

Solution. It would be quite difficult to picture the transformation T in one step. Fortunately, we don't have to do this. All we have to do is find the matrix for each rotation separately, then multiply the two matrices. We have the be careful to multiply the matrices in the correct order.

Let B be the matrix for a 30-degree rotation about the z-axis. It is given exactly as in Example 6.13:

$$B = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let C be the matrix for a 45-degree rotation about the x-axis. It is analogous to Example 6.13, except that the rotation takes place in the yz-plane instead of the xy-plane.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45^\circ \\ 0 & \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally, to apply the linear transformation T to a vector $\mathbf{v}$, we must first apply B and then C. This means that $T(\mathbf{v}) = C(B\mathbf{v})$. Therefore, the matrix corresponding to T is CB. Note that it is important that we multiply the matrices corresponding to each subsequent rotation *from right to left*.

$$A = CB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

♠

We can also consider the inverse of a linear transformation. The inverse of T, if it exists, is a linear transformation that undoes the effect of T.

Definition 6.20: Inverse of a transformation

Let $T,S:\mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Suppose that for each $\mathbf{v} \in \mathbb{R}^n$,

$$(S \circ T)(\mathbf{v}) = \mathbf{v}$$

and

$$(T \circ S)(\mathbf{v}) = \mathbf{v}.$$

Then S is called the ***inverse*** *of T, and we write $S = T^{-1}$.*

Example 6.21: Inverse of a transformation

What is the inverse of a counterclockwise rotation by the angle θ in $\mathbb{R}^2$?

Solution. The inverse is a clockwise rotation by the same angle. ♠

It is perhaps not entirely unexpected that the matrix of T^{-1} is exactly the inverse of the matrix of T, if it exists.

Theorem 6.22: Matrix of the inverse transformation

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the corresponding $n \times n$-matrix. Then T has an inverse if and only if the matrix A is invertible. In this case, the matrix of T^{-1} is A^{-1}.

Example 6.23: Matrix of the inverse transformation

Find the inverse of the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 7x+4y \end{bmatrix}.$$

Solution. The easiest way to do this is to find the matrix of T. We have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Therefore, the matrix of T is

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}.$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}.$$

Therefore, T^{-1} is the linear transformation defined by

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x-y \\ -7x+2y \end{bmatrix}.$$

♠

Exercises

Exercise 6.4.1 *Find the matrix for the linear transformation that reflects every vector in $\mathbb{R}^2$ about the x-axis and then reflects about the y-axis.*

Exercise 6.4.2 *Find the matrix for the linear transformation that rotates every vector in $\mathbb{R}^2$ by an angle of $2\pi/3$ and then reflects about the x-axis.*

Exercise 6.4.3 *Find the matrix for the linear transformation that rotates every vector in $\mathbb{R}^2$ by a counter-clockwise angle of $\pi/6$ and then reflects about the x-axis followed by a reflection about the y-axis.*

Exercise 6.4.4 *Find the matrix for the linear transformation that reflects every vector in $\mathbb{R}^2$ about the x-axis and then rotates by an angle of $\pi/4$.*

Exercise 6.4.5 *Find the matrix of the linear transformation that rotates every vector in $\mathbb{R}^3$ counterclock-wise about the z-axis when viewed from the positive z-axis by an angle of 30 degrees and then reflects about the xy-plane.*

Exercise 6.4.6 *Prove the three properties in Proposition 6.14, using only the definition of a linear transformation (i.e., the fact that it preserves addition and scalar multiplication).*

Exercise 6.4.7 *Let T be the linear transformation with matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and S the linear transformation with matrix $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find the matrix of $S \circ T$. Compute $(S \circ T)(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.*

Exercise 6.4.8 *Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is the linear transformation with matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.*

Exercise 6.4.9 *What is the inverse of a reflection? Rotation? Shearing? Scaling?*

Exercise 6.4.10 *Let T be a linear transformation with matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. Find the matrix of T^{-1}.*

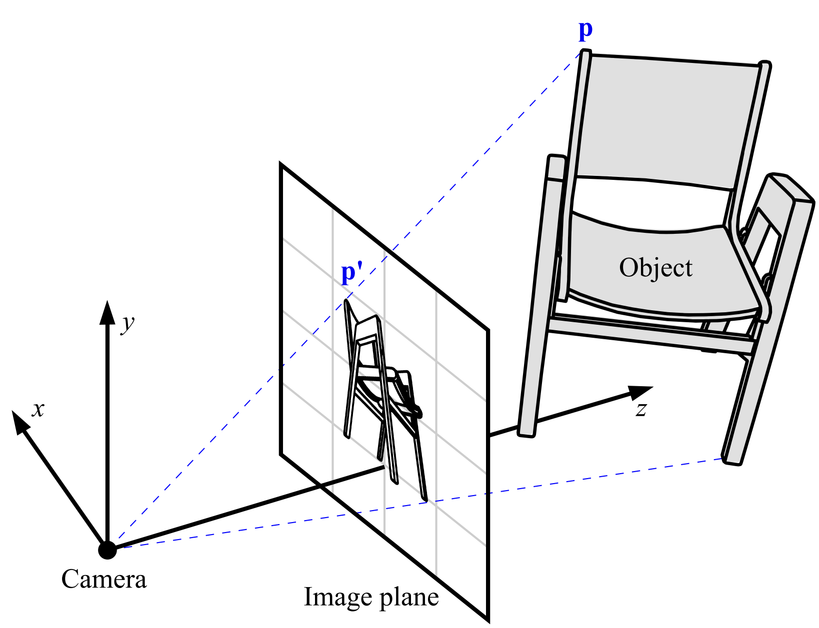
Exercise 6.4.11 *Let T be the linear transformation given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4x-3y \\ 2x-2y \end{bmatrix}$. Find the matrix of T^{-1}.*

Exercise 6.4.12 *Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Find the matrix of T and the matrix of T^{-1}.*

6.5 Application: Perspective rendering

As an application of linear transformations, we consider the problem of perspective rendering. Imagine some object has been described by coordinates in 3-dimensional space, and we wish to make an image of the object as it would be seen by a human eye or by a camera. The process of computing such an image is known as **rendering**.

Conceptually, the rendering process makes use of a **camera**, which we will assume is located at the origin of a 3-dimensional coordinate system called the **camera coordinate system**, and an **image plane**, which we will assume is the plane $z = 1$ in camera coordinates. The 3-dimensional space also contains one or more objects that we wish to render. We can consider the object to be described by a set of points. For each point $\mathbf{p}$ on the object, we draw a straight line from $\mathbf{p}$ to the camera, and let $\mathbf{p}'$ be the point where this line intersects the image plane. The point $\mathbf{p}$ of the object is rendered as the point $\mathbf{p}'$ in the image. This process is illustrated in the following figure:

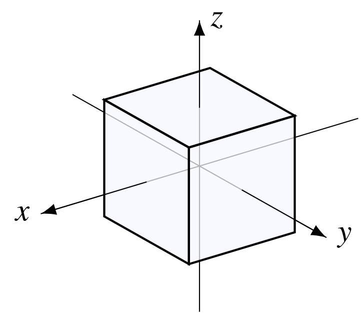


Object coordinates

It is convenient to describe each object in its own coordinate system, called the **object coordinate system**. To illustrate this concept, we will consider a cube of side length 2, centered at the origin. The 8 corners of this cube have the following coordinates in the object coordinate system:

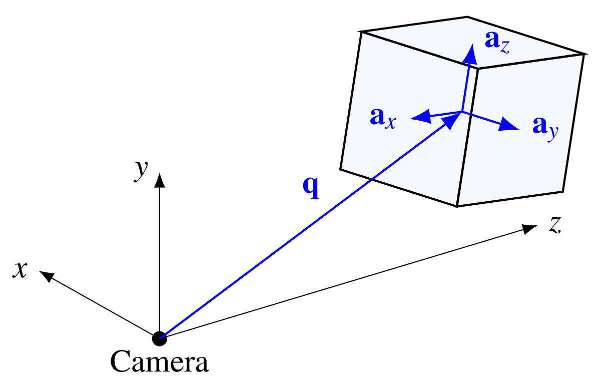
$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

For later reference, let us call this the **standard cube**. The following picture shows the standard cube within its object coordinate system:



Conversion to camera coordinates

Before we render an object, we need to place it in some appropriate location relative to the camera. We do this by specifying four vectors $\mathbf{q}$, $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ in $\mathbb{R}^3$. Here, $\mathbf{q}$ is the origin of the object coordinate system, relative to the camera coordinate system. The vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the axes of the object coordinate system, relative to the camera coordinate system, as shown in the following illustration:



Thus, given a point with object coordinates $\mathbf{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, we can find its camera coordinates $\mathbf{p} = \begin{bmatrix}p_x\\p_y\\p_z\end{bmatrix}$ by the following formula:

$$\mathbf{p} = \mathbf{q} + x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z.$$

If we write A for the 3×3-matrix whose columns are $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, we can also write this formula more succinctly as

$$\mathbf{p} = \mathbf{q} + A\mathbf{v}.$$

Example 6.24: Converting object coordinates to camera coordinates

Let

$$\mathbf{q} = \begin{bmatrix} 0 \\ 0.5 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix}.$$

Convert each of the 8 corners of the standard cube from object coordinates to camera coordinates.

Solution. Let $\mathbf{v}_1,\ldots,\mathbf{v}_8$ be the object coordinates of the 8 corners of the cube:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

We convert each of them to camera coordinates using the formula $\mathbf{p}_i = \mathbf{q} + A\mathbf{v}_i$:

$$\begin{aligned}
\mathbf{p}_1 &= \begin{bmatrix} 0 \\ 0.5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.5 \\ 4.8 \end{bmatrix}, \\
\mathbf{p}_2 &= \begin{bmatrix} 0 \\ 0.5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.4 \\ -0.5 \\ 4.8 \end{bmatrix}, \\
\mathbf{p}_3 &= \begin{bmatrix} 0 \\ 0.5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.5 \\ 6.4 \end{bmatrix},
\end{aligned}$$

and so on. Continuing in the same fashion, we find $\mathbf{p}_1,\ldots,\mathbf{p}_8$:

$$\begin{bmatrix} 1.4 \\ 1.5 \\ 4.8 \end{bmatrix}, \begin{bmatrix} 1.4 \\ -0.5 \\ 4.8 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 1.5 \\ 6.4 \end{bmatrix}, \begin{bmatrix} 0.2 \\ -0.5 \\ 6.4 \end{bmatrix}, \begin{bmatrix} -0.2 \\ 1.5 \\ 3.6 \end{bmatrix}, \begin{bmatrix} -0.2 \\ -0.5 \\ 3.6 \end{bmatrix}, \begin{bmatrix} -1.4 \\ 1.5 \\ 5.2 \end{bmatrix}, \begin{bmatrix} -1.4 \\ -0.5 \\ 5.2 \end{bmatrix}. \qquad (6.2)$$

♠

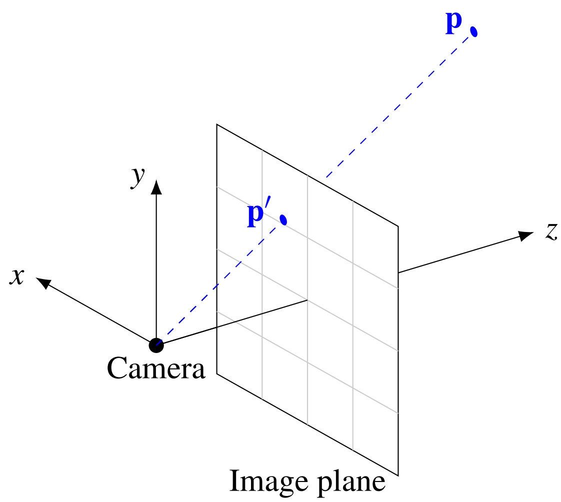
We can also write $f : \mathbb{R}^3 \to \mathbb{R}^3$ for the function that converts object coordinates to camera coordinates, i.e.,

$$f(\mathbf{v}) = \mathbf{q} + A\mathbf{v}.$$

We note that this is not a linear function, because $f(\mathbf{0}) \neq \mathbf{0}$. The function f is called an **affine function**, which means that it is a linear function $\mathbf{v} \mapsto A\mathbf{v}$ followed by a translation $\mathbf{v} \mapsto \mathbf{q} + \mathbf{v}$.

Rendering

Once we know the camera coordinates $\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ of a point, we need to render the point, i.e., find its coordinates in the image plane.



Since the camera is located at the origin, the line that passes through the camera and the point $\mathbf{p}$ has the parametric equation

$$\mathbf{r} = t\mathbf{p} = \begin{bmatrix} tp_x \\ tp_y \\ tp_z \end{bmatrix}.$$

Since the image plane is the plane $z = 1$, we must set t such that $tp_z = 1$, i.e., $t = \frac{1}{p_z}$. Therefore, the coordinates of the rendered point are

$$\mathbf{p}' = \frac{1}{p_z}\mathbf{p} = \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1 \end{bmatrix}.$$

Finally, since the image plane is 2-dimensional, we can forget the now useless z-coordinate, and render the point at the coordinates $\begin{bmatrix} p_x/p_z \\ p_y/p_z \end{bmatrix}$ in the 2-dimensional image plane.

Example 6.25: Rendering

Render the cube from Example 6.24.

Solution. We must apply the rendering function

$$g\left(\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}\right) = \begin{bmatrix} p_x/p_z \\ p_y/p_z \end{bmatrix}$$

to each of the corners of the cube from (6.2). We have

$$g\left(\begin{bmatrix}1.4\\1.5\\4.8\end{bmatrix}\right) = \begin{bmatrix}1.4/4.8\\1.5/4.8\end{bmatrix} = \begin{bmatrix}0.292\\0.312\end{bmatrix},$$

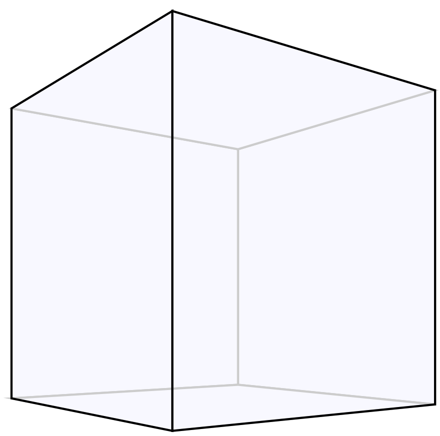
$$g\left(\begin{bmatrix}1.4\\-0.5\\4.8\end{bmatrix}\right) = \begin{bmatrix}1.4/4.8\\-0.5/4.8\end{bmatrix} = \begin{bmatrix}0.292\\-0.104\end{bmatrix},$$

$$g\left(\begin{bmatrix}0.2\\1.5\\6.4\end{bmatrix}\right) = \begin{bmatrix}0.2/6.4\\1.5/6.4\end{bmatrix} = \begin{bmatrix}0.031\\0.234\end{bmatrix},$$

and so on. The 8 rendered points are:

$$\begin{bmatrix}0.292\\0.312\end{bmatrix}, \begin{bmatrix}0.292\\-0.104\end{bmatrix}, \begin{bmatrix}0.031\\0.234\end{bmatrix}, \begin{bmatrix}0.031\\-0.078\end{bmatrix}, \begin{bmatrix}-0.056\\0.417\end{bmatrix}, \begin{bmatrix}-0.056\\-0.139\end{bmatrix}, \begin{bmatrix}-0.269\\0.288\end{bmatrix}, \begin{bmatrix}-0.269\\-0.096\end{bmatrix}.$$

Drawing these in the 2-dimensional image plane, we get the following picture, which is the final perspective-rendered image of the cube:



♠

Animation

We placed our object in the camera coordinate system using a coordinate transformation function

$$f(\mathbf{v}) = \mathbf{q} + A\mathbf{v}.$$

One of the advantages of using such a coordinate transformation (as opposed to specifying the object points directly in the camera coordinate system) is that this makes it very easy to move the objects around, rotate them, scale and shrink them, etc. For example:

1. To move the object to a different location, we only have to change the vector $\mathbf{q}$.

2. To rotate the object about its own z-axis, we only have to replace $\mathbf{v}$ by $R_\theta \mathbf{v}$, where R_θ is the matrix for a rotation about the z-axis by angle θ:

$$R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Similarly to R_θ, we can also insert other transformation matrices (for example, we could rotate the object about its x-axis instead of its z-axis, scale the object, etc). We can even make an animation by rendering the object repeatedly for different values of these parameters.

Example 6.26: An animated cube

Make an animation of a rotating, moving cube. The animation is 5 seconds long (i.e., time t ranges from 0 to 5). The location of the cube at time t, in camera coordinates, is given by

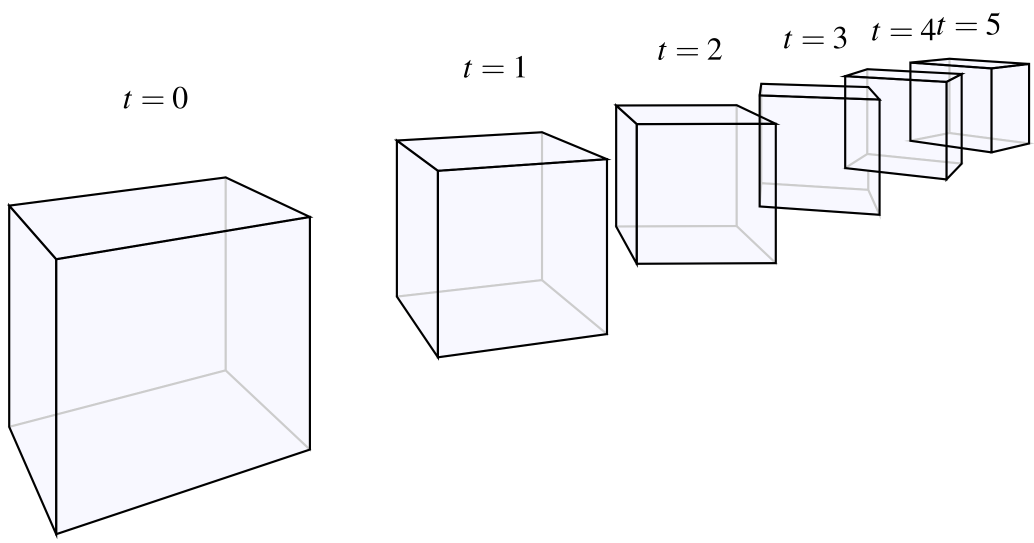
$$\mathbf{q}(t) = \begin{bmatrix} -3+3t \\ -3 \\ 8+3t \end{bmatrix}.$$

The transformation matrix A is as in Example 6.24. Moreover, the cube should make one quarter rotation about its z-axis during the time of the animation, i.e., it should be transformed by R_θ, where $\theta = \frac{\pi}{10}t$. Compute 6 frames of the animation, for $t = 0, t = 1, \dots, t = 5$.

Solution. For each of the animation frames $t \in \{0,1,2,3,4,5\}$, we do a calculation very similar to that of Examples 6.24 to convert the cube coordinates to camera coordinates, using the coordinate transformation

$$f(\mathbf{v}) = \mathbf{q}(t) + AR_\theta \mathbf{v},$$

where $\theta = \frac{\pi}{10}t$. We then render each of the frames using the same method as in Example 6.25. We skip the detailed calculations, which are best done by computer (though they could be done by hand, of course, as we did in Examples 6.24 and 6.25). The final rendered frames look like this:



Note that there is a bit of distortion in the first and last cubes. This is because the camera is very close to the image plane (the scene has been "filmed" with a wide-angle camera). The distortion goes away if you close one eye and bring the other eye very close to the page. ♠

7. Determinants

7.1 Determinants of 2×2- and 3×3-matrices

Outcomes

A. *Calculate the determinant of* 2×2*-matrices and* 3×3*-matrices.*

Let A be an $n\times n$-matrix. The **determinant** of A, denoted by $\det(A)$, is a very important number which we will explore throughout this chapter.

The determinant of a 2×2-matrix is given by the following formula.

Definition 7.1: Determinant of a 2×2-matrix

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then*

$$\det(A)=ad-bc.$$

Example 7.2: A 2×2 determinant

Find $\det(A)$ *for the matrix* $A=\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

Solution. We have $\det(A)=2\cdot 6-4\cdot(-1)=12+4=16$. ♠

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

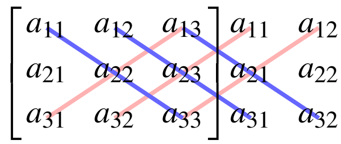
$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc.$$

Definition 7.3: Determinant of a 3×3-matrix

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. *Then*

$$\det(A)=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}.$$

The following picture may help in memorizing the formula:



Here, we have written down the matrix A, then repeated the first two columns next to it. The blue lines correspond to the positive terms of the determinant: $a_{11}a_{22}a_{33}$, $a_{12}a_{23}a_{31}$, and $a_{13}a_{21}a_{32}$. The pink lines correspond to the negative terms: $a_{31}a_{22}a_{13}$, $a_{32}a_{23}a_{11}$, and $a_{33}a_{21}a_{12}$.

Example 7.4: A 3×3 determinant

Find $\det(A)$, *where*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Solution. We have

$$\det(A) = \begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 0\cdot 1\cdot(-1)+1\cdot 0\cdot 1+2\cdot 3\cdot 1-1\cdot 1\cdot 2-1\cdot 0\cdot 0-(-1)\cdot 3\cdot 1 = 7.$$

♠

Exercises

Exercise 7.1.1 *Find the determinants of the following matrices.*

$$(a)\ \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \quad (b)\ \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \quad (c)\ \begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} \quad (d)\ \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}$$

Exercise 7.1.2 *Find the following determinants.*

$$(a)\ \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad (b)\ \begin{vmatrix} 1 & 0 & 2 \\ 2 & 5 & 3 \\ -1 & 0 & 0 \end{vmatrix} \quad (c)\ \begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \quad (d)\ \begin{vmatrix} 0 & -2 & 1 \\ 4 & 1 & -3 \\ -1 & 3 & 1 \end{vmatrix}$$

7.2 Minors and cofactors

Outcomes

A. *Compute minors and cofactors of matrices.*

B. *Use cofactor expansion to compute the determinant of an $n \times n$-matrix.*

Determinants of larger matrices can be computed in terms of the determinants of smaller matrices. We begin with the following definition.

Definition 7.5: The ij^{th} minor of a matrix

*Let A be an $n \times n$-matrix. The ij^{th} **minor** of A, denoted by M_{ij}, is the determinant of the $(n-1) \times (n-1)$-matrix that is obtained by deleting the i^{th} row and the j^{th} column of A.*

Hence, there is a minor associated with each entry of A. The following example illustrates this definition.

Example 7.6: Finding minors of a matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find the minors M_{12} and M_{33}.

Solution. First we will find M_{12}. By definition, this is the determinant of the 2×2-matrix that results when we delete the first row and the second column of A. This minor is given by

$$M_{12} = \begin{vmatrix} \not{1} & \not{2} & \not{3} \\ 4 & \not{3} & 2 \\ 3 & \not{2} & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4 \cdot 1 - 2 \cdot 3 = -2.$$

Similarly, M_{33} is the determinant of the 2×2-matrix that is obtained by deleting the third row and the third column of A. This minor is therefore

$$M_{33} = \begin{vmatrix} 1 & 2 & \not{3} \\ 4 & 3 & \not{2} \\ \not{3} & \not{2} & \not{1} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5.$$

♠

We now define the ij^{th} cofactor of a matrix A, which is either plus or minus the ij^{th} minor.

Definition 7.7: The ij^{th} cofactor of a matrix

Suppose A is an $n \times n$-matrix. The ij^{th} ***cofactor****, denoted by C_{ij}, is defined to be*

$$C_{ij} = (-1)^{i+j} M_{ij}$$

In other words, the ij^{th} cofactor is equal to the corresponding minor if $i+j$ is even, and the negative of the minor if $i+j$ is odd. For remembering the signs, the following picture is sometimes helpful:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

Example 7.8: Finding cofactors of a matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find the cofactors C_{12} and C_{33}.

Solution. We have already computed the corresponding minors in Example 7.6. For the cofactors, we have:

$$\begin{array}{rcccccccc} C_{12} & = & (-1)^{1+2} M_{12} & = & -M_{12} & = & -(-2) & = & 2, \\ C_{33} & = & (-1)^{3+3} M_{33} & = & +M_{33} & = & +(-5) & = & -5. \end{array}$$

Note that $1+2$ is odd, so $C_{12} = -M_{12}$. On the other hand, $3+3$ is even, so $C_{33} = M_{33}$. ♠

You may wish to find the remaining cofactors of the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of an $n \times n$-matrix.

Definition 7.9: The determinant of an $n \times n$-matrix

Let A be an $n \times n$-matrix. Then $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together. This process is known as ***expanding along the i^{th} row (or column)****.*

In formulas, the process of expanding along the i^{th} row is given as follows:

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}.$$

Similarly, the process of expanding along the j^{th} column is:

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj}.$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of which row or column you expand, you will always get the same number, which is the determinant of the matrix A. This method of evaluating a determinant by expanding along a row or a column is also called **cofactor expansion** or **Laplace expansion**.

Example 7.10: Finding a determinant by cofactor expansion

Find $\det(A)$ *using the method of cofactor expansion, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Solution. First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 7.9, the determinant is

$$\begin{aligned}
\det(A) &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\
&= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\
&= 1\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \\
&= 1(-1) - 4(-4) + 3(-5) \\
&= 0.
\end{aligned}$$

As mentioned in Definition 7.9, we can choose to expand along any row or column. Let's try now by expanding along the second row. The calculation is as follows.

$$\det(A) = -4\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4(-4) + 3(-8) - 2(-4) = 0.$$

You can see that for both methods, we obtained $\det(A) = 0$. ♠

You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

Theorem 7.11: The determinant is well-defined

Expanding an $n \times n$-matrix along any row or column always gives the same answer, which is the determinant.

Example 7.12: Determinant of a four by four matrix

Calculate $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 3 \\ 0 & 3 & 0 & 5 \\ 3 & 0 & 3 & 2 \end{vmatrix}$.

Solution. Using the cofactor method, we can expand this determinant along any row or column. But notice that the third row contains two zeros. This makes the cofactor expansion particularly convenient. So let us expand along the third row. We have:

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 3 \\ 0 & 3 & 0 & 5 \\ 3 & 0 & 3 & 2 \end{vmatrix} = 0 \begin{vmatrix} 2 & 3 & 0 \\ 4 & 2 & 3 \\ 0 & 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 3 \\ 3 & 0 & 2 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 0 & 3 \end{vmatrix}.$$

Note that we only need to compute two of the 3×3 determinants, since the remaining two are multiplied by 0. We can compute each 3×3 determinant using the method of Definition 7.3. We find:

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 3 \\ 0 & 3 & 0 & 5 \\ 3 & 0 & 3 & 2 \end{vmatrix} = -3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 2 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 0 & 3 \end{vmatrix} = -3 \cdot 10 - 5 \cdot (-24) = 90.$$

♠

We remark that the cofactor expansion is mainly useful for calculating determinants of small matrices, or matrices containing many zeros. Indeed, imagine calculating the determinant of a 10×10-matrix by the cofactor method. This requires calculating the determinants of ten 9×9-matrices, each of which requires calculating the determinants of nine 8×8-matrices, each of which requires calculating the determinants of eight 7×7-matrices, and so on. Calculating the determinant of a 10×10-matrix by the cofactor method would therefore require $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 2 \cdot 1 = 3628800$ steps!

In the next few sections, we will explore some important properties and characteristics of the determinant, including a much more efficient method of calculating determinants of large matrices.

Exercises

Exercise 7.2.1 *Let* $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. *Find the following minors and cofactors:*

(a) M_{11},

(b) M_{21},

(c) M_{32},

(d) C_{11},

(e) C_{21},

(f) C_{32}.

Exercise 7.2.2 *Let* $A = \begin{bmatrix} 0 & -1 & 3 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 3 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. *Find* M_{11}, M_{21}, M_{32}, C_{11}, C_{21}, *and* C_{32}.

Exercise 7.2.3 *Compute the determinants of the following matrices using cofactor expansion along any row or column.*

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 3 & -2 & 2 \\ 0 & 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -2 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Exercise 7.2.4 *Find the following determinant by expanding (a) along the first row and (b) along the second column.*

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Exercise 7.2.5 *Find the following determinant by expanding (a) along the first column and (b) along the third row.*

$$\begin{vmatrix} 2 & 3 & 1 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{vmatrix}$$

Exercise 7.2.6 *Find the following determinant by expanding (a) along the second row and (b) along the first column.*

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 4 & 0 & 2 \end{vmatrix}$$

Exercise 7.2.7 *Compute the determinant by cofactor expansion. Pick the easiest row or column to use.*

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

7.3 The determinant of a triangular matrix

Outcomes

A. *Calculate the determinant of an upper or lower triangular matrix.*

There is a certain type of matrix for which finding the determinant is a very simple procedure: a triangular matrix.

Definition 7.13: Triangular matrices

An square matrix A is ***upper triangular*** *if $a_{ij} = 0$ whenever $i > j$. In other words, a matrix is upper triangular if the entries below the main diagonal are 0. Thus, an upper triangular matrix looks as follows, where $*$ refers to any non-zero number:*

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}.$$

Similarly, a square matrix is ***lower triangular*** *if all entries above the main diagonal are 0.*

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

Theorem 7.14: Determinant of a triangular matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is equal to the product of the entries on the main diagonal. Written as a formula, we have

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}.$$

Example 7.15: Determinant of a triangular matrix

Compute $\det(A)$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 3 & 33 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Solution. By Theorem 7.14, it suffices to take the product of the elements on the main diagonal. Thus

$$\det(A) \;=\; 1 \cdot 2 \cdot 3 \cdot (-1) \;=\; -6.$$

♠

For comparison, let us compute the determinant without Theorem 7.14, i.e., by using cofactor expansion. If we expand the determinant along the first column, we get:

$$\det(A) = 1\begin{vmatrix} 2 & 6 & -7 \\ 0 & 3 & 33 \\ 0 & 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 16 \\ 0 & 3 & 33 \\ 0 & 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 16 \\ 2 & 6 & -7 \\ 0 & 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 16 \\ 2 & 6 & -7 \\ 0 & 3 & 33 \end{vmatrix}.$$

The only non-zero term in the expansion is

$$1\begin{vmatrix} 2 & 6 & -7 \\ 0 & 3 & 33 \\ 0 & 0 & -1 \end{vmatrix}.$$

We can in turns expand this 3×3 determinant by the cofactor method along the first column:

$$\begin{vmatrix} 2 & 6 & -7 \\ 0 & 3 & 33 \\ 0 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & 33 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 6 & -7 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 6 & -7 \\ 3 & 33 \end{vmatrix}.$$

Again, the only non-zero term is the first term. In summary,

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & 16 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 3 & 33 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 6 & -7 \\ 0 & 3 & 33 \\ 0 & 0 & -1 \end{vmatrix} = 1 \cdot 2\begin{vmatrix} 3 & 33 \\ 0 & -1 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot (-1) = -6.$$

Of course this is just the same as the product of the diagonal entries of A, which is the point of Theorem 7.14.

Exercises

Exercise 7.3.1 *Find the determinant of the following matrices.*

(a) $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -2 & 0 \\ 14 & 1 & 5 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7.4 Determinants and row operations

Outcomes

A. *Determine the effect of a row operation on the determinant of a matrix.*

B. *Use row operations to calculate a determinant.*

Recall that there are three kinds of elementary row operations on matrices:

1. Switch two rows.
2. Multiply a row by a non-zero number.
3. Add a multiple of one row to another row.

The following theorem examines the effect of these row operations on the determinant of a matrix.

Theorem 7.16: Effect of row operations on the determinant

Let A be an $n \times n$-matrix.

1. *If B is obtained from A by switching two rows, then*
$$\det(B) = -\det(A).$$
2. *If B is obtained from A by multiplying one row by a non-zero scalar k, then*
$$\det(B) = k\det(A).$$
3. *If B is obtained from A by adding a multiple of one row to another row, then*
$$\det(B) = \det(A).$$

Notice that the second part of this theorem is true when we multiply *one* row of the matrix by k. If we were to multiply *two* rows of A by k to obtain B, we would have $\det(B) = k^2\det(A)$.

Example 7.17: Using row operations to calculate a determinant

Use row operations to calculate the following determinant:

$$\begin{vmatrix} 1 & 5 & 5 \\ 0 & 0 & -3 \\ 0 & 2 & 7 \end{vmatrix}.$$

Solution. If we switch the second and third rows, we obtain a triangular matrix, of which the determinant is easy to compute. By Theorem 7.16, switching two rows negates the determinant. We therefore have:

$$\begin{vmatrix} 1 & 5 & 5 \\ 0 & 0 & -3 \\ 0 & 2 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & -3 \end{vmatrix} = -(1\cdot 2\cdot(-3)) = 6.$$

♠

Example 7.18: Using row operations to calculate a determinant

Use row operations to calculate the following determinant:

$$\begin{vmatrix} 1 & 4 & -2 \\ 1 & 8 & 1 \\ 2 & 4 & -9 \end{vmatrix}.$$

Solution. We can use elementary row operations to reduce this matrix to triangular form:

$$\begin{bmatrix} 1 & 4 & -2 \\ 1 & 8 & 1 \\ 2 & 4 & -9 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\simeq} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 2 & 4 & -9 \end{bmatrix} \overset{R_3 \leftarrow R_3 - 2R_1}{\simeq} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 0 & -4 & -5 \end{bmatrix} \overset{R_3 \leftarrow R_3 + R_2}{\simeq} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & -2 \end{bmatrix}.$$

Each of the row operations is of the form "add a multiple of one row to another row", and therefore does not change the determinant. We therefore have:

$$\begin{vmatrix} 1 & 4 & -2 \\ 1 & 8 & 1 \\ 2 & 4 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 2 & 4 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 0 & -4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & -2 \end{vmatrix} = 1\cdot 4\cdot(-2) = -8.$$

♠

In general, we can convert any square matrix to triangular form using elementary row operations. In fact, it is always possible to do so using only elementary operations of the first and third kind (swap two rows or add a multiple of one row to another). This gives us a very efficient way to compute determinants. If the matrices are large, this method is much more efficient than the cofactor method.

Example 7.19: Using row operations to calculate a determinant

Use elementary row operations of the first and third kind to calculate the following determinant:

$$\begin{vmatrix} 0 & 2 & 1 & 4 \\ 2 & 2 & -4 & -1 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 2 & 5 \end{vmatrix}.$$

Solution. We use elementary row operations to reduce the matrix to triangular form:

$$\begin{bmatrix} 0 & 2 & 1 & 4 \\ 2 & 2 & -4 & -1 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 2 & 5 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 2 & 2 & -4 & -1 \\ 0 & 2 & 1 & 4 \\ 1 & 3 & 2 & 5 \end{bmatrix} \overset{R_2 \leftarrow R_2-2R_1}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 1 & 3 & 2 & 5 \end{bmatrix}$$

$$\overset{R_4 \leftarrow R_4-R_1}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix} \overset{R_4 \leftarrow R_4-R_3}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix} \overset{R_3 \leftrightarrow R_4}{\simeq} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem 7.16, the determinant changes signs each time we swap two rows. The determinant is unchanged when we add a multiple of one row to another. Therefore, we have

$$\begin{vmatrix} 0 & 2 & 1 & 4 \\ 2 & 2 & -4 & -1 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 2 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & -2 & -1 \\ 2 & 2 & -4 & -1 \\ 0 & 2 & 1 & 4 \\ 1 & 3 & 2 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 1 & 3 & 2 & 5 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 2 & 4 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$= + \begin{vmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -6.$$

In practice, the last calculation could have been done in a single step. All we had to do is count the number of swap operations we performed during the row operations. If there is an odd number of swap operations, the sign of the determinant changes; otherwise, it stays the same. ♠

Exercises

Exercise 7.4.1 *Use row operations to calculate the following determinants:*

$$(a) \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}, \quad (b) \begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}, \quad (c) \begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}, \quad (d) \begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix}.$$

7.5 Properties of determinants

Outcomes

A. *Use the determinant of a square matrix to decide whether the matrix is invertible.*

B. *From the determinants of two matrices, calculate the determinant of their product.*

C. *From the determinant of a matrix, calculate the determinant of its inverse.*

D. *From the determinant of a matrix, calculate the determinant of its transpose.*

E. *Calculate the determinant of kA, if the determinant of A is known.*

F. *Without calculation, find the determinant of a matrix containing a row or column of zeros, or a matrix containing a row (or column) that is a scalar multiple of another row (or column).*

G. *Use algebraic properties to reason about determinants.*

One reason that the determinant is such an important quantity is that it permits us to tell whether a square matrix is invertible.

Theorem 7.20: Determinants and invertible matrices

Let A be an $n \times n$-matrix. Then A is invertible if and only if $\det(A) \neq 0$.

Proof. We know that every matrix A can be converted to echelon form by elementary row operations. We also know from Theorem 7.16 that no elementary row operation changes whether the determinant is zero or not. Let R be an echelon form of A. Because R is an echelon form, it is also an upper triangular matrix. Case 1: A is invertible. In that case, the rank of R is n, and every diagonal entry of R is a pivot entry (therefore non-zero). It follows that $\det(R) \neq 0$, which implies $\det(A) \neq 0$. Case 2: A is not invertible. In that case, the triangular matrix R contains a row of zeros. It follows that $\det(R) = 0$, and therefore $\det(A) = 0$. ♠

Example 7.21: Determinants and invertible matrices

Determine which of the following matrices are invertible by computing their determinants.

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}.$$

Solution. We have $\det(A) = 3 \cdot 4 - 2 \cdot 6 = 0$ and $\det(B) = 2 \cdot 1 - 5 \cdot 3 = -13$. Therefore, B is invertible and A is not invertible. A quick way to compute the determinant of C is to expand it along the third row. We have

$$\det(C) = 3\begin{vmatrix} 2 & -5 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -5 \\ 2 & 2 \end{vmatrix} = 3 \cdot 4 - 1 \cdot 12 = 0.$$

Therefore, C is not invertible. ♠

As an application of Theorem 7.20, we note that the determinant of an $n \times n$-matrix can be used to predict whether a homogeneous system of equations has non-trivial solutions.

Corollary 7.22: Determinants and homogeneous systems

Let A be an $n \times n$-matrix. Then the homogeneous system $A\mathbf{v} = \mathbf{0}$ has non-trivial solutions if and only if $\det(A) = 0$.

Proof. We know from Theorem 1.35 that the homogeneous system has a non-trivial solution if and only if $\operatorname{rank}(A) < n$. This is the case if and only if A is not invertible, i.e., if and only if $\det(A) = 0$. ♠

Another reason the determinant is important is that it compatible with matrix product.

Theorem 7.23: Determinant of a product

Let A and B be $n \times n$-matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Proof. We first prove this in case $A = E$ is an elementary matrix. Remember from Section 4.6 that elementary matrices correspond to elementary row operations.

1. If E is an elementary matrix for swapping two rows, then $\det(E) = -1$. Also, by Theorem 7.16(1), $\det(EB) = -\det(B)$. Therefore $\det(EB) = \det(E)\det(B)$.

2. If E is an elementary matrix for multiplying a row by a non-zero scalar k, then $\det(E) = k$. Also, by Theorem 7.16(2), $\det(EB) = k\det(B)$. Therefore $\det(EB) = \det(E)\det(B)$.

3. If E is an elementary matrix for adding a multiple of one row to another, then $\det(E) = 1$. Also, by Theorem 7.16(3), $\det(EB) = \det(B)$. Therefore $\det(EB) = \det(E)\det(B)$.

Now consider the case where A is an arbitrary matrix. Case 1: A is invertible. Then by Theorem 4.61, we can write A as a product of elementary matrices $A = E_1E_2\cdots E_k$. By repeatedly using the formula $\det(EB) = \det(E)\det(B)$ that we proved above, we have

$$\det(AB) = \det(E_1E_2\cdots E_kB) = \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) = \det(A)\det(B).$$

Case 2: A is not invertible. Then AB is also not invertible (because if C were an inverse of AB, we would have $ABC = I$, and therefore, BC would be an inverse of A). Therefore, by Theorem 7.20, we have $\det(A) = 0$ and $\det(AB) = 0$. It follows that $\det(AB) = \det(A)\det(B)$. ♠

Example 7.24: The determinant of a product

Compare $\det(AB)$ *and* $\det(A)\det(B)$, *where*

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}.$$

Solution. We first compute AB:

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}.$$

The three determinants are

$$\det(AB) = \begin{vmatrix} 11 & 4 \\ -1 & -4 \end{vmatrix} = -40, \quad \det(A) = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 8, \quad \text{and} \quad \det(B) = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5.$$

Therefore $\det(A)\det(B) = 8 \cdot (-5) = -40 = \det(AB)$. ♠

The following proposition summarizes some properties of determinants we have discussed so far, as well as additional properties.

Proposition 7.25: Properties of determinants

Let A, B *be* $n \times n$*-matrices. Then:*

1. $\det(AB) = \det(A)\det(B)$.
2. $\det(I) = 1$.
3. *A is invertible if and only if* $\det(A) \neq 0$. *Moreover, if this is the case, then*
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$
4. $\det(kA) = k^n \det(A)$.
5. $\det(A^T) = \det(A)$.

Proof. Property 1 is a restatement of Theorem 7.23. Property 2 follows from Theorem 7.14, because the identity matrix is an upper triangular matrix. Property 3: The first part is Theorem 7.20. For the second part, assume A is invertible. Then by properties 1 and 2, $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$. The claim follows by dividing both sides of the equation by $\det(A)$. Property 4 follows from Theorem 7.16(2), because kA is obtained from A by multiplying all n rows by k. Each time we multiple one row by k, the determinant is multiplied by k. Property 5 follows because expanding $\det(A)$ along columns amounts to the same thing as expanding $\det(A^T)$ along rows. ♠

We end this section with a few useful ways of spotting matrices of determinant 0.

Theorem 7.26: Special matrices with zero determinant

Let A be an $n \times n$-matrix.

1. *If A has a row consisting only of zeros, or a column consisting only of zeros, then* $\det(A) = 0$.
2. *If A has a row that is a scalar multiple of another row, or a column that is a scalar multiple of another column, then* $\det(A) = 0$.

Proof. The first property follows by cofactor expansion: simply expand the determinant along the row or column that consists only of zeros. For the second property, assume that A has a row that is a scalar multiple of another row. We can then perform an elementary row operation to create a row of zeros. By Theorem 7.16(3), the determinant is unchanged, so that $\det(A) = 0$. In the case that A has a column that is a scalar multiple of another column, we apply the same reasoning to A^T and use the fact that $\det(A) = \det(A^T)$. ♠

Exercises

Exercise 7.5.1 *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

An operation is done to get from A to a matrix B. In each case, identify which operation was done and explain how it will affect the value of the determinant.

(a)
$$B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

(b)
$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

(c)
$$B = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

(d)
$$B = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

(e)
$$B = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Exercise 7.5.2 *Let A be an $n \times n$-matrix and suppose there are $n-1$ rows such that the remaining row is a linear combinations of these $n-1$ rows. Show* $\det(A) = 0$.

Exercise 7.5.3 *Let A be an $n \times n$-matrix. Show that if* $\det(A) \neq 0$ *and* $A\mathbf{x} = \mathbf{0}$, *then* $\mathbf{x} = \mathbf{0}$.

Exercise 7.5.4 *Using only Theorems 7.14 and 7.23, show that* $\det(kA) = k^n \det(A)$ *for an $n \times n$-matrix A and scalar k.*

Exercise 7.5.5 *Construct two random 2×2-matrices A and B and verify that* $\det(A)\det(B) = \det(AB)$.

Exercise 7.5.6 *Is it true that* $\det(A+B) = \det(A) + \det(B)$*? If this is so, explain why. If it is not so, give a counterexample.*

Exercise 7.5.7 *An $n \times n$-matrix is called **nilpotent** if there exists some positive integer k such that* $A^k = 0$. *If A is a nilpotent matrix, what are the possible values of* $\det(A)$*?*

Exercise 7.5.8 *A square matrix is said to be **orthogonal** if $A^T A = I$. Thus the inverse of an orthogonal matrix is its transpose. What are the possible values of* $\det(A)$ *if A is an orthogonal matrix?*

Exercise 7.5.9 *Let A and B be two $n \times n$-matrices. We say that A is **similar** to B, in symbols $A \sim B$, if there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then* $\det(A) = \det(B)$.

Exercise 7.5.10 *Find the determinant of*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{bmatrix}.$$

For which values of a and b is this matrix invertible? Hint: after you compute the determinant, you can factor out $(a-1)$ *and* $(b-1)$ *from it.*

Exercise 7.5.11 *Assume A, B, and C are $n \times n$-matrices and ABC is invertible. Use determinants to show that each of A, B, and C is invertible.*

Exercise 7.5.12 *Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non-zero. Is it true that A^{-1} will also be upper triangular? Explain. Could the same be concluded for lower triangular matrices?*

Exercise 7.5.13 *Specify whether each statement is true or false. If true, provide a proof. If false, provide a counterexample.*

(a) *If A is a 3×3-matrix with determinant zero, then one column must be a multiple of some other column.*

(b) *If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.*

(c) *For two $n \times n$-matrices A and B,* $\det(A+B) = \det(A) + \det(B)$.

(d) *For an $n \times n$-matrix A,* $\det(3A) = 3\det(A)$.

(e) *If A^{-1} exists, then* $\det(A^{-1}) = \det(A)^{-1}$.

(f) *If B is obtained by multiplying a single row of A by 4, then* $\det(B) = 4\det(A)$.

(g) *For an $n \times n$-matrix A, we have* $\det(-A) = (-1)^n \det(A)$.

(h) *If A is a real $n \times n$-matrix, then* $\det(A^T A) \geq 0$.

(i) *If $A^k = 0$ for some positive integer k, then* $\det(A) = 0$.

(j) *If $A\mathbf{x} = 0$ for some $\mathbf{x} \neq 0$, then* $\det(A) = 0$.

7.6 Application: A formula for the inverse of a matrix

Outcomes

A. *Find the cofactor matrix and the adjugate of a matrix.*

B. *Find the inverse of a matrix using the adjugate formula.*

The determinant of a matrix also provides a way to find the inverse of a matrix. Recall the definition of the inverse of a matrix from Definition 4.36. If A is an $n \times n$-matrix, we say that A^{-1} is the inverse of A if $AA^{-1} = I$ and $A^{-1}A = I$.

We now define a new matrix called the **cofactor matrix** of A. The cofactor matrix of A is the matrix whose ij^{th} entry is the ij^{th} cofactor of A.

Definition 7.27: The cofactor matrix

Let A be an $n \times n$-matrix. Then the ***cofactor matrix of*** A*, denoted* $\operatorname{cof}(A)$*, is defined by*

$$\operatorname{cof}(A) = \left[C_{ij}\right] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix},$$

where C_{ij} is the ij^{th} cofactor of A.

We will use the cofactor matrix to create a formula for the inverse of A. First, we define the **adjugate** of A, denoted $\operatorname{adj}(A)$, to be the transpose of the cofactor matrix:

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T.$$

The adjugate is also sometimes called the **classical adjoint** of A.

Example 7.28: Cofactor matrix and adjugate

Find the cofactor matrix and the adjugate of A, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution. We first find $\operatorname{cof}(A)$. To do so, we need to compute the cofactors of A. We have:

$$\begin{aligned}
C_{11} &= +M_{11} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2, \\
C_{12} &= -M_{12} = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2, \\
C_{13} &= +M_{13} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 6, \\
C_{21} &= -M_{21} = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 4,
\end{aligned}$$

and so on. Continuing in this way, we find the cofactor matrix

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}.$$

Finally, the adjugate is the transpose of the cofactor matrix:

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}.$$

♠

The following theorem provides a formula for A^{-1} using the determinant and the adjugate of A.

Theorem 7.29: Formula for the inverse

Let A be an $n \times n$-matrix. Then

$$A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I.$$

Moreover, A is invertible if and only if $\det(A) \neq 0$. In this case, we have:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

*We call this the **adjugate formula** for the matrix inverse.*

Proof. Recall that the (i,j)-entry of $\operatorname{adj}(A)$ is equal to C_{ji}. Thus the (i,j)-entry of $B = A \operatorname{adj}(A)$ is:

$$\begin{aligned} B_{ij} &= a_{i1} \operatorname{adj}(A)_{1j} + a_{i2} \operatorname{adj}(A)_{2j} + \ldots + a_{in} \operatorname{adj}(A)_{nj} \\ &= a_{i1} C_{j1} + a_{i2} C_{j2} + \ldots + a_{in} C_{jn}. \end{aligned}$$

By the cofactor expansion theorem, we see that this expression for B_{ij} is equal to the determinant of the matrix obtained from A by replacing its jth row by $[a_{i1}, a_{i2}, \ldots a_{in}]$, i.e., by its ith row.

If $i = j$ then this matrix is A itself and therefore $B_{ii} = \det(A)$. If on the other hand $i \neq j$, then this matrix has its ith row equal to its jth row, and therefore $B_{ij} = 0$ in this case. Thus we obtain:

$$A \operatorname{adj}(A) = \det(A) I.$$

By a similar argument (using columns instead of rows), we can verify that:

$$\operatorname{adj}(A) A = \det(A) I.$$

This proves the first part of the theorem. For the second part, assume that A is invertible. Then by Theorem 7.20, $\det(A) \neq 0$. Dividing the formula from the first part of the theorem by $\det(A)$, we obtain

$$A \left(\frac{1}{\det(A)} \operatorname{adj}(A) \right) = \left(\frac{1}{\det(A)} \operatorname{adj}(A) \right) A = I,$$

and therefore

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

This completes the proof. ♠

Example 7.30: Finding the inverse using a formula

Use the adjugate formula to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution. We must compute

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

We will start by computing the determinant. We expand along the second row:

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -3 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -3(-4) - 1(0) = 12.$$

We have already calculated the adjugate adj(A) in Example 7.28:

$$\operatorname{adj}(A) = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}.$$

Therefore, the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{12} \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Since it is very easy to make a mistake in this calculation, we double-check our answer by computing $A^{-1}A$:

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

♠

Example 7.31: Finding the inverse using a formula

Use the adjugate formula to find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix}.$$

Solution. We start by calculating the determinant:

$$\det(A) = \begin{vmatrix} 0 & 2 & 1 \\ -1 & 2 & 2 \\ 2 & -2 & -2 \end{vmatrix} = -2\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} = -2\cdot(-2)+1\cdot(-2) = 2.$$

Next, we compute the cofactor matrix:

$$\operatorname{cof}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 2 & -2 & 4 \\ 2 & -1 & 2 \end{bmatrix}.$$

The adjugate is the transpose of the cofactor matrix:

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & -1 \\ -2 & 4 & 2 \end{bmatrix}.$$

We therefore have

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{2}\begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & -1 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{2} \\ -1 & 2 & 1 \end{bmatrix}.$$

Once again, we double-check our work by computing $A^{-1}A$:

$$A^{-1}A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{2} \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ -1 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

♠

It is always a good idea to double-check your work. At the end of the calculation, it is very easy to compute $A^{-1}A$ and check whether it is equal to I. If they are not equal, be sure to go back and double-check each step. One common mistake is to forget to take the transpose of the cofactor matrix, so be sure not to forget this step.

In practice, it is usually much faster to compute the inverse by the method of Section 4.5.2, because this only requires solving a single system of equations, rather than computing a large number of cofactors. However, there are some situations where the adjugate formula is useful. One such situation is when the matrix has complicated entries that are functions rather than numbers. The following example illustrates this.

Example 7.32: Inverse for non-constant matrix

Let

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Show that $A(t)^{-1}$ exists and find it.

Solution. First note that

$$\det(A(t)) \;=\; e^t(\cos^2 t + \sin^2 t) \;=\; e^t \neq 0.$$

Therefore $A(t)^{-1}$ exists for all values of the variable t. The cofactor matrix is

$$\operatorname{cof}(A(t)) \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}.$$

The adjugate is the transpose of the cofactor matrix, and therefore the inverse is

$$A(t)^{-1} \;=\; \frac{1}{\det(A(t))}\operatorname{adj}(A(t)) \;=\; \frac{1}{e^t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & -e^t \sin t \\ 0 & e^t \sin t & e^t \cos t \end{bmatrix} \;=\; \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

♠

Another situation where the adjugate formula is useful is the case of a 2×2-matrix. In this case both the determinant and the adjugate are especially easy to compute. For a 2×2-matrix

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$\begin{aligned} \det(A) &\;=\; ad - bc \\ \operatorname{adj}(A) &\;=\; \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \end{aligned}$$

Therefore, A is invertible if and only if $ad - bc \neq 0$, and in that case, the inverse is given by

$$A^{-1} \;=\; \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{7.1}$$

Example 7.33: Inverse of a 2×2-matrix

Find the inverse of

$$A \;=\; \begin{bmatrix} 7 & 5 \\ 2 & 2 \end{bmatrix}.$$

Solution. We use formula (7.1) to compute the inverse:

$$A^{-1} = \frac{1}{7 \cdot 2 - 2 \cdot 5}\begin{bmatrix} 2 & -5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ -\frac{1}{2} & \frac{7}{4} \end{bmatrix}.$$

♠

Exercises

Exercise 7.6.1 *Find the cofactor matrix and the adjugate of each of the following matrices.*

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Exercise 7.6.2 *For each of the following matrices, determine whether it is invertible by checking whether the determinant is non-zero. If the determinant is non-zero, use the adjugate formula to find the inverse.*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

Exercise 7.6.3 *Determine whether each of the following matrices is invertible. If so, use the adjugate formula to find the inverse. If the inverse does not exist, explain why.*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Exercise 7.6.4 *Use the adjugate formula to find the inverse of the matrix*

$$A = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & -3 \\ -5 & 4 & -3 \end{bmatrix}.$$

Exercise 7.6.5 *Use the adjugate formula to find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ 2 & -2 & 5 \end{bmatrix}.$$

Exercise 7.6.6 *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Does there exist a value of t for which this matrix fails to be invertible? Explain.

Exercise 7.6.7 *Consider the matrix*

$$A = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}.$$

Does there exist a value of t for which this matrix fails to be invertible? Explain.

Exercise 7.6.8 *Consider the matrix*

$$A = \begin{bmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{bmatrix}.$$

Does there exist a value of t for which this matrix fails to be invertible? Explain.

Exercise 7.6.9 *Consider the matrix*

$$A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}.$$

Does there exist a value of t for which this matrix fails to be invertible? Explain.

Exercise 7.6.10 *Use the adjugate formula to find the inverse of the matrix*

$$A = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & \cos t - \sin t & \cos t + \sin t \end{bmatrix}.$$

Exercise 7.6.11 *Find the inverse, if it exists, of the matrix*

$$A = \begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}.$$

7.7 Application: Cramer's rule

Outcomes

A. *Use Cramer's rule to solve a system of equations with invertible coefficient matrix.*

Another application of determinants is **Cramer's rule** for solving a system of equations. Recall that we can represent a system of linear equations in the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is a vector of variables. Cramer's rule gives a formula for the solutions $\mathbf{x}$ in the special case that the coefficient matrix A is a square invertible matrix. Note that Cramer's rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Theorem 7.34: Cramer's rule

Suppose A is an invertible $n \times n$-matrix and we wish to solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = [x_1, \ldots, x_n]$. Then x_i can be computed by the rule

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where A_i is the matrix obtained by replacing the i^{th} column of A with $\mathbf{b}$.

Proof. Since A is invertible, the solution to the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$. By Theorem 7.29, we have

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

and therefore

$$\mathbf{x} = \frac{1}{\det(A)} \operatorname{adj}(A)\mathbf{b}.$$

Let x_i be the i^{th} component of $\mathbf{x}$ and b_j be the j^{th} component of $\mathbf{b}$. Recall that the ij^{th} entry of $\operatorname{adj}(A)$ is C_{ji}, the ji^{th} cofactor of A. By definition of matrix multiplication, we have

$$x_i = \frac{1}{\det(A)}(C_{1i}b_1 + \ldots + C_{ni}b_n).$$

By the formula for the expansion of a determinant along a column, this is equal to

$$x_i = \frac{1}{\det(A)} \begin{vmatrix} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{vmatrix},$$

where the i^{th} column of A is replaced with the column vector $\mathbf{b}$. But this last formula is exactly Cramer's rule. ♠

Example 7.35: Using Cramer's rule

Use Cramer's rule to solve the system of equations

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

Solution. The matrices A_1, A_2, and A_3 are obtained by respectively replacing the first, second, and third column of A by $\mathbf{b}$. We compute

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 4, \qquad \det(A_1) = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 2 & 1 \\ 6 & 4 & 1 \end{vmatrix} = 4$$

$$\det(A_2) = \begin{vmatrix} 1 & 3 & 1 \\ 3 & 5 & 1 \\ 1 & 6 & 1 \end{vmatrix} = 6, \qquad \det(A_3) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \\ 1 & 4 & 6 \end{vmatrix} = -4.$$

Then by Cramer's rule,

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{4} = 1, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{6}{4} = \frac{3}{2}, \quad \text{and} \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-4}{4} = -1.$$

Thus, the solution is $(x,y,z) = (1, \frac{3}{2}, -1)$. ♠

Cramer's rule is sometimes useful in situations where row operations would be difficult to do. One such situation is when a system of equations involves functions rather than numbers, as in the following example.

Example 7.36: Using Cramer's rule for non-constant matrix

Solve the following system of equations for z.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}.$$

Solution. We are asked to find the value of z in the solution. By Cramer's rule, we have

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = \frac{e^t(t^2 \cos t - t \sin t)}{e^{2t}} = e^{-t}(t^2 \cos t - t \sin t).$$

♠

Exercises

Exercise 7.7.1 *True or false? "Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions."*

Exercise 7.7.2 *Use Cramer's rule to find the solution to*

$$\begin{aligned} x+2y &= 1 \\ 2x-y &= 2 \end{aligned}$$

Exercise 7.7.3 *Use Cramer's rule to find the solution to*

$$\begin{aligned} x+2y+z &= 3 \\ 2x-y-z &= 1 \\ x+z &= 1 \end{aligned}$$

Exercise 7.7.4 *Use Cramer's rule to solve the system of equations*

$$\left[\begin{array}{rrr} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 2 & 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right]=\left[\begin{array}{r} 3 \\ -2 \\ 5 \end{array}\right].$$

Exercise 7.7.5 *Find the value of y in the following system of equations:*

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & s & s^2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right]=\left[\begin{array}{c} t \\ s \\ 1 \end{array}\right].$$

8. Eigenvalues, eigenvectors, and diagonalization

In this chapter, we introduce the theory of eigenvalues and eigenvectors, and the technique of diagonalization. In the same way that Gaussian elimination is a fundamental tool that permits us to solve many different kinds of problems, diagonalization is also one of the fundamentals tool of linear algebra. It has a great number of applications in every field of mathematics, science, and engineering.

8.1 Eigenvectors and eigenvalues

Outcomes

A. *Determine whether a vector is an eigenvector of a matrix.*

B. *Given an eigenvector, find the corresponding eigenvalue.*

C. *Given an eigenvalue, find the corresponding eigenvectors.*

D. *Find a basis for the eigenspace of a given eigenvalue.*

When we multiply a square matrix A by a non-zero vector $\mathbf{v}$, we obtain another vector $A\mathbf{v}$. Most of the time, the vectors $A\mathbf{v}$ and $\mathbf{v}$ are unrelated; they could point in completely different directions. However, sometimes it can happen that $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. In that case, $\mathbf{v}$ is called an **eigenvector** of A. We will see later in this chapter that we can learn a lot about the matrix A by considering its eigenvectors.

Definition 8.1: Eigenvalues and eigenvectors

Let A be an $n\times n$-matrix. Suppose that $\mathbf{v}\in\mathbb{R}^n$ is a non-zero vector such that $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$. In other words, suppose that there exists a scalar λ such that

$$A\mathbf{v}=\lambda\mathbf{v}.$$

*Then $\mathbf{v}$ is called an **eigenvector** of A, and λ is called the corresponding **eigenvalue**.*

Example 8.2: Eigenvalues and eigenvectors

Consider the matrix

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Which of the following vectors are eigenvectors of A? Find the corresponding eigenvalues.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution. We compute

$$A\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix}, \quad A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \quad A\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \quad A\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- We see that $A\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_1$, namely $A\mathbf{v}_1 = 2\mathbf{v}_1$. Therefore, $\mathbf{v}_1$ is an eigenvector of A with corresponding eigenvalue $\lambda = 2$.
- Similarly, $A\mathbf{v}_2 = 3\mathbf{v}_2$, so $\mathbf{v}_2$ is an eigenvector of A with corresponding eigenvalue $\lambda = 3$.
- On the other hand, $A\mathbf{v}_3$ is not a scalar multiple of $\mathbf{v}_3$. Hence, $\mathbf{v}_3$ is not an eigenvector of A.
- Finally, although $A\mathbf{v}_4$ is a scalar multiple of $\mathbf{v}_4$, the zero vector is not considered an eigenvector.

♠

Example 8.3: Find eigenvectors for the given eigenvalue

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

Find the eigenvectors corresponding to the eigenvalue $\lambda = 2$.

Solution. We have to solve the equation $A\mathbf{v} = 2\mathbf{v}$. We can use algebra to rewrite this as

$$\begin{aligned} A\mathbf{v} = 2\mathbf{v} \quad &\Longleftrightarrow \quad A\mathbf{v} - 2\mathbf{v} = \mathbf{0} \\ &\Longleftrightarrow \quad (A - 2I)\mathbf{v} = \mathbf{0} \\ &\Longleftrightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & -2 & -2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

This is a homogeneous system of equations with general solution

$$\mathbf{v} \;=\; s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

where s and t are parameters. These (except the zero vector) are exactly the eigenvectors corresponding to the eigenvalue $\lambda = 2$. ♠

As the last example shows, the eigenvectors for a given eigenvalue λ, plus the zero vector, form a subspace of $\mathbb{R}^n$. This is called the **eigenspace** of λ.

Definition 8.4: Eigenspace

Let A be an $n \times n$-matrix, and let λ be an eigenvalue of A. The ***eigenspace*** *of λ is the set*

$$E_\lambda = \{\mathbf{v} \mid A\mathbf{v} = \lambda\mathbf{v}\}.$$

It is a subspace of $\mathbb{R}^n$.

Instead of finding *all* eigenvectors for a given eigenvalue, it is often sufficient to find a basis for the eigenspace. We also sometimes call the basis vectors of the eigenspace **basic eigenvectors**.

Example 8.5: Basis of eigenspace

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

The matrix A has eigenvalues $\lambda = 1$ and $\lambda = 2$. Find a basis for each eigenspace.

Solution. We already found a basis for the eigenspace E_2 in Example 8.3.

$$\text{Basis of } E_2\text{:} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To find a basis for the eigenspace E_1, we proceed analogously. We must solve the equation $A\mathbf{v} = 1\mathbf{v}$. We have:

$$\begin{aligned} A\mathbf{v} = 1\mathbf{v} \quad &\iff \quad A\mathbf{v} - \mathbf{v} = \mathbf{0} \\ &\iff \quad (A - I)\mathbf{v} = \mathbf{0} \\ &\iff \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & -2 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

This is a homogeneous system of rank 2, with general solution

$$\mathbf{v} = t \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

Thus, the following is a basis for the eigenspace E_1:

$$\text{Basis of } E_1\text{:} \quad \left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

♠

Exercises

Exercise 8.1.1 *Consider the matrix*

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}.$$

Which of the following vectors are eigenvectors of A? Find the corresponding eigenvalues.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Exercise 8.1.2 *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & -1 & 5 \\ -3 & 0 & 4 \end{bmatrix}.$$

Find the eigenvectors corresponding to the eigenvalue $\lambda = 4$.

Exercise 8.1.3 *Let*

$$A = \begin{bmatrix} 7 & -4 & 8 \\ -1 & 4 & -2 \\ -2 & 2 & -1 \end{bmatrix}.$$

Find the eigenvectors corresponding to the eigenvalue $\lambda = 3$.

Exercise 8.1.4 *Let*

$$A = \begin{bmatrix} 4 & 0 & 3 \\ -3 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix has eigenvalues $\lambda = 1$ and $\lambda = 4$. Find a basis for each eigenspace.

Exercise 8.1.5 *Let*

$$A = \begin{bmatrix} 2 & 4 & -4 \\ -1 & 6 & -9 \\ 0 & 0 & -3 \end{bmatrix}.$$

This matrix has eigenvalues $\lambda = -3$ and $\lambda = 4$. Find a basis for each eigenspace.

Exercise 8.1.6 *Suppose A is a 3×3-matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_3 = 2$ and corresponding eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix}.$$

(By "corresponding", we mean that $\mathbf{v}_1$ corresponds to λ_1, $\mathbf{v}_2$ corresponds to λ_2, and so on). Find

$$A \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}.$$

Exercise 8.1.7 *Let A be an $n\times n$-matrix, and assume λ is an eigenvalue of A. Show that λ^2 is an eigenvalue of A^2.*

Exercise 8.1.8 *Let A be an invertible $n\times n$-matrix, and assume λ is an eigenvalue of A. Show that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.*

Exercise 8.1.9 *If A is an $n\times n$-matrix and c is a non-zero constant, compare the eigenvalues of A and cA.*

Exercise 8.1.10 *Let A, B be invertible $n\times n$-matrices which commute. That is, $AB = BA$. Suppose $\mathbf{v}$ is an eigenvector of B. Show that then $A\mathbf{v}$ must also be an eigenvector for B.*

Exercise 8.1.11 *Suppose A is an $n\times n$-matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then λ equals either 0, 1, or -1.*

Exercise 8.1.12 *Show that if $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \lambda\mathbf{w}$, then whenever k, p are scalars,*

$$A(k\mathbf{v} + p\mathbf{w}) = \lambda(k\mathbf{v} + p\mathbf{w})$$

Does this imply that $k\mathbf{v} + p\mathbf{w}$ is an eigenvector? Explain.

8.2 Finding eigenvalues

Outcomes

A. *Find the characteristic polynomial, eigenvalues, and eigenvectors of a matrix.*

B. *Find the eigenvalues of a triangular matrix.*

In the previous section, we saw how to find the eigenvectors corresponding to a given eigenvalue λ, if λ is already known. But we have not yet seen how to find the eigenvalues of a matrix. However, the calculations in Examples 8.3 and 8.5 suggest a way forward. We can see that the following are equivalent:

1. λ is an eigenvalue of A.
2. There exists a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
3. The homogeneous system of equations $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a non-trivial solution.

Indeed, the equivalence between 1 and 2 is just the definition of an eigenvalue, and the equivalence between 2 and 3 is just algebra. By Corollary 7.22, we know that the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a non-trivial solution if and only if $\det(A - \lambda I) = 0$. Therefore, we have proved the following theorem:

Theorem 8.6: Eigenvalues

Let A be a square matrix, and let λ be a scalar. Then λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0.$$

Example 8.7: Finding the eigenvalues

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}.$$

Solution. By Theorem 8.6, a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. We calculate the determinant:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} \\ &= (-5-\lambda)(4-\lambda) + 14 \\ &= \lambda^2 + \lambda - 6. \end{aligned}$$

Therefore, λ is an eigenvalue if and only if $\lambda^2 + \lambda - 6 = 0$. We can find the roots of this equation using the quadratic formula, or equivalently, by factoring the left-hand side:

$$\lambda^2 + \lambda - 6 = 0 \iff (\lambda + 3)(\lambda - 2) = 0.$$

Therefore, the eigenvalues are $\lambda = -3$ and $\lambda = 2$. ♠

Example 8.8: Finding the eigenvalues

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 2 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. Once again, we calculate $\det(A-\lambda I)$:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 5-\lambda & -4 & 4 \\ 2 & -1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (5-\lambda)(-1-\lambda)(2-\lambda)-2(-4)(2-\lambda) \\ &= -\lambda^3+6\lambda^2-11\lambda+6 \\ &= (3-\lambda)(1-\lambda)(2-\lambda).\end{aligned}$$

The eigenvalues are the roots of this polynomial, i.e., the solutions of the equation $(\lambda-3)(\lambda-1)(2-\lambda)=0$. Therefore, the eigenvalues of A are $\lambda=1$, $\lambda=2$, and $\lambda=3$. ♠

Example 8.9: No real eigenvalue

Find the eigenvalues of the matrix

$$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Solution. We have

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1.$$

Since $\lambda^2+1=0$ does not have any solutions in the real numbers, the matrix A has no real eigenvalues. (However, if we were working over the field of complex numbers rather than real numbers, this matrix would have eigenvalues $\lambda=\pm i$). ♠

As the examples show, the quantity $\det(A-\lambda I)$ is always a polynomial in the variable λ. A **polynomial** is an expression of the form

$$p(\lambda)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_1\lambda+a_0,$$

where $a_0,\ldots,a_n$ are constants called the **coefficients** of the polynomial. The polynomial $\det(A-\lambda I)$ has a special name:

Definition 8.10: Characteristic polynomial

Let A be a square matrix. The expression

$$p(\lambda)=\det(A-\lambda I)$$

is called the ***characteristic polynomial*** *of A.*

Example 8.11: Characteristic polynomial

Find the characteristic polynomial of the matrix

$$A=\begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 3-\lambda & 0 & 2 \\ 6 & 4-\lambda & 3 \\ -4 & 0 & -3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 4-\lambda & 3 \\ 0 & -3-\lambda \end{vmatrix} + 2\begin{vmatrix} 6 & 4-\lambda \\ -4 & 0 \end{vmatrix} \\
&= (3-\lambda)(4-\lambda)(-3-\lambda)+8(4-\lambda) \\
&= -\lambda^3+4\lambda^2+\lambda-4.
\end{aligned}$$

♠

It is time to summarize the method for finding the eigenvalues and eigenvectors of a matrix.

Procedure 8.12: Finding eigenvalues and eigenvectors

Let A be an $n\times n$-matrix. To find the eigenvalues and eigenvectors of A:

1. *Calculate the characteristic polynomial* $\det(A-\lambda I)$.
2. *The eigenvalues are the roots of the characteristic polynomial.*
3. *For each eigenvalue λ, find a basis for the eigenvectors by solving the homogeneous system*

$$(A-\lambda I)\mathbf{v}=\mathbf{0}.$$

To double-check your work, make sure that $A\mathbf{v}=\lambda\mathbf{v}$ for each eigenvalue λ and associated eigenvector $\mathbf{v}$.

Example 8.13: Finding eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}.$$

Solution. We already found the characteristic polynomial in Example 8.11. It is

$$p(\lambda)=\det(A-\lambda I)=-\lambda^3+4\lambda^2+\lambda-4.$$

Finding the roots of a cubic polynomial can be a bit tricky, but with some trial and error, we can find that $\lambda=1$ is a root. We can therefore factor out $(\lambda-1)$:

$$p(\lambda)=(\lambda-1)(-\lambda^2+3\lambda+4).$$

Then we can use the quadratic formula to find the remaining two roots:

$$\lambda=\frac{-3\pm\sqrt{9+16}}{-2},$$

which yields the two roots $\lambda = -1$ and $\lambda = 4$. Therefore, we have

$$p(\lambda) = -(\lambda - 1)(\lambda + 1)(\lambda - 4),$$

and the eigenvalues of A are $\lambda = 1$, $\lambda = -1$, and $\lambda = 4$. We now find the eigenvectors for each eigenvalue.

- **For $\lambda = 1$:** We must solve $(A - I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 2 & 0 & 2 \\ 6 & 3 & 3 \\ -4 & 0 & -4 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

The basic solution is

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

- **For $\lambda = -1$:** We must solve $(A + I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 4 & 0 & 2 \\ 6 & 5 & 3 \\ -4 & 0 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

The basic solution is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

- **For $\lambda = 4$:** We must solve $(A - 4I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} -1 & 0 & 2 \\ 6 & 0 & 3 \\ -4 & 0 & -7 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

The basic solution is

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

♠

Example 8.14: A zero eigenvalue

Let

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of A.

Solution. To find the eigenvalues of A, we first compute the characteristic polynomial.

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix}=-\lambda^3+6\lambda^2-8\lambda.$$

You can verify that the roots of this polynomial are $\lambda_1=0$, $\lambda_2=2$, $\lambda_3=4$. Notice that while eigenvectors can never equal $\mathbf{0}$, it is possible to have an eigenvalue equal to 0. Now we will find the basic eigenvectors.

- **For $\lambda_1=0$:** We must solve the equation $(A-0I)\mathbf{v}=\mathbf{0}$. This equation becomes $A\mathbf{v}=\mathbf{0}$. We write the augmented matrix for this system and reduce to echelon form:

$$\left[\begin{array}{rrr|r} 2 & 2 & -2 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic solution is

$$\mathbf{v}_1=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

- **For $\lambda_2=2$:** We solve the equation $(A-2I)\mathbf{v}=\mathbf{0}$:

$$\left[\begin{array}{rrr|r} 0 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic solution is

$$\mathbf{v}_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- **For $\lambda_3=4$:** We solve the equation $(A-4I)\mathbf{v}=\mathbf{0}$:

$$\left[\begin{array}{rrr|r} -2 & 2 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic solution is

$$\mathbf{v}_3=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus we have found the eigenvectors $\mathbf{v}_1$ for λ_1, $\mathbf{v}_2$ for λ_2, and $\mathbf{v}_3$ for λ_3. We can double-check our answers by checking the equation $A\mathbf{v}=\lambda\mathbf{v}$ in each case:

$$A\mathbf{v}_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}_1,$$

$$A\mathbf{v}_2 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_2,$$

$$A\mathbf{v}_3 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\mathbf{v}_3.$$

Therefore, our eigenvectors and eigenvalues are correct. ♠

We conclude this section by considering the eigenvalues of a triangular matrix. Recall from Definition 7.13 that a matrix is **upper triangular** if all entries below the main diagonal are zero, and **lower triangular** if all entries above the main diagonal are zero.

Example 8.15: Eigenvalues of a triangular matrix

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}.$$

Solution. We calculate $\det(A-\lambda I) = 0$ as follows:

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda).$$

Solving the equation $(1-\lambda)(4-\lambda)(6-\lambda) = 0$ results in the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of A. ♠

Clearly, the same is true for any (upper or lower) triangular matrix. We therefore have the following proposition:

Proposition 8.16: Eigenvalues of a triangular matrix

Let A be an upper or lower triangular matrix. Then the eigenvalues of A are the entries on the main diagonal.

Exercises

Exercise 8.2.1 *Find the characteristic polynomial of the matrix*

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Use the quadratic formula to find the eigenvalues.

Exercise 8.2.2 *Find the characteristic polynomial, eigenvalues, and basic eigenvectors of the matrix*

$$\begin{bmatrix} 9 & 10 \\ -5 & -6 \end{bmatrix}.$$

Exercise 8.2.3 *Find the characteristic polynomial, eigenvalues, and basic eigenvectors of the matrix*

$$\begin{bmatrix} 0 & 3 & -1 \\ -2 & 4 & -2 \\ 2 & -3 & 3 \end{bmatrix}.$$

One eigenvalue is 1.

Exercise 8.2.4 *Find the characteristic polynomial, eigenvalues, and basic eigenvectors of the matrix*

$$\begin{bmatrix} 3 & 0 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

One eigenvalue is 3.

Exercise 8.2.5 *Find the characteristic polynomial, eigenvalues, and basic eigenvectors of the matrix*

$$\begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}.$$

One eigenvalue is -3.

Exercise 8.2.6 *Which of the following matrices have no real eigenvalue?*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Exercise 8.2.7 *Find the eigenvalues and eigenvectors of the following triangular matrix:*

$$\begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Exercise 8.2.8 *Is it possible for a non-zero matrix to have only* 0 *as an eigenvalue?*

8.3 Geometric interpretation of eigenvectors

Outcomes

A. Visualize the effect of a linear transformation by considering its eigenvectors and eigenvalues.

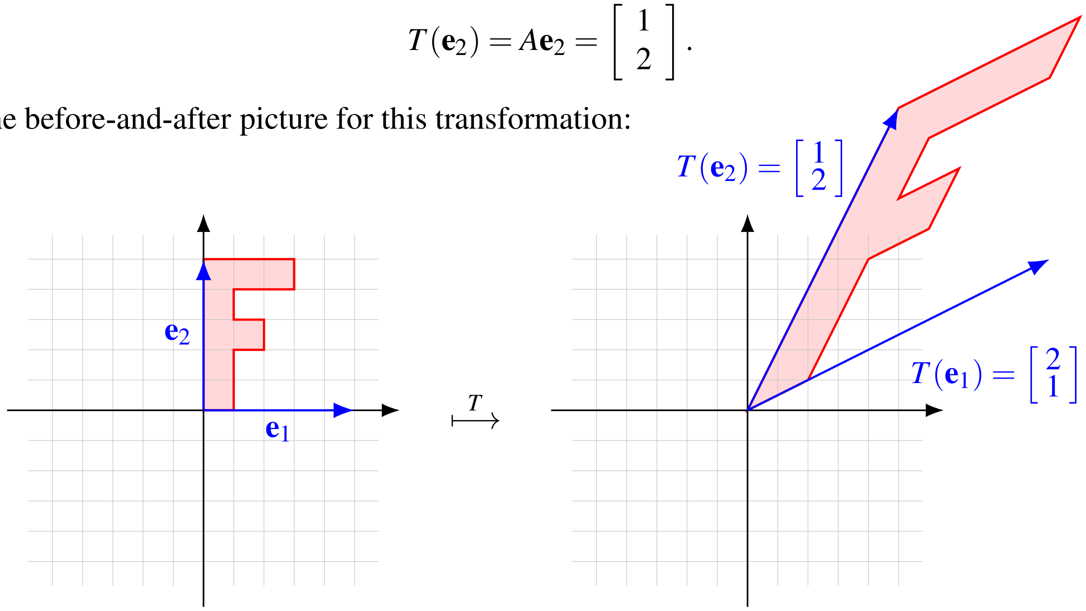
Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In Chapter 6, we saw that this matrix corresponds to a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$. We also saw how to visualize this linear transformation as a before-and-after picture. For this, we considered the images of the first and second standard basis vectors:

$$T(\mathbf{e}_1) = A\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
$$T(\mathbf{e}_2) = A\mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Here is the before-and-after picture for this transformation:



Although we can see from this picture that the letter "F" is being distorted somehow, it is perhaps not very obvious what exactly this linear transformation does.

We can get a much better idea by computing the eigenvectors and eigenvalues of A. A short calculation shows that the basic eigenvectors are

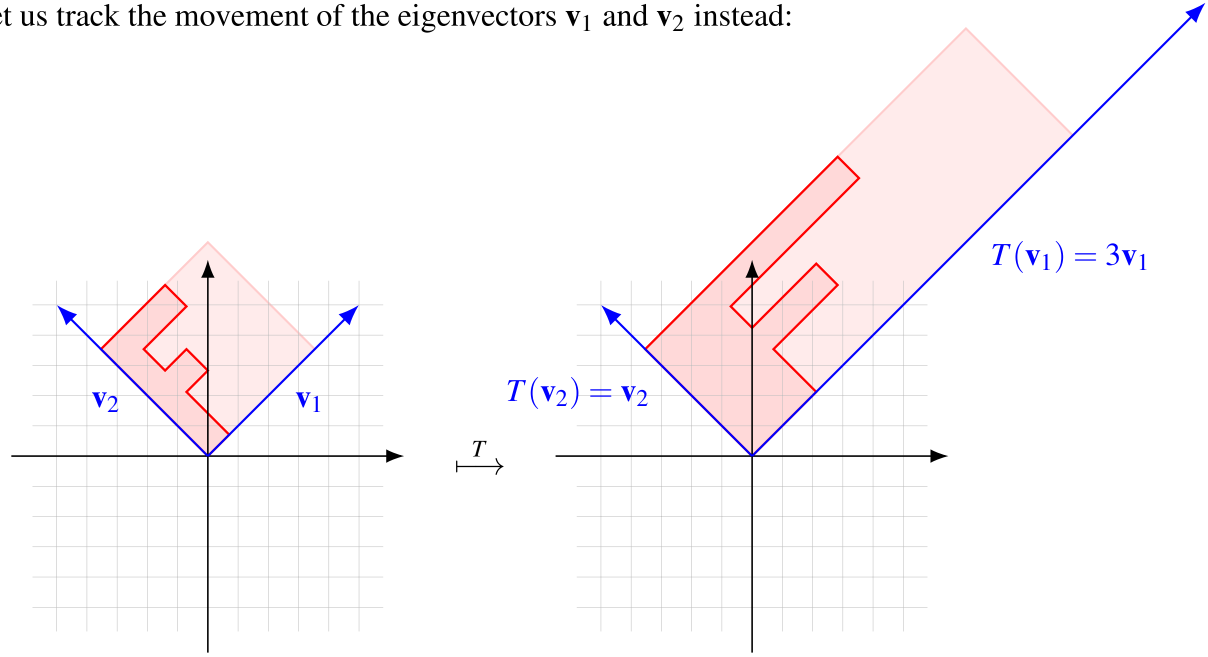
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

with corresponding eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. Consider the effect of the linear transformation T on the eigenvectors:

$$T(\mathbf{v}_1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}_1,$$

$$T(\mathbf{v}_2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{v}_2.$$

So each eigenvector is mapped to a scalar multiple of itself. This gives us a hint for how to draw a more useful before-and-after picture. Rather than tracking the movement of the standard basis vectors $\mathbf{e}_1$ and $\vec{e}_2$, let us track the movement of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ instead:



Thus, the linear transformation described by the matrix A is revealed to be just a scaling by a factor of 3 along the direction of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In summary, the geometric meaning of an eigenvector is that it is mapped to a multiple of itself. Thus, when viewed from the point of view of its action on the eigenvectors, a linear transformation behaves like a scaling of each eigenvector. We can say that each eigenvector describes a direction of scaling, and each corresponding eigenvalue giving the corresponding (positive or negative) scaling factor.

Example 8.17: Visualize a linear transformation

Visualize the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is described by the matrix

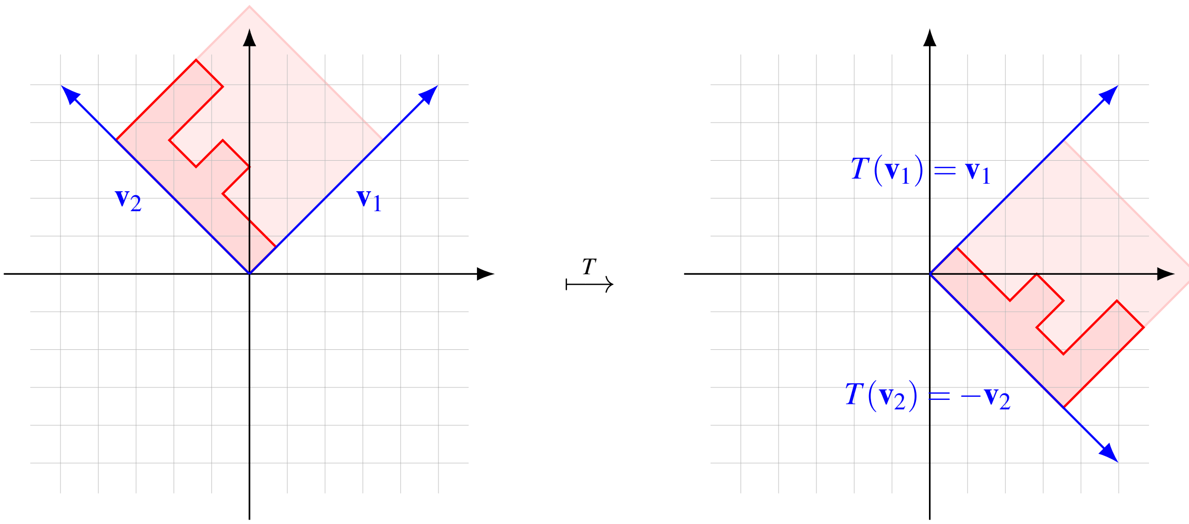
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

by considering the eigenvectors and eigenvalues.

Solution. The characteristic polynomial is $\lambda^2 - 1$, and so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. By solving each equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$, we find that the corresponding basic eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(check this!). We get the following before-and-after picture:



We see that this linear transformation is a reflection about the vector $\mathbf{v}_1$. ♠

Example 8.18: Visualize a linear transformation

Visualize the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is described by the matrix

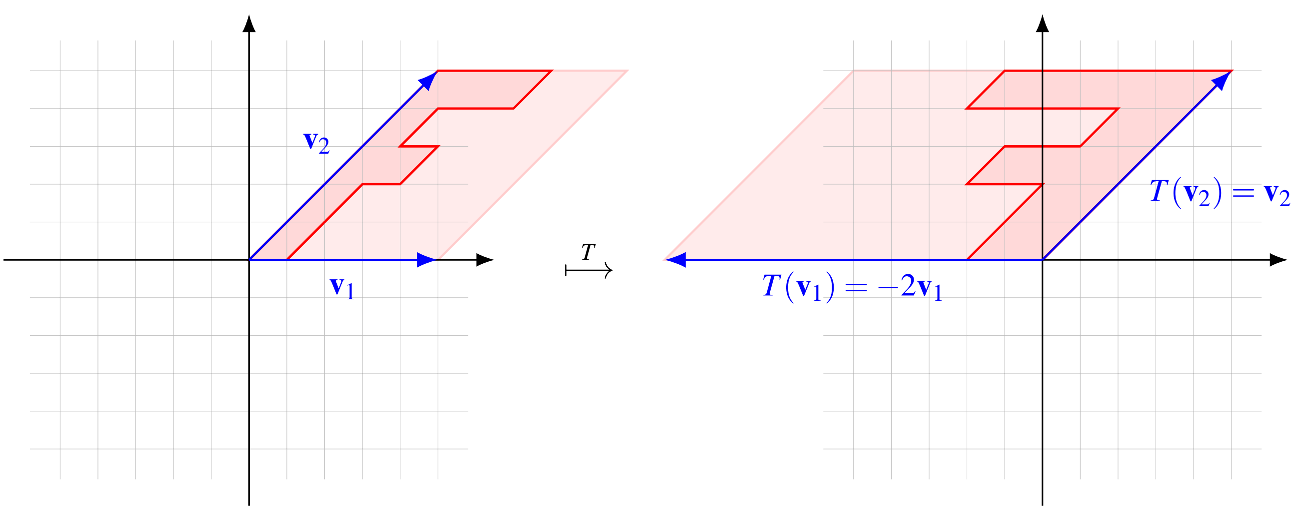
$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$$

by considering the eigenvectors and eigenvalues.

Solution. The characteristic polynomial is $(-2-\lambda)(1-\lambda)$, and therefore, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$. We find that the corresponding basic eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

respectively. We get the following before-and-after picture:



This particular linear transformation keeps the vector $\mathbf{v}_2$ fixed, while scaling by a factor of -2 in the direction of $\mathbf{v}_1$. It could be described as a kind of slanted reflection with scaling. ♠

Exercises

Exercise 8.3.1 *For each of the following matrices, find the eigenvectors and eigenvalues. Use this to visualize the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *that is described by the matrix.*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Exercise 8.3.2 *If A is the matrix of a linear transformation that rotates all vectors in $\mathbb{R}^2$ by 60°, explain why A cannot have any real eigenvalues. Is there an angle such that rotation by this angle would have a real eigenvalue? What eigenvalues would be obtainable in this way?*

Exercise 8.3.3 *Let T be the linear transformation that reflects vectors about the x-axis. Find a matrix for T and then find its eigenvalues and eigenvectors.*

Exercise 8.3.4 *Let T be the linear transformation that reflects vectors about the line $x = y$. Find a matrix of T and then find eigenvalues and eigenvectors.*

Exercise 8.3.5 *Let T be the linear transformation that reflects all vectors in $\mathbb{R}^3$ about the xy-plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.*

8.4 Diagonalization

Outcomes

A. *Compute sums, products, and powers of diagonal matrices.*

B. *Determine whether a square matrix is diagonalizable, and diagonalize it if possible.*

A square matrix D is called a **diagonal matrix** if all entries except those on the main diagonal are zero. Such matrices look like the following:

$$\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}.$$

Diagonal matrices are particularly easy to work with. For example, the sum of two diagonal matrices is diagonal. Also, the product of two diagonal matrices is diagonal, and is computed by taking the product of corresponding diagonal entries.

Example 8.19: Sums, products, and powers of diagonal matrices

Compute $A+B$, AB, and A^4, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. We have

$$A+B = \begin{bmatrix} 2+1 & 0 & 0 \\ 0 & 3-2 & 0 \\ 0 & 0 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

$$AB = \begin{bmatrix} 2\cdot 1 & 0 & 0 \\ 0 & 3\cdot(-2) & 0 \\ 0 & 0 & 4\cdot 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

and

$$A^4 = \begin{bmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 4^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}.$$

Notice that all operations are computed componentwise on the diagonal. Therefore, multiplication of diagonal matrices is much simpler than multiplication of general matrices. ♠

One of the most important problem solving techniques in linear algebra is **diagonalization**. In a nutshell, the point of diagonalization is to simplify a problem by replacing an arbitrary matrix by a diagonal matrix. We say that two square matrices A and B are **similar** if there exists an invertible matrix P such that $P^{-1}AP = B$. A matrix is **diagonalizable** if it is similar to a diagonal matrix. This is summarized in the following definition.

Definition 8.20: Diagonalizable matrix

Let A be an $n\times n$-matrix. Then A is said to be ***diagonalizable*** *if there exists an invertible matrix P and a diagonal matrix D such that*

$$P^{-1}AP = D.$$

The key connection between diagonalizability, eigenvectors, and eigenvalues is the following theorem.

Theorem 8.21: Diagonalization and eigenvectors

An $n\times n$-matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover, in this case, let P be the invertible matrix whose columns are n linearly independent eigenvectors of A, and let D be the diagonal matrix whose diagonal entries are the corresponding eigenvalues. Then $P^{-1}AP = D$.

Proof. Assume that A has n linearly independent eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$. Let $\lambda_1,\ldots,\lambda_n$ be the corresponding eigenvalues, so that

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{8.1}$$

for all $i = 1,\ldots,n$. Let P be the matrix that has $\mathbf{v}_1,\ldots,\mathbf{v}_n$ as its columns. Then P is invertible because $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are linearly independent. Let D be the diagonal matrix that has $\lambda_1,\ldots,\lambda_n$ as its diagonal entries. By the column method of matrix multiplication, the i^{th} column of AP is $A\mathbf{v}_i$. Also by the column method of matrix multiplication, the i^{th} column of PD is $\lambda_i\mathbf{v}_i$. Therefore, by (8.1), the matrices AP and PD have the same columns, i.e.,

$$AP = PD.$$

It follows that $P^{-1}AP = D$, as desired.

Conversely, assume that A is diagonalizable. Then there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, or equivalently, $AP = PD$. Let $\mathbf{v}_1,\ldots,\mathbf{v}_n$ be the columns of P and let $\lambda_1,\ldots,\lambda_n$ be the diagonal entries of D. Again we find that the i^{th} column of AP is $A\mathbf{v}_i$ and the i^{th} column of PD is $\lambda_i\mathbf{v}_i$, and therefore $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ holds for all i. It follows that $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are eigenvectors of A. Since P is invertible, $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are linearly independent, so A has n linearly independent eigenvectors. ♠

Example 8.22: Diagonalizing a matrix

Diagonalize the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}.$$

In other words, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution. By Theorem 8.21, we use the eigenvectors of A as the columns of P and the corresponding eigenvalues as the diagonal entries of D. We already found the eigenvectors and -values of A in Example 8.13. They were

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

with corresponding eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 4$. Therefore we can use

$$P = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

To double-check that $P^{-1}AP$ is indeed equal to D, we first compute the inverse of P:

$$P^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Then

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D.$$

Alternatively, we could have checked that $AP = PD$, which would not have required computing P^{-1}. ♠

Example 8.23: Diagonalizing a matrix

Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}.$$

Solution. First, we will find the characteristic polynomial of A:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 4-\lambda & -1 \\ -2 & -4 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)\left((4-\lambda)(4-\lambda)-4\right) \\ &= (2-\lambda)(12-8\lambda+\lambda^2) \\ &= (2-\lambda)(2-\lambda)(6-\lambda). \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 2$ and $\lambda = 6$. Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda = 2$. We solve $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ -2 & -4 & 2 & 0 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

where t, s are parameters. Thus, the basic eigenvectors for $\lambda = 2$ are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Doing a similar calculation, we find that the basic eigenvector for $\lambda = 6$ is

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

By Theorem 8.21, we use the eigenvectors of A as the columns of P and the corresponding eigenvalues as the diagonal entries of D. Therefore,

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

We can double-check this answer by computing

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D.$$

Notice that the eigenvalues on the main diagonal of D *must* be in the same order as the corresponding eigenvectors in P. Since the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are both for the eigenvalue $\lambda = 2$, the entry 2 appears twice in the matrix D. ♠

The following example shows that not all matrices are diagonalizable.

Example 8.24: A matrix that cannot be diagonalized

Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *cannot be diagonalized.*

Solution. Through the usual procedure, we find that the characteristic polynomial is $(1-\lambda)^2$, and therefore the only eigenvalue is $\lambda = 1$. To find the eigenvectors, we solve the equation $(A-I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$\mathbf{v} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Because the solution space is 1-dimensional, there is only one basic eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since the matrix A has only one basic eigenvector, we cannot find two linearly independent eigenvectors. Therefore, by Theorem 8.21, A cannot be diagonalized. ♠

Exercises

Exercise 8.4.1 *Let* $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ *and* $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. *Find* $D+E$, DE, *and* D^7.

Exercise 8.4.2 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}.$$

One eigenvalue is 3*. Diagonalize if possible.*

Exercise 8.4.3 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}.$$

One eigenvalue is 1*. Diagonalize if possible.*

Exercise 8.4.4 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 8 & 0 & 10 \\ -6 & -3 & -6 \\ -5 & 0 & -7 \end{bmatrix}.$$

One eigenvalue is -3*. Diagonalize if possible.*

Exercise 8.4.5 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} -1 & -2 & 2 \\ 0 & 5 & -8 \\ 0 & 4 & -7 \end{bmatrix}.$$

One eigenvalue is 1*. Diagonalize if possible.*

Exercise 8.4.6 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 2 & -1 & 6 \\ -4 & -1 & -6 \\ -2 & 1 & -6 \end{bmatrix}.$$

One eigenvalue is 0*. Diagonalize if possible.*

Exercise 8.4.7 *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

One eigenvalue is 3*. Diagonalize if possible.*

8.5 Application: Matrix powers

Outcomes

A. *Use diagonalization to raise a matrix to a high power.*

B. *Use diagonalization to compute a square root of a matrix.*

Suppose we have a matrix A and we want to find A^{50}. One could try to multiply A with itself 50 times, but this is a lot of work (try it!). However, diagonalization allows us to compute high powers of a matrix relatively easily. Suppose A is diagonalizable, so that $P^{-1}AP = D$. We can rearrange this equation to write $A = PDP^{-1}$. Now, consider A^2. Since $A = PDP^{-1}$, it follows that

$$A^2 = (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}.$$

Similarly,

$$A^3 = (PDP^{-1})^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}.$$

In general,

$$A^n = (PDP^{-1})^n = PD^nP^{-1}.$$

Therefore, we have reduced the problem to finding D^n. But as we saw in Example 8.19, computing a power of a diagonal matrix is easy. To compute D^n, we only need to raise every entry on the diagonal to the power of n. Through this method, we can compute large powers of matrices.

Example 8.25: Raising a matrix to a high power

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. *Find* A^{50}.

Solution. First, we will diagonalize A. Following the usual steps, we find that the eigenvalues are $\lambda = 1$ and $\lambda = 2$. The basic eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

and the basic eigenvector corresponding to $\lambda = 2$ is

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Now we construct P by using the basic eigenvectors of A as the columns of P. Thus

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The inverse of P is

$$P^{-1} = \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right].$$

Then

$$P^{-1}AP = \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]\left[\begin{array}{rrr} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array}\right]\left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right] = D.$$

Now it follows by rearranging the equation that $A = PDP^{-1}$, and therefore, as noted above,

$$\begin{aligned} A^{50} = PD^{50}P^{-1} &= \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{array}\right]\left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \\ &= \left[\begin{array}{ccc} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{array}\right]. \end{aligned}$$

♠

Thus, through diagonalization, we have efficiently computed a high power of A. The following example shows that we can also use the same technique for finding a square root of a matrix.

Example 8.26: Square root of a matrix

Let $A = \left[\begin{array}{rrr} 1 & 3 & 3 \\ -1 & 5 & 3 \\ 1 & -1 & 1 \end{array}\right]$. *Find a square root of* A*, i.e., find a matrix* B *such that* $A = B^2$.

Solution. We first diagonalize A. The characteristic polynomial is

$$\begin{aligned} \left|\begin{array}{ccc} 1-\lambda & 3 & 3 \\ -1 & 5-\lambda & 3 \\ 1 & -1 & 1-\lambda \end{array}\right| &= (1-\lambda)(5-\lambda)(1-\lambda)+9+3-3(5-\lambda)+3(1-\lambda)+3(1-\lambda) \\ &= -\lambda^3+7\lambda^2-14\lambda+8, \end{aligned}$$

with roots $\lambda = 1$, $\lambda = 2$, and $\lambda = 4$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right], \quad \mathbf{v}_2 = \left[\begin{array}{r} 0 \\ 1 \\ -1 \end{array}\right], \quad \text{and} \quad \mathbf{v}_3 = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right],$$

respectively. Therefore we have $P^{-1}AP = D$, where

$$P = \left[\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{array}\right], \quad P^{-1} = \left[\begin{array}{rrr} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right], \quad \text{and} \quad D = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{array}\right].$$

We can equivalently write $A = PDP^{-1}$. Finding a square root of a diagonal matrix is easy:

$$D^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

If we now define $B = PD^{\frac{1}{2}}P^{-1}$, we clearly have $B^2 = PD^{\frac{1}{2}}P^{-1}PD^{\frac{1}{2}}P^{-1} = PDP^{-1} = A$. So the desired square root of A is

$$B = PD^{\frac{1}{2}}P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & 1-\sqrt{2} & 1 \end{bmatrix}.$$

Finally, we verify that we have computed B correctly by squaring it and double-checking that we really get A.

$$B^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & 1-\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & 1-\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 5 & 3 \\ 1 & -1 & 1 \end{bmatrix} = A.$$

We note that the square root of a matrix is not unique. In fact, D has 8 different square roots, all of the form

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm\sqrt{2} & 0 \\ 0 & 0 & \pm 2 \end{bmatrix}.$$

It follows that A has 8 different square roots as well. We leave it as an exercise to compute them all. ♠

The same method can also be used to compute other powers of a matrix, for example a cube root.

Exercises

Exercise 8.5.1 *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Find* A^{10} *by diagonalization.*

Exercise 8.5.2 *Let* $A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. *Find* A^{50} *by diagonalization.*

Exercise 8.5.3 *Let* $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. *Find* A^{100} *by diagonalization.*

Exercise 8.5.4 *Let* $A = \begin{bmatrix} -5 & -6 \\ 9 & 10 \end{bmatrix}$. *Find a square root of A, i.e., find a matrix B such that* $B^2 = A$.

Exercise 8.5.5 *Let* $A = \begin{bmatrix} -2 & 0 & 6 \\ -3 & 1 & 6 \\ -3 & 0 & 7 \end{bmatrix}$. *Find a square root of A.*

8.6 Application: Solving recurrences

Outcomes

A. *Solve a linear recurrence relation using diagonalization.*

Consider the following sequence of integers, called the **Fibonacci sequence**:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

The first two Fibonacci numbers are 0 and 1. Every subsequent Fibonacci number is the sum of the previous two numbers. For example, $0+1=1$, $1+1=2$, $1+2=3$, $2+3=5$, and so on. Thus, if we write F_n for the n^{th} Fibonacci number, then the Fibonacci sequence is given by the following conditions:

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_{n+2} &= F_n + F_{n+1}, \quad \text{for all } n \geq 0. \end{aligned}$$

The condition $F_{n+2} = F_n + F_{n+1}$ is known as a **recurrence relation**, or simply as a **recurrence**, because we compute each member of the sequence from previous members (the word "recurrence" comes from Latin "recurrere", which means "to go back"). The conditions $F_0 = 0$ and $F_1 = 1$ are known as the **base cases** of the recurrence. Note that we start counting from zero, i.e., we call $F_0 = 0$ the "zeroth Fibonacci number", $F_1 = 1$ the "first Fibonacci number", and so on. Counting from zero will help simplify our calculations later.

Example 8.27: Computing a Fibonacci number

Compute the 10^{th} *Fibonacci number* F_{10}.

Solution. To compute the 10^{th} Fibonacci number using the recurrence, we have to compute all the previous Fibonacci numbers as well.

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_2 &= F_0 + F_1 = 0+1 = 1, \\ F_3 &= F_1 + F_2 = 1+1 = 2, \\ F_4 &= F_2 + F_3 = 1+2 = 3, \\ F_5 &= F_3 + F_4 = 2+3 = 5, \end{aligned}$$

$$\begin{aligned}
F_6 &= F_4 + F_5 = 3+5 = 8,\\
F_7 &= F_5 + F_6 = 5+8 = 13,\\
F_8 &= F_6 + F_7 = 8+13 = 21,\\
F_9 &= F_7 + F_8 = 13+21 = 34,\\
F_{10} &= F_8 + F_9 = 21+34 = 55.
\end{aligned}$$

Thus, the 10^{th} Fibonacci number is 55. ♠

Suppose we want to compute the 100^{th} Fibonacci number. As the previous example shows, computing this by using the recurrence relation would be lot of work. We will therefore explore how to use linear algebra, and in particular diagonalization, to find a closed formula for the n^{th} Fibonacci number. By a **closed formula**, we mean a formula to calculate F_n directly in one step, i.e., without using a recurrence. The process of finding a closed formula is called **solving the recurrence**.

The first step in solving the recurrence is to replace the recurrence relation $F_{n+2} = F_n + F_{n+1}$, which requires *two* previous terms of the sequence, by another recurrence relation requiring only *one* previous term. To that end, we define $\mathbf{v}_n$ to be the vector consisting of the n^{th} and $n+1^{\text{st}}$ Fibonacci numbers, for all $n \geq 0$:

$$\mathbf{v}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Using the recurrence relation for F_{n+2}, we have

$$\mathbf{v}_{n+1} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{v}_n.$$

Therefore, to compute $\mathbf{v}_{n+1}$, we only need to know $\mathbf{v}_n$ (and not $\mathbf{v}_{n-1}$). Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $\mathbf{v}_n$ is obtained from $\mathbf{v}_0$ by applying the matrix A n times, we have $\mathbf{v}_n = A^n \mathbf{v}_0$ for all $n \geq 0$. We can therefore get a formula for $\mathbf{v}_n$, and thus for F_n, by diagonalizing the matrix A.

Problem 8.28: Diagonalizing A

Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

The eigenvalues are the roots of the characteristic polynomial. We find them by using the quadratic formula. The eigenvalues are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

To simplify later calculations, we note that $\lambda_1 + \lambda_2 = 1$, or equivalently,

$$1 - \lambda_1 = \lambda_2. \tag{8.2}$$

We also note that $\lambda_1 \lambda_2 = -1$, or equivalently,

$$\lambda_2 = -\frac{1}{\lambda_1}. \tag{8.3}$$

To find the eigenvector corresponding to the eigenvalue λ_1, we solve the equation

$$\begin{bmatrix} -\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{bmatrix} \mathbf{u} = \mathbf{0}.$$

By (8.2) and (8.3), this is equivalent to

$$\begin{bmatrix} -\lambda_1 & 1 \\ 1 & -\frac{1}{\lambda_1} \end{bmatrix} \mathbf{u} = \mathbf{0},$$

and we find the basic solution

$$\mathbf{u} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}.$$

By a similar method, we find that the basic eigenvector corresponding to the eigenvalue λ_2 is

$$\mathbf{w} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Therefore, by Theorem 8.21, A is diagonalizable, and we have $P^{-1}AP = D$, where

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

For later reference, we note that the inverse of P is given by

$$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{bmatrix}.$$

♠

We are now ready to find our formula for the n^{th} Fibonacci number.

Problem 8.29: Solving the recurrence

Find a formula for the n^{th} Fibonacci number.

Solution. Since

$$\mathbf{v}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix},$$

we know that the n^{th} Fibonacci number is the first component of $\mathbf{v}_n$, i.e.,

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}_n.$$

Putting together all of the above calculations, we then have:

$$\begin{aligned}
F_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}_n \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} A^n \mathbf{v}_0 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} PD^nP^{-1} \mathbf{v}_0 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n).
\end{aligned}$$

So the n^{th} Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

♠

Example 8.30: Computing the 100^{th} Fibonacci number

Calculate the 100^{th} Fibonacci number without using the recurrence.

Solution. Note that this calculation requires about 25 digits of precision to give the correct result. We have

$$\begin{aligned}
F_{100} &= \frac{1}{\sqrt{5}}(\lambda_1^{100} - \lambda_2^{100}) \\
&= \frac{1}{\sqrt{5}}(1.6180339887498948482045868^{100} - (-0.6180339887498948482045868)^{100}) \\
&= 354224848179261915075.
\end{aligned}$$

♠

We can use the same method to solve other linear recurrences. Here is another example:

Example 8.31: Solving a recurrence

Consider the sequence of numbers defined by the recurrence

$$\begin{aligned}
b_0 &= 1, \\
b_1 &= 2, \\
b_{n+2} &= 6b_n + b_{n+1}, \quad \text{for all } n \geq 0.
\end{aligned}$$

Find the first 5 members of this sequence. Then solve the recurrence and find b_{20}.

Solution. The first 5 members of the sequence are:

$$\begin{aligned} b_0 &= 1, \\ b_1 &= 2, \\ b_2 &= 6b_0 + b_1 \;=\; 6+2 \;=\; 8, \\ b_3 &= 6b_1 + b_2 \;=\; 12+8 \;=\; 20, \\ b_4 &= 6b_2 + b_3 \;=\; 48+20 \;=\; 68. \end{aligned}$$

To solve the recurrence, let

$$\mathbf{w}_n = \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix},$$

so that for all $n \geq 0$,

$$\mathbf{w}_{n+1} = \begin{bmatrix} b_{n+1} \\ b_{n+2} \end{bmatrix} = \begin{bmatrix} b_{n+1} \\ 6b_n + b_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \mathbf{w}_n.$$

We then diagonalize the matrix

$$B = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda - 6,$$

with roots $\lambda = -2$ and $\lambda = 3$. The respective eigenvectors are

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{for the eigenvalue } \lambda = -2,$$
$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{for the eigenvalue } \lambda = 3.$$

Therefore, $B = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}.$$

The inverse of P is

$$P^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}.$$

Finally, we use this information to solve the recurrence:

$$\begin{aligned} b_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{w}_n \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} B^n \mathbf{w}_0 \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} PD^nP^{-1} \mathbf{w}_0 \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ &= \frac{(-2)^n + 4 \cdot 3^n}{5} \end{aligned}$$

Finally, we calculate

$$b_{20} = \frac{(-2)^{20} + 4 \cdot 3^{20}}{5} = \frac{1048576 + 4 \cdot 3486784401}{5} = 2789637236.$$

♠

Exercises

Exercise 8.6.1 *Consider the sequence of numbers defined by the recurrence*

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_{n+2} &= 2a_{n+1} + 3a_n, \quad \textit{for all } n \geq 0. \end{aligned}$$

Find the first 5 members of this sequence. Then solve the recurrence and find a_{20}.

Exercise 8.6.2 *Consider the sequence of numbers defined by the recurrence*

$$\begin{aligned} b_0 &= 1, \\ b_1 &= 2, \\ b_2 &= 3, \\ b_{n+3} &= 2b_{n+2} + b_{n+1} - 2b_n, \quad \textit{for all } n \geq 0. \end{aligned}$$

Find the first 5 members of this sequence. Then solve the recurrence and find b_{20}.

Exercise 8.6.3 *Consider two sequences of numbers $c_0, c_1, c_2, \ldots$ and $d_0, d_1, d_2, \ldots$, defined by the following mutual recurrence relation.*

$$\begin{aligned} c_0 &= 0, & d_0 &= 1 \\ c_{n+1} &= 4c_n - 3d_n, & d_{n+1} &= 2c_n - d_n, \quad \textit{for all } n \geq 0. \end{aligned}$$

Find the first 5 members of both sequences. Then solve the recurrence. Hint: first find a matrix A such that

$$\begin{bmatrix} c_{n+1} \\ d_{n+1} \end{bmatrix} = A \begin{bmatrix} c_n \\ d_n \end{bmatrix}.$$

8.7 Application: Systems of linear differential equations

Outcomes

A. *Solve a system of second order linear differential equations.*

8.7.1. Differential equations

Recall from calculus that if $y = f(x)$ is a function, then $y' = f'(x)$ is its derivative and $y'' = f''(x)$ is its second derivative. For example,

$$\begin{aligned} y &= \sin(x), \\ y' &= \cos(x), \\ y'' &= -\sin(x). \end{aligned}$$

Note that if $y = \sin(x)$, the second derivative y'' is exactly the negative of y, i.e.,

$$y'' = -y.$$

This last equation is called a **differential equation**. Unlike an ordinary equation, which is about an unknown *number*, a differential equation is about an unknown *function*. Typically, a differential equation mentions the function and one or more of its derivatives. A differential equation that only mentions the first derivative y' is called a **first-order** differential equation. A differential equation that also mentions the second derivative y'' is called a **second-order** differential equation. If the equation is a linear function of y and its derivatives, it is called a **linear differential equation**.

We say that the function $y = \sin(x)$ is a **solution** of the differential equation $y'' = -y$. It is not the only solution. Another solution is $y = \cos(x)$, because in that case, $y'' = -\cos(x)$, and therefore $y'' = -y$. From calculus, we know that the **general solution** to the differential equation $y'' = -y$ is given by

$$y = a\sin(x) + b\cos(x),$$

where a and b are any real numbers, i.e., parameters. Using the terminology of linear algebra, we can say that the general solution of the equation $y'' = -y$ is a **linear combination** of the **basic solutions** $y = \sin(x)$ and $y = \cos(x)$. More generally, we have the following theorem from calculus:

Theorem 8.32: Solutions of $y'' = qy$

Let q be a positive real number. The differential equation

$$\begin{array}{lll} y'' = -qy & \text{has basic solutions} & y = \sin(\sqrt{q}x) \text{ and } y = \cos(\sqrt{q}x), \\ y'' = 0 & \text{has basic solutions} & y = 1 \text{ and } y = x, \\ y'' = qy & \text{has basic solutions} & y = e^{\sqrt{q}x} \text{ and } y = e^{-\sqrt{q}x}. \end{array}$$

Proof. By taking derivatives, it is easy to check that each of the six functions is a solution of the corresponding differential equation. For example, for $y = \sin(\sqrt{q}x)$, we have $y' = \sqrt{q}\cos(\sqrt{q}x)$ and $y'' = -q\sin(\sqrt{q}x)$. Therefore, $y'' = -qy$.

Note that we can obtain the general solution of each of the differential equations as a linear combination of the basic solutions. Thus, the general solution of $y'' = -qy$ is

$$y = a\sin(\sqrt{q}x) + b\cos(\sqrt{q}x),$$

the general solution of $y'' = 0$ is

$$y = a + bx,$$

and the general solution of $y'' = qy$ is

$$y = ae^{\sqrt{q}x} + be^{-\sqrt{q}x},$$

where a and b are parameters. The fact that each of these solutions is indeed the most general one is proved in a calculus course. ♠

8.7.2. Systems of linear differential equations

In the same way that a system of linear equations consists of several linear equations about several variables, a **system of differential equations** consists of several differential equations about several unknown functions and their derivatives. For example, the following is a system of second order linear differential equations:

$$\begin{aligned} y'' &= 4y - 3z, \\ z'' &= 6y - 5z. \end{aligned}$$

In reading these equations, it is important to understand that we are looking for two unknown *functions* $y = f(x)$ and $z = g(x)$ such that their second derivatives satisfy both of the equations $y'' = 4y - 3z$ and $z'' = 6y - 5z$. The reason that this is in principle a difficult problem is that the equation for y'' mentions not only y, but also z, and the equation for z'' mentions not only z, but also y. Therefore, it is not possible to solve this system one function at a time. We say that the variables y and z are **coupled**.

The following example shows how we can use diagonalization to decouple the variables in a system of differential equations. This makes it possible to solve the equations.

Example 8.33: A system of linear differential equations

Solve the following system of second order linear differential equations:

$$\begin{aligned} y'' &= 4y - 3z, \\ z'' &= 6y - 5z. \end{aligned}$$

Solution. We start by writing the system in matrix form:

$$\begin{bmatrix} y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

Let us write

$$\mathbf{v} = \begin{bmatrix} y \\ z \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}.$$

With these notations, the system of differential equations take the form

$$\mathbf{v}'' = A\mathbf{v}. \tag{8.4}$$

Our next step is to diagonalize the matrix A. Following the usual diagonalization procedure, we find that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

With this, our equation takes the form $\mathbf{v}'' = PDP^{-1}\mathbf{v}$, which we can also write as $P^{-1}\mathbf{v}'' = DP^{-1}\mathbf{v}$. We now introduce a **change of variables**. Let $\mathbf{w} = P^{-1}\mathbf{v}$. Then our system of differential equations can be written as

$$\mathbf{w}'' = D\mathbf{w}. \tag{8.5}$$

Note that the equation (8.5) is of exactly the same form as the equation (8.4), but with the crucial difference that the matrix in (8.5) is diagonal. Let us give a name to the components of $\mathbf{w}$:

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Then the equation (8.5) can be written as

$$\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

or equivalently,

$$\begin{aligned} u'' &= u, \\ v'' &= -2v. \end{aligned}$$

Note that the variables u and v are not coupled! This happened because the matrix D is diagonal. We can therefore use Theorem 8.32 to solve the equations for u and for v separately. By Theorem 8.32, the general solution for the equation $u'' = u$ is

$$u = ae^x + be^{-x},$$

and the general solution for the equation $v'' = -2v$ is

$$v = c\sin(\sqrt{2}x) + d\cos(\sqrt{2}x).$$

Here, a, b, c, and d are parameters. Therefore, the general solution for (8.5) is

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ae^x + be^{-x} \\ c\sin(\sqrt{2}x) + d\cos(\sqrt{2}x) \end{bmatrix} = a\begin{bmatrix} e^x \\ 0 \end{bmatrix} + b\begin{bmatrix} e^{-x} \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ \sin(\sqrt{2}x) \end{bmatrix} + d\begin{bmatrix} 0 \\ \cos(\sqrt{2}x) \end{bmatrix}.$$

We have therefore found the four basic solution of (8.5). But what about our original equation (8.4)? We can undo our change of variables. Since $\mathbf{w} = P^{-1}\mathbf{v}$, we have $\mathbf{v} = P\mathbf{w}$. Therefore, the general solution to our original system of differential equations is

$$\begin{aligned} \mathbf{v} \;=\; P\mathbf{w} &= aP\begin{bmatrix} e^x \\ 0 \end{bmatrix} + bP\begin{bmatrix} e^{-x} \\ 0 \end{bmatrix} + cP\begin{bmatrix} 0 \\ \sin(\sqrt{2}x) \end{bmatrix} + dP\begin{bmatrix} 0 \\ \cos(\sqrt{2}x) \end{bmatrix} \\ &= ae^x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\sin(\sqrt{2}x)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\cos(\sqrt{2}x)\begin{bmatrix} 1 \\ 2 \end{bmatrix}. \end{aligned}$$

♠

8.7.3. Example: coupled train cars

One of the reasons that differential equations are important is that the **laws of nature** often take the form of differential equations. For example, **Newton's second law of motion** asserts that the acceleration of an object is equal to the total force on the object divided by the mass of the object. In physics, it is common to use t instead of x for the independent variable, and x instead of y for the dependent variable, so that we write $x = f(t)$ instead of $y = f(x)$. If x is the position of the object at time t, then the object's acceleration is x'', and Newton's second law takes the form

$$x'' = \frac{F}{m}.$$

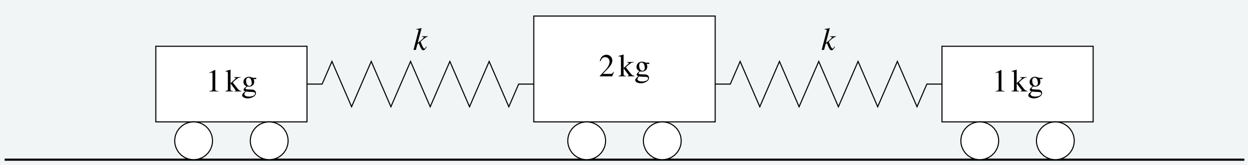
This is a differential equation. In the following example, we will need another law of physics, namely **Hooke's law** about the force exerted by a spring. A **spring** is an object made from an elastic material (often in the shape of a coil), which returns to its original shape after being stretched or compressed. Hooke's law states that the force exerted by a spring to both of its ends is equal to

$$F = kx.$$

Here x is the **extension** of the spring, i.e., the change in length of the spring, relative to its relaxed (natural) length. Also, k is a constant called the **spring constant**, measured in units of $\frac{\text{N}}{\text{m}}$ (Newtons per meter). Of course the direction of the force on one end of the spring is the opposite of the direction on the other end. Hooke's law is not a differential equation, because it does not mention any derivatives. It is just an ordinary equation. Nevertheless, both the force F and the extension x can vary with time, i.e., they can both be functions of t.

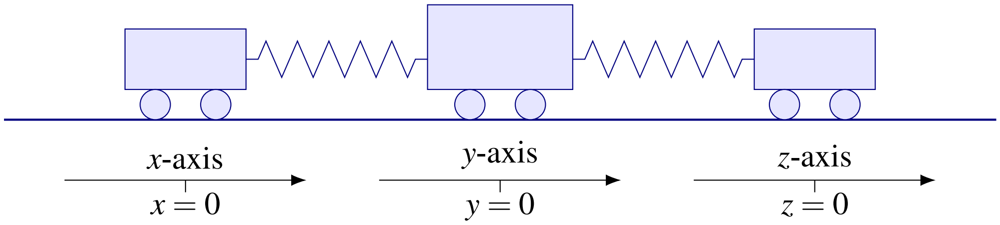
Example 8.34: Coupled train cars

Consider a train made up of three cars of mass 1 kg, 2 kg, *and* 1 kg, *which are aligned on a linear track and connected by springs. Assume that each spring has the same spring constant* $k = 2\frac{\text{N}}{\text{m}}$.



Find and solve the equations of motion of this system.

Solution. Let us start by defining appropriate coordinates. Let x be the position of the first car, y the position of the second car, and z the position of the third car, measured in meters from left to right, relative to each car's natural resting position. The x-, y-, and z-axes are shown in the following picture. All three axes are parallel to the train tracks, but they have their origins in different places. The coordinates are chosen so that when $x = 0$, $y = 0$, and $z = 0$, then all three cars are at rest and both springs are in their natural relaxed state.



Then the extension of the left spring is $y-x$, and therefore the left spring's contracting force is

$$F_1 = k(y-x).$$

Similarly, the extension of the right spring is $z-y$, and therefore its contracting force is

$$F_2 = k(z-y).$$

The total force acting on the left car is F_1, the total force acting on the middle car is $F_2 - F_1$, and the total force acting on the right car is $-F_2$. By Newton's second law, the acceleration of each car is given by $x'' = \frac{F_1}{m_1}$, $y'' = \frac{F_2-F_1}{m_2}$, and $z'' = \frac{-F_2}{m_3}$. We therefore have the following equations of motion:

$$\begin{aligned} x'' &= \frac{k}{m_1}(y-x), \\ y'' &= \frac{k}{m_2}(x-2y+z), \\ z'' &= \frac{k}{m_3}(z-y). \end{aligned}$$

Let us ignore the physical units and plug in the masses $m_1 = 1$, $m_2 = 2$, and $m_3 = 1$ and the spring constant $k = 2$. Then the equations of motion are:

$$\begin{aligned} x'' &= 2(y-x), \\ y'' &= x-2y+z, \\ z'' &= 2(z-y), \end{aligned}$$

or equivalently in matrix form:

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We can also write this as $\mathbf{v}'' = A\mathbf{v}$, where

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

To solve the equation, we diagonalize the matrix A. Using the usual method for diagonalization, we find that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equation $\mathbf{v}'' = A\mathbf{v}$ then becomes $\mathbf{v}'' = PDP^{-1}\mathbf{v}$, or equivalently $P^{-1}\mathbf{v}'' = DP^{-1}\mathbf{v}$. We then diagonalize the equation by performing the change of variables $\mathbf{w} = P^{-1}\mathbf{v}$. The equation becomes

$$\mathbf{w}'' = D\mathbf{w}.$$

If the components of $\mathbf{w}$ are called u, v, and w, we can write this as

$$\begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

or equivalently,

$$\begin{aligned} u'' &= -2u, \\ v'' &= -4v, \\ w'' &= \ \ 0. \end{aligned}$$

Since the variables are now decoupled, we can solve each differential equation individually.

- **Solutions for $\lambda = -2$:** By Theorem 8.32, the basic solutions for $u'' = -2u$ are

  $$u = \sin(\sqrt{2}t) \quad \text{and} \quad u = \cos(\sqrt{2}t).$$

  This translates into the following basic solutions for $\mathbf{w}$:

  $$\mathbf{w} = \sin(\sqrt{2}t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \cos(\sqrt{2}t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

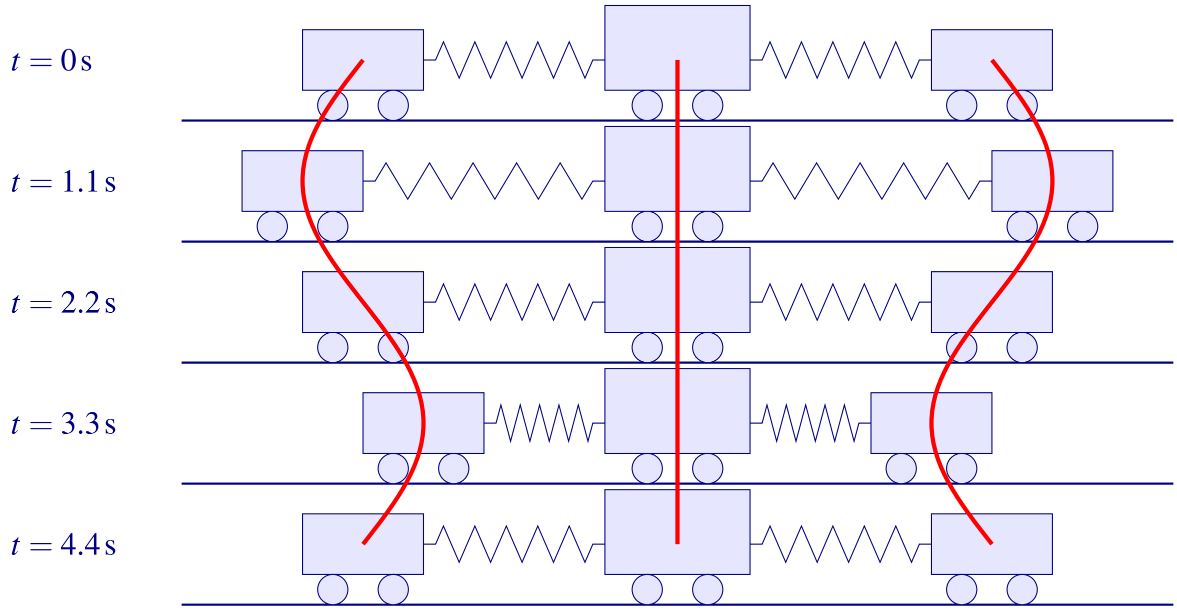
  Using $\mathbf{v} = P\mathbf{w}$ to change to the original variables, we get the basic solutions

  $$\mathbf{v} = \sin(\sqrt{2}t)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \cos(\sqrt{2}t)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

  For example, the first of these basic solutions, written in the coordinates x, y, and z, gives

  $$\begin{aligned} x &= -\sin(\sqrt{2}t), \\ y &= \ \ 0, \\ z &= \ \ \sin(\sqrt{2}t). \end{aligned}$$

  This corresponds to a periodic oscillation of the train where the middle car is stationary, and the left car moves left when the right car moves right. Each oscillation takes $2\pi/\sqrt{2} \approx 4.4$ seconds. Here is a "movie" showing one oscillation:



The other basic solution, with cos instead of sin, is the same motion, just starting at a different offset in time. Note how the eigenvector of A,

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

describes the relative motion of the three cars, i.e., the first and last cars are moving in opposite directions, whereas the middle car is stationary. The corresponding eigenvalue $\lambda = -2$ determines the frequency. The frequency, which is $\sqrt{2}/2\pi$ oscillations per second, is also called an **eigenfrequency** or **resonance frequency** of the system.

- **Solutions for $\lambda = -4$:** By Theorem 8.32, the basic solutions for $v'' = -4v$ are

$$v = \sin(2t) \quad \text{and} \quad v = \cos(2t).$$

This translates into the following basic solutions for $\mathbf{w}$:

$$\mathbf{w} = \sin(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \cos(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

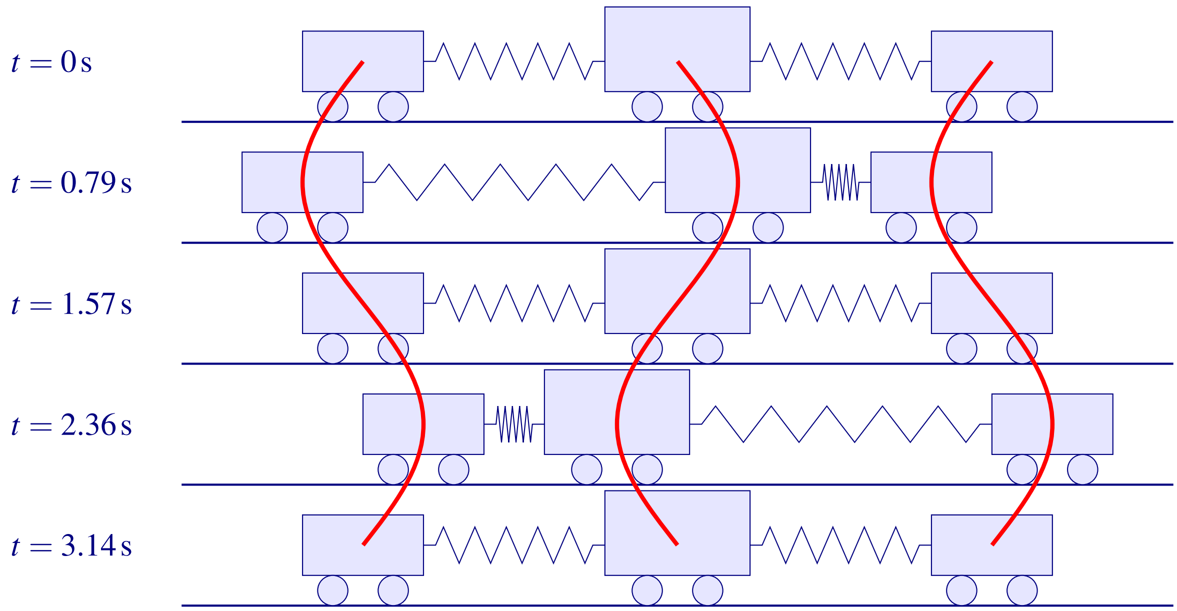
We change this to the original variables using $\mathbf{v} = P\mathbf{w}$, and get the basic solutions

$$\mathbf{v} = \sin(2t)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \cos(2t)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Writing the first of these basic solutions in the coordinates x, y, and z, we get

$$\begin{aligned} x &= \sin(2t), \\ y &= -\sin(2t), \\ z &= \sin(2t). \end{aligned}$$

This corresponds to a periodic oscillation of the train where the outer cars move right at the same time that the middle car moves left. Each oscillation takes $2\pi/2 \approx 3.14$ seconds. Here is a "movie" of the motion:



As before, the other basic solution, using cos instead of sin, is the same motion, but shifted in time. Also, the eigenvector

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

describes the relative motion of the three cars; here the first and last car move in the same direction while the middle car moves in the opposite direction. The eigenfrequency of this oscillation, at $2/2\pi$ oscillations per second, is slightly higher than the first one, due to the larger magnitude of the eigenvalue $\lambda = -4$.

- **Solutions for $\lambda = 0$:** The last eigenvalue is zero. By Theorem 8.32, the general solution of $w'' = 0$ is

$$w = a + bt,$$

where a and b are arbitrary constants. This translates into the following solution for $\mathbf{w}$:

$$\mathbf{w} = (a+bt)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

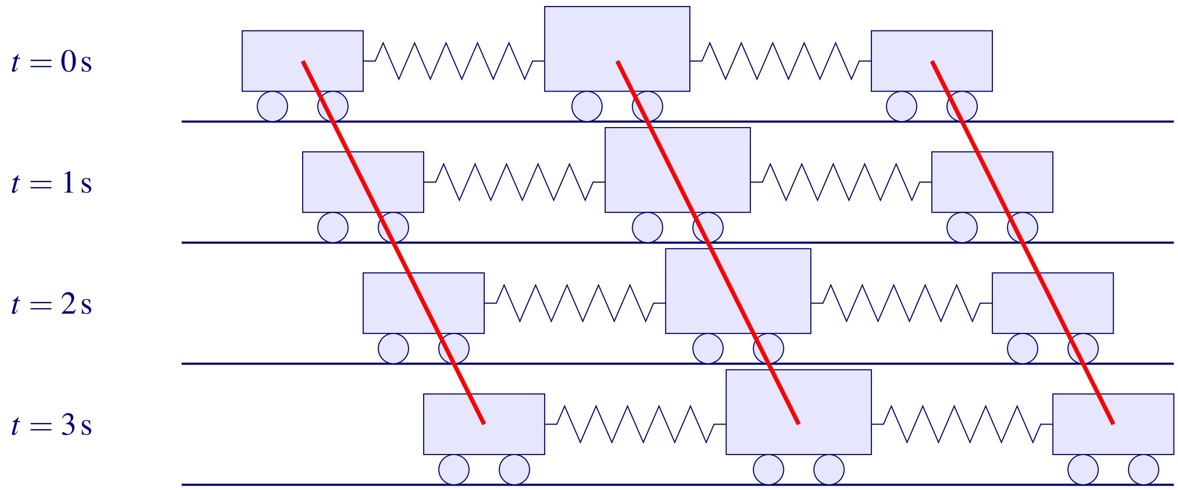
and by the change of variables $\mathbf{v} = P\mathbf{w}$, we get

$$\mathbf{v} = (a+bt)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Writing this in terms of the coordinates x, y, and z, we get

$$\begin{aligned} x &= a+bt, \\ y &= a+bt, \\ z &= a+bt. \end{aligned}$$

This simply describes a linear motion: the three cars are moving down the track at constant speed b from some initial starting position a.



As before, the eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

describes the relative motion of the three cars (in this case all moving in the same direction). The eigenvalue $\lambda = 0$ indicates linear motion.

What we have described here are the *basic* solutions of the system. The solutions corresponding to each of the eigenvalues are also called **modes** of the system. Thus, the train has three modes, each corresponding to a particular eigenvalue of the matrix A. In the first mode, the middle car is stationary and the other two cars oscillate in opposite directions. In the second mode, the middle car oscillates in the opposite direction of the two outer cars. The third mode is a linear movement along the track.

As always, the general solution is a linear combination of basic solutions; for example, the cars might be oscillating in both the first and second modes at their respective frequencies, while also moving down the tracks. ♠

Exercises

Exercise 8.7.1 *Solve the following system of second order linear differential equations:*

$$\begin{aligned} y'' &= -5y - 6z, \\ z'' &= 3y + 4z. \end{aligned}$$

Exercise 8.7.2 *Solve the following system of second order linear differential equations:*

$$\begin{aligned} y'' &= 4y - 3z, \\ z'' &= 2y - z. \end{aligned}$$

Exercise 8.7.3 *Solve the following system of second order linear differential equations:*

$$\begin{aligned} x'' &= -2y+2z, \\ y'' &= x-z, \\ z'' &= x-2y+z. \end{aligned}$$

Exercise 8.7.4 *Solve the following system of first order linear differential equations.*

$$\begin{aligned} y' &= 4y + 6z, \\ z' &= -3y - 5z. \end{aligned}$$

Hint: The method is similar to that of second-order equations. Use the fact, known from calculus, that the equation $f' = kf$ has basic solution $f(x) = e^{kx}$. Here k is any constant (positive, negative, or zero).

Exercise 8.7.5 *Consider three coupled train cars as in Example 8.34, except that all three cars have mass* 1 kg *and both spring have spring constant $k = 1\frac{\mathrm{N}}{\mathrm{m}}$. Find and solve the equations of motion.*

8.8 Application: The matrix exponential

Outcomes

A. *Compute e^A, $\sin A$, and $\cos A$, for a square matrix A.*

B. *Apply any analytic function to a square matrix.*

From calculus, recall that a function is called **analytic** if it can be defined by a power series. For example:

$$\begin{aligned} e^x &= 1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots \\ \sin x &= x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7\pm\ldots \\ \cos x &= 1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6\pm\ldots \end{aligned}$$

We know from calculus that the above power series converge for all real numbers x. Since it makes sense to compute the n^{th} power of a square matrix, in principle it also makes sense to plug a matrix into a power series. For a square matrix A, we can define

$$\begin{aligned} e^A &= I+A+\frac{1}{2}A^2+\frac{1}{3!}A^3+\frac{1}{4!}A^4+\ldots \\ \sin A &= A-\frac{1}{3!}A^3+\frac{1}{5!}A^5-\frac{1}{7!}A^7\pm\ldots \\ \cos A &= I-\frac{1}{2}A^2+\frac{1}{4!}A^4-\frac{1}{6!}A^6\pm\ldots \end{aligned}$$

The goal of this section is to investigate whether these power series converge, and if yes, how to compute the sum of the series. We begin with the case of a diagonal matrix.

Example 8.35: Exponential of a diagonal matrix

Compute e^D, where

$$D = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$

Solution. By definition,

$$\begin{aligned} e^D &= I + D + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + \ldots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} x^3 & 0 \\ 0 & y^3 \end{bmatrix} + \ldots \\ &= \begin{bmatrix} 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots & 0 \\ 0 & 1 + y + \frac{1}{2}y^2 + \frac{1}{3!}y^3 + \ldots \end{bmatrix} \\ &= \begin{bmatrix} e^x & 0 \\ 0 & e^y \end{bmatrix}. \end{aligned}$$

Therefore, the exponential of a diagonal matrix is computed by taking the exponential of each diagonal entry. Note that this proves, in particular, that the sum converges. ♠

The same argument also works for applying other analytic functions to diagonal matrices, for example:

$$\begin{aligned} \sin \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} &= \begin{bmatrix} \sin x & 0 \\ 0 & \sin y \end{bmatrix}, \\ \cos \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} &= \begin{bmatrix} \cos x & 0 \\ 0 & \cos y \end{bmatrix}. \end{aligned}$$

But how can we compute the matrix exponential of a non-diagonal matrix? This can be done by diagonalization. The following theorem shows how:

Theorem 8.36: Matrix functions by diagonalization

Suppose A is a diagonalizable square matrix, with $A = PDP^{-1}$. Then

$$\begin{aligned} e^A &= Pe^D P^{-1}, \\ \sin A &= P(\sin D)P^{-1}, \\ \cos A &= P(\cos D)P^{-1}. \end{aligned}$$

Proof. We have

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \ldots \\ &= I + PDP^{-1} + \frac{1}{2}(PDP^{-1})^2 + \frac{1}{3!}(PDP^{-1})^3 + \ldots \end{aligned}$$

$$\begin{aligned}
&= PIP^{-1} + PDP^{-1} + \frac{1}{2}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\
&= P(I + D + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + \dots)P^{-1} \\
&= Pe^DP^{-1}.
\end{aligned}$$

The proof for sin and cos is similar. Indeed, the same method works for any analytic function. ♠

Example 8.37: A matrix exponential

Compute e^A, where

$$A = \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix}.$$

Solution. We first diagonalize A. Following the usual method, we find that the eigenvalues are $\lambda = -3$ and $\lambda = 2$, with corresponding eigenvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, we can diagonalize A as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

By Theorem 8.36, we have

$$e^A = Pe^DP^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3} & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^{-3} - e^2 & -2e^{-3} + 2e^2 \\ e^{-3} - e^2 & -e^{-3} + 2e^2 \end{bmatrix}.$$

♠

Exercises

Exercise 8.8.1 *Compute e^A and $\cos A$ for the following matrices:*

$$(a) \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad (b) \quad A = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}, \quad (c) \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Exercise 8.8.2 *The cube root is an analytic function. Compute the cube root of the matrix*

$$A = \begin{bmatrix} 22 & -21 \\ 14 & -13 \end{bmatrix}.$$

Exercise 8.8.3 *Use matrix exponentials to find the solution to the first-order linear differential equation*

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with initial value

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Hint: form the matrix exponential e^{At} and then the solution is $e^{At}\,\mathbf{v}_0$, where $\mathbf{v}_0$ is the initial vector.

8.9 Properties of eigenvectors and eigenvalues

Outcomes

A. *Know that eigenvectors corresponding to distinct eigenvalues are linearly independent.*

B. *Compute the algebraic and geometric multiplicity of an eigenvalue.*

C. *Determine whether a matrix is diagonalizable from the geometric multiplicities of its eigenvalues.*

In this section, we state some useful properties of eigenvectors and eigenvalues. The first question we consider is whether eigenvectors for different eigenvalues are linearly independent. This is indeed the case, as the following proposition shows:

Proposition 8.38: Eigenvectors for different eigenvalues are linearly independent

Let A be a square matrix, and suppose that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Proof. Suppose, for the sake of obtaining a contradiction, that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent. Let m the the smallest index such that $\mathbf{v}_m$ is redundant, i.e., such that $\mathbf{v}_m$ is a linear combination of previous vectors. Say

$$\mathbf{v}_m = a_1\mathbf{v}_1 + \ldots + a_{m-1}\mathbf{v}_{m-1}. \tag{8.6}$$

Multiplying the equation by A, we get

$$A\mathbf{v}_m = a_1 A\mathbf{v}_1 + \ldots + a_{m-1} A\mathbf{v}_{m-1},$$

and therefore, since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are eigenvectors,

$$\lambda_m\mathbf{v}_m = a_1\lambda_1\mathbf{v}_1 + \ldots + a_{m-1}\lambda_{m-1}\mathbf{v}_{m-1}. \tag{8.7}$$

Subtracting λ_m times equation (8.6) from (8.7), we get

$$\mathbf{0} = a_1(\lambda_1 - \lambda_m)\mathbf{v}_1 + \ldots + a_{m-1}(\lambda_{m-1} - \lambda_m)\mathbf{v}_{m-1}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}$ are, by assumption, linearly independent (because $\mathbf{v}_m$ was the leftmost redundant vector), it follows that $a_1(\lambda_1 - \lambda_m) = 0, \ldots, a_{m-1}(\lambda_{m-1} - \lambda_{m-1}) = 0$. Since the eigenvalues $\lambda_1, \ldots, \lambda_m$ are, by assumption, distinct, it follows that $a_1, \ldots, a_{m-1} = 0$. But then (8.6) implies that $\mathbf{v}_m = \mathbf{0}$, contradicting the assumption that $\mathbf{v}_m$ is an eigenvector (and therefore non-zero). ♠

An immediate consequence of this proposition is that an $n \times n$-matrix with n distinct eigenvalues is diagonalizable.

Corollary 8.39: Distinct eigenvalues

Let A be an $n \times n$-matrix and suppose it has n distinct eigenvalues. Then A is diagonalizable.

Proof. Each of the n eigenvalues has an eigenvector, and by Proposition 8.38, they are linearly independent. Then A is diagonalizable by Theorem 8.21. ♠

The next issue we consider is that of "repeated" eigenvalues. There are two senses in which an eigenvalue can occur "more than once". The first is if the eigenvalue appears as a repeated root of the characteristic polynomial. For example, if the characteristic polynomial is $p(\lambda) = (1-\lambda)(1-\lambda)(3-\lambda)$, then we say that the root $\lambda = 1$ appears with multiplicity two, and the root $\lambda = 3$ appears with multiplicity one. We call this the **algebraic multiplicity** of the eigenvalue.

The second sense in which an eigenvalue can occur "more than once" is when an eigenvalue has more than one linearly independent eigenvector. In other words, when the eigenspace has dimension greater than 1. We call this the **geometric multiplicity** of the eigenvalue.

The following definition summarizes these concepts.

Definition 8.40: Algebraic and geometric multiplicity

*Let $\hat{\lambda}$ be an eigenvalue of a square matrix A. Then the **algebraic multiplicity** of $\hat{\lambda}$ is the largest power k such that $(\hat{\lambda} - \lambda)^k$ is a factor of the characteristic polynomial. The **geometric multiplicity** of $\hat{\lambda}$ is the dimension of its eigenspace $E_{\hat{\lambda}}$.*

One would hope that the algebraic and geometric multiplicities are always equal. Unfortunately, this is not the case, as the following example shows.

Example 8.41: Algebraic and geometric multiplicity

Let

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Find the algebraic and geometric multiplicity of each eigenvalue of A.

Solution. The characteristic polynomial is $(3-\lambda)^2(4-\lambda)^2(5-\lambda)$. Therefore, the eigenvalues are 3, 4, and 5, with algebraic multiplicity 2, 2, and 1, respectively. To compute the geometric multiplicity, we need to find each eigenspace. For $\lambda = 3$, we must solve $(A-3I)\mathbf{v} = \mathbf{0}$, or equivalently,

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right].$$

This system has rank 4, and the only basic solution is $[1,0,0,0,0]^T$. Thus, the eigenspace E_3 is 1-dimensional, and the geometric multiplicity of $\lambda = 3$ is 1. A similar calculation show that $\lambda = 4$ has geometric multiplicity 2 and $\lambda = 5$ has geometric multiplicity 1. The information is summarized in the following table:

Eigenvalue	$\lambda=3$	$\lambda=4$	$\lambda=5$
Algebraic multiplicity	2	2	1
Geometric multiplicity	1	2	1

♠

In the example, the geometric multiplicity is either smaller than or equal to the algebraic multiplicity. The following proposition states that this is always the case.

Proposition 8.42: Algebraic and geometric multiplicity

Let $\hat{\lambda}$ be an eigenvalue of a matrix A, with algebraic multiplicity k and geometric multiplicity m. Then

$$1 \leq m \leq k.$$

Proof. It is clear that $m \geq 1$, because each eigenvalue, by definition, must have at least one associated eigenvector. Therefore, the eigenspace is at least 1-dimensional. We must show $m \leq k$. Assume that A is an $n \times n$-matrix. By assumption, the geometric multiplicity of $\hat{\lambda}$ is m, so the eigenspace $E_{\hat{\lambda}}$ has dimension m. So there exist m linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ for the eigenvalue $\hat{\lambda}$. Extend $\mathbf{v}_1, \ldots, \mathbf{v}_m$ to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$, and let P be the invertible matrix that has $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as its columns. Let $B = P^{-1}AP$. Since the first m columns of P are eigenvectors of A for the eigenvalue $\hat{\lambda}$, it follows that B is of the form

$$\begin{bmatrix} \hat{\lambda} & 0 & \cdots & 0 & * & \cdots & * \\ 0 & \hat{\lambda} & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \hat{\lambda} & * & \cdots & * \\ 0 & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{bmatrix},$$

i.e., the first m columns of B are like those of a diagonal matrix. Then from the cofactor method for computing determinants, we know that $\det(B-\lambda I)$ contains the factor $(\hat{\lambda}-\lambda)^m$. But since A and B are similar matrices, they have the same characteristic polynomial. Therefore, $\det(A-\lambda I)$ also has $(\hat{\lambda}-\lambda)^m$ as a factor. It follows, by definition of algebraic multiplicity, that $m \leq k$, as desired. ♠

We know from Theorem 8.21 that an $n\times n$-matrix is diagonalizable if and only if it has n linearly independent eigenvectors. We can re-state this in terms of geometric multiplicity as follows.

Proposition 8.43: Geometric multiplicity and diagonalization

An $n\times n$-matrix A is diagonalizable if and only if the sum of the geometric multiplicities of all the eigenvalues of A is n.

Proof. By Proposition 8.38, eigenvectors corresponding to different eigenvalues are linearly independent. Therefore, by taking a basis of each eigenspace, we can obtain exactly as many linearly independent eigenvectors as the sum of the dimensions of all the eigenspaces, i.e., the sum of the geometric multiplicities of all eigenvalues. By Theorem 8.21, A is diagonalizable if and only if this number is n. ♠

Exercises

Exercise 8.9.1 *Find the algebraic and geometric multiplicity of each eigenvalue of the following matrices. Which of the matrices are diagonalizable?*

$$(a)\quad A=\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix},\quad (b)\quad B=\begin{bmatrix} -7 & 8 \\ -4 & 5 \end{bmatrix},$$

$$(c)\quad C=\begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 2 \\ 0 & -1 & 1 \end{bmatrix},\quad (d)\quad D=\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix},\quad (e)\quad E=\begin{bmatrix} -2 & 0 & 1 \\ -1 & -1 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

Exercise 8.9.2 *Determine which of the following matrices are diagonalizable.*

(a) A is a 3×3-matrix with eigenvalues -1 and 3. The eigenvalue -1 has algebraic multiplicity 2 and geometric multiplicity 1. The eigenvalue 3 has algebraic and geometric multiplicity 1.

(b) B is a 4×4-matrix with eigenvalues 2 and -2. The eigenvalue 2 has algebraic and geometric multiplicity 1. The eigenvalue -2 has algebraic and geometric multiplicity 3.

(c) C is a 5×5-matrix with eigenvalues 1 and 3. The eigenvalue 1 has algebraic and geometric multiplicity 2, and the eigenvalue 3 has algebraic and geometric multiplicity 1.

8.10 The Cayley-Hamilton Theorem

Outcomes

A. *For a square matrix A, find a polynomial $p(x)$ such that $p(A) = 0$.*

In this section, we will consider the so-called **Cayley-Hamilton theorem**. It states that every square matrix is a root of its own characteristic polynomial. We use the following notation. If

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

is a polynomial, we denote by $p(A)$ the matrix defined by

$$p(A) = a_nA^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I.$$

The explanation for the last term is that A^0 is interpreted as I, the identity matrix.

Theorem 8.44: Cayley-Hamilton theorem

Let A be a square matrix and let $p(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial. Then $p(A) = 0$.

Before we prove this theorem, we consider an example.

Example 8.45: Cayley-Hamilton theorem

Let

$$A = \left[\begin{array}{rr} 3 & 4 \\ -1 & 2 \end{array} \right].$$

Find the characteristic polynomial $p(\lambda)$, and compute $p(A)$.

Solution. The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \left| \begin{array}{cc} 3-\lambda & 4 \\ -1 & 2-\lambda \end{array} \right| = (3-\lambda)(2-\lambda) - (-1)4 = \lambda^2 - 5\lambda + 10.$$

Applying the characteristic polynomial to A, we get

$$\begin{aligned} p(A) = A^2 - 5A + 10I &= \left[\begin{array}{rr} 3 & 4 \\ -1 & 2 \end{array} \right]^2 - 5 \left[\begin{array}{rr} 3 & 4 \\ -1 & 2 \end{array} \right] + \left[\begin{array}{rr} 10 & 0 \\ 0 & 10 \end{array} \right] \\ &= \left[\begin{array}{rr} 5 & 20 \\ -5 & 0 \end{array} \right] - \left[\begin{array}{rr} 15 & 20 \\ -5 & 10 \end{array} \right] + \left[\begin{array}{rr} 10 & 0 \\ 0 & 10 \end{array} \right] = \left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \end{array} \right], \end{aligned}$$

just as predicted by the Cayley-Hamilton theorem. ♠

The remainder of this section is devoted to the proof of the Cayley-Hamilton theorem. Readers who are not interested in the proof can skip this material. We begin with a lemma:

Lemma 8.46: Polynomials with matrix coefficients

Let $A_0, \ldots, A_m$ be $n \times n$-matrices and assume that for all scalars λ,

$$A_0 + A_1\lambda + \ldots + A_m\lambda^m = 0.$$

Then each $A_i = 0$.

Proof. Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \ldots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \ldots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \to \infty$ to obtain $A_i = 0$ for all i. ♠

The following is a simple consequence of the lemma.

Corollary 8.47:

Let A_i and B_i be $n \times n$-matrices and suppose that

$$A_0 + A_1\lambda + \ldots + A_m\lambda^m = B_0 + B_1\lambda + \ldots + B_m\lambda^m$$

for all λ. Then for any $n \times n$-matrix C,

$$A_0 + A_1C + \ldots + A_mC^m = B_0 + B_1C + \ldots + B_mC^m.$$

Proof. Subtracting the right-hand side from the left-hand side and using Lemma 8.46, we get that $A_i = B_i$ for all i. But then the conclusion immediately follows. ♠

With this preparation, it is now relatively easy to prove the Cayley-Hamilton theorem.

Proof of the Cayley-Hamilton Theorem. Let A be an $n \times n$-matrix, and let $p(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial. Let $\operatorname{adj}(A - \lambda I)$ be the adjugate of the matrix $A - \lambda I$ (see Section 7.6 for the definition of the adjugate). Since each of the entries of the adjugate is a cofactor of $A - \lambda I$, the entries are polynomials in λ of degree at most $n-1$. Therefore, the adjugate can be written in the form

$$\operatorname{adj}(A - \lambda I) = C_0 + C_1\lambda + \ldots + C_{n-1}\lambda^{n-1}.$$

By Theorem 7.29, we have

$$\det(A - \lambda I)\, I = (A - \lambda I)\, \operatorname{adj}(A - \lambda I),$$

or equivalently,

$$p(\lambda)\, I = (A - \lambda I)\,(C_0 + C_1\lambda + \ldots + C_{n-1}\lambda^{n-1}).$$

Since this equation holds for all λ, Corollary 8.47 may be used. Therefore, if λ is replaced with A, the two sides will be equal. Thus

$$p(A)I = (A - A)(C_0 + C_1 A + \ldots + C_{n-1}A^{n-1}) = 0.$$

It follows that $p(A) = 0$, concluding the proof of the Cayley-Hamilton Theorem. ♠

Exercises

Exercise 8.10.1 *Let*

$$A = \begin{bmatrix} 5 & 7 \\ -4 & 3 \end{bmatrix}.$$

Find the characteristic polynomial $p(\lambda)$*, and compute* $p(A)$*.*

Exercise 8.10.2 *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 4 \end{bmatrix}.$$

Find the characteristic polynomial $p(\lambda)$*, and compute* $p(A)$*.*

Exercise 8.10.3

(a) Let A be a 2×2*-matrix. Prove that* A^2 *is a linear combination of A and I.*

(b) Given an example of a 3×3*-matrix A such that* A^2 *is not a linear combination of A and I.*

8.11 Complex eigenvalues and eigenvectors

Outcomes

A. Find the complex eigenvalues and eigenvectors of a matrix.

B. Diagonalize a matrix over the complex numbers.

An $n \times n$-matrix is diagonalizable if and only if it has n linearly independent eigenvectors. But as we saw in Example 8.9, if we work over the real numbers, it can sometimes happen that a matrix has no eigenvalues, and therefore no eigenvectors, at all. For example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has characteristic polynomial λ^2+1. Since the equation λ^2+1 does not have any roots in the real numbers, there are no real eigenvalues.

On the other hand, the fundamental theorem of algebra tell us that over the *complex* numbers, every non-constant polynomial has a root. In fact, every polynomial of degree n factors into n linear factors. Therefore, every matrix has at least one eigenvalue over the complex numbers. Some matrices are diagonalizable over the complex numbers but not over the real numbers. An introduction to complex numbers and the fundamental theorem of algebra can be found in Appendix A.

Example 8.48: Complex eigenvalues and eigenvectors

Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

over the complex numbers. Diagonalize A if possible.

Solution. The characteristic polynomial is λ^2+1. This has no roots in the real numbers, but it has two roots $\lambda = i$ and $\lambda = -i$ in the complex numbers. To find the eigenvectors for $\lambda = i$, we solve $(A-iI)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\simeq} \left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right] \overset{R_2 \leftarrow R_2 + iR_1}{\simeq} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus, the basic eigenvector for $\lambda = i$ is

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Similarly, to find the eigenvectors for $\lambda = -i$, we solve $(A+iI)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\simeq} \left[\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right] \overset{R_2 \leftarrow R_2 - iR_1}{\simeq} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus, the basic eigenvector for $\lambda = -i$ is

$$\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Since we have found two linearly independent eigenvectors, the matrix A is diagonalizable. We have $A = PDP^{-1}$, where

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

♠

Example 8.49: Complex eigenvalues and eigenvectors

Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

over the complex numbers. Diagonalize A if possible.

Solution. The characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}=(1-\lambda)^2+1=\lambda^2-2\lambda+2.$$

To find the roots, we use the quadratic formula. The roots are given by:

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{-4}}{2}=1\pm i.$$

Note that since the discriminant b^2-4ac is negative, there are no real solutions. However, we find two complex solutions $\lambda=1+i$ and $\lambda=1-i$. To find the eigenvectors for $\lambda=1+i$, we solve the equation $(A-(1+i)I)\mathbf{v}=\mathbf{0}$:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \simeq \cdots \simeq \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The basic eigenvector for $\lambda=1+i$ is

$$\mathbf{v}_1=\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Similarly, the basic eigenvector for $\lambda=1-i$ is

$$\mathbf{v}_2=\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Since we have found two linearly independent eigenvectors, the matrix A is diagonalizable. We have $A=PDP^{-1}$, where

$$P=\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad\text{and}\quad D=\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

♠

Example 8.50: Diagonalize a matrix over the complex numbers

Diagonalize the matrix

$$A=\begin{bmatrix} -3 & -2 & 4 \\ 2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$p(\lambda)=(-3-\lambda)(1-\lambda)(3-\lambda)-16+8(1-\lambda)+4(3-\lambda)=-\lambda^3+\lambda^2-3\lambda-5.$$

By trial and error, we find that $\lambda=-1$ is one of the roots. We factor out $(\lambda+1)$:

$$p(\lambda)=(\lambda+1)(-\lambda^2+2\lambda-5).$$

We then use the quadratic formula to find the other two eigenvalues, i.e., the roots of $-\lambda^2+2\lambda-5$. They are:

$$\lambda=\frac{-2\pm\sqrt{-16}}{-2}=1\pm 2i.$$

Eigenvectors:

- For $\lambda = -1$, we solve $(A-(-1)I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} -2 & -2 & 4 & 0 \\ 2 & 2 & 0 & 0 \\ -2 & -2 & 4 & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

- For $\lambda = 1+2i$, we solve $(A-(1+2i)I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} -4-2i & -2 & 4 & 0 \\ 2 & -2i & 0 & 0 \\ -2 & -2 & 2-2i & 0 \end{array}\right] \overset{\substack{R_1 \leftarrow -R_1/2 \\ R_2 \leftarrow R_2/2 \\ R_3 \leftarrow -R_3/2}}{\simeq} \left[\begin{array}{ccc|c} 2+i & 1 & -2 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 1 & i-1 & 0 \end{array}\right]$$

$$\overset{R_1 \leftrightarrow R_2}{\simeq} \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 2+i & 1 & -2 & 0 \\ 1 & 1 & i-1 & 0 \end{array}\right]$$

$$\overset{\substack{R_2 \leftarrow R_2-(2+i)R_1 \\ R_3 \leftarrow R_3-R_1}}{\simeq} \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & 2i & -2 & 0 \\ 0 & 1+i & i-1 & 0 \end{array}\right]$$

$$\overset{\substack{R_2 \leftarrow R_2/2i \\ R_3 \leftarrow R_3/(1+i)}}{\simeq} \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 1 & i & 0 \end{array}\right]$$

$$\overset{\substack{R_1 \leftarrow R_1+iR_2 \\ R_3 \leftarrow R_3-R_2}}{\simeq} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}.$$

- For $\lambda = 1-2i$, we solve $(A-(1-2i)I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} -4+2i & -2 & 4 & 0 \\ 2 & 2i & 0 & 0 \\ -2 & -2 & 2+2i & 0 \end{array}\right] \simeq \ldots \simeq \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The basic eigenvector is

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}.$$

Therefore, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -i & i \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}.$$

♠

In Example 8.48, the complex eigenvalues were i and $-i$. In Example 8.49, the complex eigenvalues were $1+i$ and $1-i$. In Example 8.50, the complex eigenvalues were $1+2i$ and $1-2i$, and there was also a real eigenvalue of -1. Is it a coincidence that the complex eigenvalues always come in conjugate pairs? The following proposition states that this is always the case.

Proposition 8.51: Complex conjugate eigenvalues

Let A be a square matrix whose entries are real numbers. If λ is an eigenvalue of A, then so is $\overline{\lambda}$.

Proof. Assume λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then $A\mathbf{v} = \lambda\mathbf{v}$. Taking complex conjugates of both sides of the equation, we have $\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}}$, and therefore $\overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. Since A is matrix with real entries, we have $\overline{A} = A$, and therefore $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. It follows that $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{v}}$. ♠

It is important to note that even over the complex numbers, not all matrices are diagonalizable. On the one hand, the characteristic polynomial of an $n \times n$-matrix always factors into n linear factors over the complex numbers. Therefore, the sum of the algebraic multiplicities of the eigenvalues is always n. However, it can still happen that the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity. In that case, the matrix is not diagonalizable, even over the complex numbers. We have:

Proposition 8.52: Diagonalizability criterion

A square matrix A is diagonalizable over the complex numbers if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

Proof. Let A be an $n \times n$-matrix. By the fundamental theorem of algebra, the characteristic polynomial factors into n linear factors. Therefore, the sum of the algebraic multiplicities of all the eigenvalues is n. We know by Proposition 8.42 that the geometric multiplicity of each eigenvalue less than or equal to its algebraic multiplicity. If the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, then the sum of the geometric multiplicities is n, and therefore A is diagonalizable by Proposition 8.43. On the other hand, if the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity, then the sum of the geometric multiplicities is less than n, and A is not diagonalizable. ♠

Example 8.53: Non-diagonalizable matrix

Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ cannot be diagonalized, even over the complex numbers.

Solution. The characteristic polynomial is $(1-\lambda)^2$, and therefore the only eigenvalue is $\lambda = 1$, with algebraic multiplicity 2. On the other hand, the eigenspace for $\lambda = 1$ is 1-dimensional:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a 1-dimensional solution space. Therefore, we can find only one basic eigenvector, and the matrix is not diagonalizable. ♠

To finish up this chapter, we will consider an application of complex eigenvalues. We will solve a recurrence as in Section 8.6. But this time, although the recurrence relation only uses real numbers, complex numbers will be required to solve it.

Example 8.54: Solving a recurrence using complex eigenvalues

Consider the sequence of numbers defined by the recurrence

$$\begin{aligned} f_0 &= 1, \\ f_1 &= 3, \\ f_{n+2} &= 2f_{n+1} - 2f_n, \quad \text{for all } n \geq 0. \end{aligned}$$

Solve the recurrence, i.e., find a closed formula for f_n.

Solution. The first few members of the sequence are:

$$1, 3, 4, 2, -4, -12, -16, -8, 16, \ldots$$

To solve the recurrence, let

$$\mathbf{v}_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix},$$

so that for all $n \geq 0$,

$$\mathbf{v}_{n+1} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ 2f_{n+1} - 2f_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \mathbf{v}_n.$$

We then diagonalize the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 2\lambda + 2.$$

The eigenvalues are the roots of the characteristic polynomial. We compute them using the quadratic formula:

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

To find the eigenvectors for $\lambda_1 = 1+i$, we solve the equation $(A - (1+i)I)\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{array}\right] \overset{R_1 \leftarrow (1-i)R_1}{\simeq} \left[\begin{array}{cc|c} -2 & 1-i & 0 \\ -2 & 1-i & 0 \end{array}\right] \simeq \left[\begin{array}{cc|c} -2 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basic eigenvector for $\lambda = 1+i$ is

$$\mathbf{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

Similarly, a basic eigenvector for $\lambda = 1-i$ is

$$\mathbf{u} = \left[\begin{array}{c} 1+i \\ 2 \end{array}\right].$$

Since we have found two linearly independent eigenvectors, the matrix A is diagonalizable. We have $A = PDP^{-1}$, where

$$P = \left[\begin{array}{cc} 1-i & 1+i \\ 2 & 2 \end{array}\right] \quad \text{and} \quad D = \left[\begin{array}{cc} 1+i & 0 \\ 0 & 1-i \end{array}\right].$$

The inverse of P is

$$P^{-1} = \frac{1}{4}\left[\begin{array}{cc} 2i & 1-i \\ -2i & 1+i \end{array}\right].$$

Finally, we use this information to solve the recurrence:

$$\begin{aligned}
f_n &= \left[\begin{array}{cc} 1 & 0 \end{array}\right]\left[\begin{array}{c} f_n \\ f_{n+1} \end{array}\right] \\
&= \left[\begin{array}{cc} 1 & 0 \end{array}\right]\mathbf{v}_n \\
&= \left[\begin{array}{cc} 1 & 0 \end{array}\right]A^n\,\mathbf{v}_0 \\
&= \left[\begin{array}{cc} 1 & 0 \end{array}\right]PD^nP^{-1}\,\mathbf{v}_0 \\
&= \left[\begin{array}{cc} 1 & 0 \end{array}\right]\left[\begin{array}{cc} 1-i & 1+i \\ 2 & 2 \end{array}\right]\left[\begin{array}{cc} (1+i)^n & 0 \\ 0 & (1-i)^n \end{array}\right]\frac{1}{4}\left[\begin{array}{cc} 2i & 1-i \\ -2i & 1+i \end{array}\right]\left[\begin{array}{c} 1 \\ 3 \end{array}\right] \\
&= \frac{1}{4}\left[\begin{array}{cc} 1-i & 1+i \end{array}\right]\left[\begin{array}{cc} (1+i)^n & 0 \\ 0 & (1-i)^n \end{array}\right]\left[\begin{array}{c} 3-i \\ 3+i \end{array}\right] \\
&= \frac{1}{4}((1-i)(1+i)^n(3-i)+(1+i)(1-i)^n(3+i)) \\
&= \frac{1}{4}((2-4i)(1+i)^n+(2+4i)(1-i)^n) \\
&= \frac{1}{2}((1-2i)(1+i)^n+(1+2i)(1-i)^n).
\end{aligned}$$

We can use this, for example, to calculate the 8^{th} element of the sequence:

$$f_8 = \frac{1}{2}\left((1-2i)(1+i)^8+(1+2i)(1-i)^8\right) = \frac{1}{2}((1-2i)16+(1+2i)16) = \frac{32}{2} = 16.$$

♠

Exercises

Exercise 8.11.1 *Find the (real or complex) eigenvalues and eigenvectors of the following matrices. Diagonalize each matrix if possible.*

$$A = \left[\begin{array}{cc} 2 & 1 \\ -1 & 2 \end{array}\right], \quad B = \left[\begin{array}{cc} -1 & -2 \\ 1 & 1 \end{array}\right], \quad C = \left[\begin{array}{ccc} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{array}\right], \quad D = \left[\begin{array}{ccc} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{array}\right]$$

Exercise 8.11.2 *I know a certain real* 2×2*-matrix* A*. My matrix has complex eigenvalue* $\lambda = 1 + 2i$ *and corresponding eigenvector* $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

(a) Find another eigenvalue and corresponding eigenvector of A*.*

(b) Diagonalize A*.*

(c) What is the secret matrix A*?*

Exercise 8.11.3 *Solve the recurrence*

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_{n+2} &= 4a_{n+1} - 5a_n, \quad \textit{for all } n \geq 0. \end{aligned}$$

9. Vector spaces

In Chapter 2, we considered $\mathbb{R}^n$, the set of n-dimensional column vectors. We now introduce a more general concept of "vector", as an element of an abstract vector space. Basically, vectors are entities that can be added and scaled. While some vectors look like lists of numbers (for example, column vectors, row vectors), other kinds of vectors don't look like lists of numbers at all (for example, functions, polynomials). Part of the power of linear algebra comes from our ability to find vector spaces in many unexpected places.

Much of the content of this chapter will be a repetition of things we have already seen in Chapter 2 in the context of $\mathbb{R}^n$. For example, we will be talking about linear combinations, linear independence, spanning sets, bases, subspaces, linear transformations, and so on. We initially introduced these concepts in the context of the vector space $\mathbb{R}^n$, so that they would be easier to understand. We will now see that they in fact apply to *all* vector spaces.

9.1 Definition of vector spaces

Outcomes

A. *Develop the concept of a vector space through axioms.*

B. *Use the vector space axioms to determine if a set and its operations constitute a vector space.*

C. *Encounter several examples of vector spaces.*

Definition 9.1: Vector space

*Let K be a field. A **vector space** over K is a set V equipped with two operations of **addition** and **scalar multiplication**, such that the following properties hold:*

(A1) *Commutative law of addition:* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

(A2) *Associative law of addition:* $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(A3) *The existence of an additive unit: there exists an element* $\mathbf{0}\in V$ *such that for all* $\mathbf{u}$, $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

(A4) *The law of additive inverses:* $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(SM1) *The distributive law over vector addition:* $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

(SM2) *The distributive law over scalar addition:* $(k+\ell)\mathbf{u}=k\mathbf{u}+\ell\mathbf{u}$.

(SM3) *The associative law for scalar multiplication:* $k(\ell\mathbf{u})=(k\ell)\mathbf{u}$.

(SM4) *The rule for multiplication by one:* $1\mathbf{u}=\mathbf{u}$.

The above definition is concerned about two operations: vector addition, denoted by $\mathbf{v}+\mathbf{w}$, and scalar multiplication, denoted by $k\mathbf{v}$ or sometimes $k\cdot\mathbf{v}$. In the law of additive inverses, we have written $-\mathbf{u}$ for $(-1)\mathbf{u}$. Often, the scalars will be real numbers, but it is also possible to use scalars from a different field K. We also use the term **K-vector space** to refer to a vector space over a field K. When $K=\mathbb{R}$, we also speak of a **real vector space**, and when $K=\mathbb{C}$, we speak of a **complex vector space**. If the field is clear from the context, we often don't mention it at all, and just speak of a "vector space". The elements of a vector space are called **vectors**.

Our first example of a vector space is of course $\mathbb{R}^n$.

Example 9.2: $\mathbb{R}^n$ is a vector space

The set $\mathbb{R}^n$ of n-dimensional real column vectors, with the usual operations of vector addition and scalar multiplication, is a vector space.
More generally, if K is a field, the set K^n of n-dimensional column vectors with components in K is a K-vector space.

Proof. Properties (A1)–(A4) hold by Proposition 2.8, and properties (SM1)–(SM4) hold by Proposition 2.11. ♠

We now consider some other examples of vector spaces.

Example 9.3: Vector space of polynomials of degree 2

Let $\mathbf{P}_2$ be the set of all polynomials of degree at most 2 with coefficients from a field K, i.e., expressions of the form

$$p(x)=ax^2+bx+c,$$

where $a,b,c\in K$. Define addition and scalar multiplication of polynomials in the usual way, i.e.,

$$\begin{array}{rcl}(ax^2+bx+c)+(a'x^2+b'x+c') &=& (a+a')x^2+(b+b')x+(c+c')\\ k(ax^2+bx+c) &=& kax^2+kbx+kc.\end{array}$$

Then $\mathbf{P}_2$ is a vector space.

Proof. To show that $\mathbf{P}_2$ is a vector space, we verify the 8 vector space axioms. Let

$$\begin{array}{rcl}p(x) &=& a_2x^2+a_1x+a_0,\\ q(x) &=& b_2x^2+b_1x+b_0,\\ r(x) &=& c_2x^2+c_1x+c_0\end{array}$$

be polynomials in $\mathbf{P}_2$ and let k,ℓ be scalars.

(A1) We prove the commutative law of addition.

$$\begin{array}{rcl}p(x)+q(x) &=& (a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0)\\ &=& (a_2+b_2)x^2+(a_1+b_1)x+(a_0+b_0)\end{array}$$

$$\begin{aligned} &= (b_2+a_2)x^2+(b_1+a_1)x+(b_0+a_0) \\ &= (b_2x^2+b_1x+b_0)+(a_2x^2+a_1x+a_0) \\ &= q(x)+p(x). \end{aligned}$$

(A2) We prove the associative law of addition.

$$\begin{aligned} (p(x)+q(x))+r(x) &= ((a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0))+(c_2x^2+c_1x+c_0) \\ &= ((a_2+b_2)x^2+(a_1+b_1)x+(a_0+b_0))+(c_2x^2+c_1x+c_0) \\ &= ((a_2+b_2)+c_2)x^2+((a_1+b_1)+c_1)x+((a_0+b_0)+c_0) \\ &= (a_2+(b_2+c_2))x^2+(a_1+(b_1+c_1))x+(a_0+(b_0+c_0)) \\ &= (a_2x^2+a_1x+a_0)+((b_2+c_2)x^2+(b_1+c_1)x+(b_0+c_0)) \\ &= (a_2x^2+a_1x+a_0)+((b_2x^2+b_1x+b_0)+(c_2x^2+c_1x+c_0)) \\ &= p(x)+(q(x)+r(x)). \end{aligned}$$

(A3) To prove the existence of an additive unit, let $0(x)=0x^2+0x+0$, the so-called **zero polynomial**. Then

$$\begin{aligned} p(x)+0(x) &= (a_2x^2+a_1x+a_0)+(0x^2+0x+0) \\ &= (a_2+0)x^2+(a_1+0)x+(a_0+0) \\ &= a_2x^2+a_1x+a_0 \\ &= p(x). \end{aligned}$$

(A4) We prove the law of additive inverses.

$$\begin{aligned} p(x)+(-p(x)) &= (a_2x^2+a_1x+a_0)+(-a_2x^2-a_1x-a_0) \\ &= (a_2-a_2)x^2+(a_1-a_1)x+(a_0-a_0) \\ &= 0x^2+0x+0 \\ &= 0(x). \end{aligned}$$

(SM1) We prove the distributive law over vector addition.

$$\begin{aligned} k(p(x)+q(x)) &= k((a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0)) \\ &= k((a_2+b_2)x^2+(a_1+b_1)x+(a_0+b_0)) \\ &= k(a_2+b_2)x^2+k(a_1+b_1)x+k(a_0+b_0) \\ &= (ka_2+kb_2)x^2+(ka_1+kb_1)x+(ka_0+kb_0) \\ &= (ka_2x^2+ka_1x+ka_0)+(kb_2x^2+kb_1x+kb_0) \\ &= kp(x)+kq(x). \end{aligned}$$

(SM2) We prove the distributive law over scalar addition.

$$\begin{aligned} (k+\ell)p(x) &= (k+\ell)(a_2x^2+a_1x+a_0) \\ &= (k+\ell)a_2x^2+(k+\ell)a_1x+(k+\ell)a_0 \\ &= (ka_2x^2+ka_1x+ka_0)+(\ell a_2x^2+\ell a_1x+\ell a_0) \\ &= kp(x)+\ell p(x). \end{aligned}$$

(SM3) We prove the associative law for scalar multiplication.

$$\begin{aligned} k(\ell p(x)) &= k(\ell(a_2x^2+a_1x+a_0)) \\ &= k(\ell a_2x^2+\ell a_1x+\ell a_0) \\ &= k\ell a_2x^2+k\ell a_1x+k\ell a_0 \\ &= (k\ell)(a_2x^2+a_1x+a_0) \\ &= (k\ell)p(x). \end{aligned}$$

(SM4) Finally, we prove the rule for multiplication by one.

$$\begin{aligned} 1p(x) &= 1(a_2x^2+a_1x+a_0) \\ &= 1a_2x^2+1a_1x+1a_0 \\ &= a_2x^2+a_1x+a_0 \\ &= p(x). \end{aligned}$$

Since the operations of addition and scalar multiplication on $\mathbf{P}_2$ satisfy the 8 vector space axioms, $\mathbf{P}_2$ is a vector space. ♠

Our next example of a vector space is the set of all $m \times n$-matrices.

Example 9.4: Vector space of matrices

Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$-matrices with entries in a field K, together with the usual operations of matrix addition and scalar multiplication. Then $\mathbf{M}_{m,n}$ is a vector space.

Proof. The properties (A1)–(A4) hold by Proposition 4.11, and the properties (SM1)–(SM4) hold by Proposition 4.14. ♠

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore *not* a vector space.

Example 9.5: Not a vector space

Let V denote the set of 2×3-matrices. Let us define a non-standard addition in V by $A \oplus B = A$ for all matrices $A, B \in V$. Let scalar multiplication in V be the usual scalar multiplication of matrices. Show that V is not a vector space.

Solution. In order to show that V is not a vector space, it suffices to find one of the 8 axioms that is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. In fact, for this example, the very first axiom fails. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $A \oplus B = A$ and $B \oplus A = B$. Since $A \neq B$, we have $A \oplus B \neq B \oplus A$ for these two matrices, so property (A1) is false. ♠

Our next example looks a little different.

Example 9.6: Vector space of functions

Let X be a nonempty set, K a field, and define $\mathbf{Func}_{X,K}$ to be the set of functions defined on X and valued in K. In other words, the elements of $\mathbf{Func}_{X,K}$ are functions $f : X \to K$. The sum of two functions is defined by

$$(f+g)(x) = f(x)+g(x),$$

and the scalar multiplication is defined by

$$(kf)(x) = k(f(x)).$$

Then $\mathbf{Func}_{X,K}$ is a vector space.

Proof. To verify that $\mathbf{Func}_{X,K}$ is a vector space, we must prove the 8 axioms of vector spaces. Let f,g,h be functions in $\mathbf{Func}_{X,K}$, and let k,ℓ be scalars. Recall that two functions f,g are **equal** if for all $x \in X$, we have $f(x) = g(x)$.

(A1) We prove the commutative law of addition. For all $x \in X$, we have

$$(f+g)(x) \;=\; f(x)+g(x) \;=\; g(x)+f(x) \;=\; (g+f)(x).$$

Therefore, $f+g = g+f$.

(A2) We prove the associative law of addition. For all $x \in X$, we have

$$\begin{aligned} ((f+g)+h)(x) \;&=\; (f+g)(x)+h(x) \;=\; (f(x)+g(x))+h(x) \\ &=\; f(x)+(g(x)+h(x)) \;=\; (f(x)+(g+h)(x)) \;=\; (f+(g+h))(x). \end{aligned}$$

Therefore, $(f+g)+h = f+(g+h)$.

(A3) To prove the existence of an additive unit, let 0 denote the function that is given by $0(x) = 0$. This is called the **zero function**. It is an additive unit because for all x,

$$(f+0)(x) \;=\; f(x)+0(x) \;=\; f(x),$$

and so $f+0 = f$.

(A4) We prove the law of additive inverses. Let $-f = (-1)f$ be the function that satisfies $(-f)(x) = -f(x)$. Then for all x,

$$(f+(-f))(x) \;=\; f(x)+(-f)(x) \;=\; f(x)+-f(x) \;=\; 0.$$

Therefore $f+(-f) = 0$.

(SM1) We prove the distributive law over vector addition. For all x, we have

$$\begin{aligned} (k(f+g))(x) \;&=\; k(f+g)(x) \;=\; k(f(x)+g(x)) \\ &=\; kf(x)+kg(x) \;=\; (kf+kg)(x), \end{aligned}$$

and so $k(f+g) = kf+kg$.

(SM2) We prove the distributive law over scalar addition.

$$((k+\ell)f)(x) \;=\; (k+\ell)f(x) \;=\; kf(x)+\ell f(x) \;=\; (kf+\ell f)(x),$$

and so $(k+\ell)f = kf+\ell f$.

(SM3) We prove the associative law for scalar multiplication.

$$((k\ell)f)(x) \;=\; (k\ell)f(x) \;=\; k(\ell f(x)) \;=\; (k(\ell f))(x),$$

so $(k\ell f) = k(\ell f)$.

(SM4) Finally, we prove the rule for multiplication by one. For all $x \in X$, we have

$$(1f)(x) \;=\; 1f(x) \;=\; f(x),$$

and therefore $1f = f$.

It follows that $\mathbf{Func}_{X,K}$ satisfies all the required axioms and is a vector space. ♠

For the next two examples of vector spaces, we leave the proofs as an exercise.

Example 9.7: Infinite sequences

Let K be a field. A ***sequence*** *of elements of K is an infinite list*

$$(a_0, a_1, a_2, a_3 \ldots),$$

where $a_i \in K$ for all i. We also use the notation $(a_i)_{i\in\mathbb{N}}$, or occasionally (a_i), to denote such a sequence. Let $\mathbf{Seq}_K$ be the set of sequences of elements of K. We add two sequences by adding their i^{th} elements:

$$(a_i)_{i\in\mathbb{N}} + (b_i)_{i\in\mathbb{N}} = (a_i+b_i)_{i\in\mathbb{N}}.$$

We scale a sequence by scaling each of its elements:

$$k(a_i)_{i\in\mathbb{N}} = (ka_i)_{i\in\mathbb{N}}.$$

Then $\mathbf{Seq}_K$ is a vector space.

Example 9.8: Vector space of polynomials of unbounded degree

Let K be a field, and let $\mathbf{P}$ be the set of all polynomials (of any degree) with coefficients from K, i.e., expressions of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $n \geq 0$ and $a_0, \ldots, a_n \in K$. Addition and scalar multiplication of polynomials are defined in the usual way . Then $\mathbf{P}$ is a vector space.

We conclude this section by deriving some initial consequences of the vector space axioms.

Proposition 9.9: Elementary consequences of the vector space axioms

In any vector space, the following are true:

(a) *The additive unit is unique. In other words, whenever* $\mathbf{u}+\mathbf{v}=\mathbf{u}$*, then* $\mathbf{v}=\mathbf{0}$.

(b) *Additive inverses are unique. In other words, whenever* $\mathbf{u}+\mathbf{v}=\mathbf{0}$*, then* $\mathbf{v}=-\mathbf{u}$.

(c) $0\mathbf{u}=\mathbf{0}$ *for all vectors* $\mathbf{u}$.

(d) *The following* ***cancellation law*** *holds: if* $\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$*, then* $\mathbf{u}=\mathbf{v}$.

Proof. We prove the first three properties, and leave the last one as an exercise. Assume V is any vector space over a field K.

(a) Consider arbitrary vectors $\mathbf{u},\mathbf{v}\in V$ and assume

$$\mathbf{u}+\mathbf{v}=\mathbf{u}.$$

Applying the law (A1) (commutative law) to the left-hand side, we have

$$\mathbf{v}+\mathbf{u}=\mathbf{u}.$$

Adding $-\mathbf{u}$ to both sides of the equation, we have

$$(\mathbf{v}+\mathbf{u})+(-\mathbf{u})=\mathbf{u}+(-\mathbf{u}).$$

Applying the law (A2) (associative law) to the left-hand side, we have

$$\mathbf{v}+(\mathbf{u}+(-\mathbf{u}))=\mathbf{u}+(-\mathbf{u}).$$

Applying the law (A4) (additive inverse law) to both sides of the equation, we have

$$\mathbf{v}+\mathbf{0}=\mathbf{0}.$$

Applying the law (A3) (additive unit law) to the left-hand side, we have

$$\mathbf{v}=\mathbf{0}.$$

This proves that whenever $\mathbf{u}+\mathbf{v}=\mathbf{u}$, then $\mathbf{v}=\mathbf{0}$, or in other words, $\mathbf{v}=\mathbf{0}$ is the only element acting as an additive unit.

(b) Consider arbitrary vectors $\mathbf{u},\mathbf{v}\in V$ and assume

$$\mathbf{u}+\mathbf{v}=\mathbf{0}.$$

Applying the law (A1) (commutative law) to the left-hand side, we have

$$\mathbf{v}+\mathbf{u}=\mathbf{0}.$$

Adding $-\mathbf{u}$ to both sides of the equation, we have

$$(\mathbf{v}+\mathbf{u})+(-\mathbf{u}) = \mathbf{0}+(-\mathbf{u}).$$

Applying the law (A2) (associative law) to the left-hand side, we have

$$\mathbf{v}+(\mathbf{u}+(-\mathbf{u})) = \mathbf{0}+(-\mathbf{u}).$$

Applying the law (A4) (additive inverse law) to the left-hand side, we have

$$\mathbf{v}+\mathbf{0} = \mathbf{0}+(-\mathbf{u}).$$

Applying the law (A1) (commutative law) to the right-hand side, we have

$$\mathbf{v}+\mathbf{0} = -\mathbf{u}+\mathbf{0}.$$

Applying the law (A3) (additive unit law) to both sides of the equation, we have

$$\mathbf{v} = -\mathbf{u}.$$

This proves that whenever $\mathbf{u}+\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = -\mathbf{u}$, or in other words, $\mathbf{v} = -\mathbf{u}$ is the only element acting as an additive inverse of $\mathbf{u}$.

(c) First, note that the scalar $0 \in K$ satisfies the property $0+0 = 0$, by property (A3) of the definition of a field. Now let $\mathbf{u} \in V$ be any vector. Using the vector space law (SM2) (distributive law over scalar addition) and $0+0 = 0$, we have

$$0\mathbf{u}+0\mathbf{u} = (0+0)\mathbf{u} = 0\mathbf{u}.$$

Next, we use a small trick: add $-(0\mathbf{u})$ to both sides of the equation. This gives

$$(0\mathbf{u}+0\mathbf{u})+(-(0\mathbf{u})) \quad = \quad 0\mathbf{u}+(-(0\mathbf{u})).$$

Applying the additional laws (A2), (A4), and (A3), we have

$$\begin{aligned} 0\mathbf{u}+(0\mathbf{u}+(-(0\mathbf{u}))) &= 0\mathbf{u}+(-(0\mathbf{u})), \\ 0\mathbf{u}+\mathbf{0} &= \mathbf{0}, \\ 0\mathbf{u} &= \mathbf{0}. \end{aligned}$$

This proves that $0\mathbf{u} = \mathbf{0}$ holds for all vectors $\mathbf{u}$, as desired.

(d) This is left as an exercise.

♠

Exercises

Exercise 9.1.1 *Consider the set $\mathbb{R}^2$ with the following non-standard addition operation $\oplus$:*

$$(a,b) \oplus (c,d) = (a+d, b+c).$$

Scalar multiplication is defined in the usual way. Is this a vector space? Explain why or why not.

Exercise 9.1.2 *Consider $\mathbb{R}^2$ with the following non-standard addition operation $\oplus$:*

$$(a,b) \oplus (c,d) = (0, b+d).$$

Scalar multiplication is defined in the usual way. Is this a vector space? Explain why or why not.

Exercise 9.1.3 *Consider $\mathbb{R}^2$ with the following non-standard scalar multiplication:*

$$c \odot (a,b) = (a, cb).$$

Vector addition is defined as usual. Is this a vector space? Explain why or why not.

Exercise 9.1.4 *Consider $\mathbb{R}^2$ with the following non-standard addition operation $\oplus$:*

$$(a,b) \oplus (c,d) = (a-c, b-d).$$

Scalar multiplication is defined as usual. Is this a vector space? Explain why or why not.

Exercise 9.1.5 *Prove that the set $\mathbf{Seq}_K$ from Example 9.7 is a vector space. Hint: this is a special case of Example 9.6, if you realize that a sequence $(a_i)_{i\in\mathbb{N}}$ is the same thing as a function $a : \mathbb{N} \to K$.*

Exercise 9.1.6 *Prove that the set **P** from Example 9.8 is a vector space.*

Exercise 9.1.7 *Let V be the set of functions defined on a set X that have values in a vector space W. Is this a vector space? Explain.*

Exercise 9.1.8 *Consider the set $\mathbb{R}^2$ with the following non-standard operations of addition and scalar multiplication:*

$$\begin{aligned} (a,b) \oplus (c,d) &= (a+c-1, b+d-1), \\ k \odot (c,d) &= (kc+(1-k), kd+(1-k)). \end{aligned}$$

Show that $\mathbb{R}^2$ is a vector space with these operations. Hint: the zero vector is not $(0,0)$, but $(1,1)$.

Exercise 9.1.9 *Consider the set $\mathbb{R}$ of real numbers. Addition of real numbers is defined in the usual way, and scalar multiplication is just multiplication of one real number by another. In other words, $x+y$ means to add the two numbers and xy means to multiply them. Show that $\mathbb{R}$, with these operations, is a real vector space.*

Exercise 9.1.10 *Let $K = \mathbb{Q}$ be the field of rational numbers, and let V be the set of real numbers of the form $a+b\sqrt{2}$, where a and b are rational numbers. Show that with the usual operations, V is a $\mathbb{Q}$-vector space.*

Exercise 9.1.11 *Let $K = \mathbb{Q}$ be the field of rational numbers, and let $V = \mathbb{R}$ be the set of real numbers. Show that with the usual operations, V is a $\mathbb{Q}$-vector space.*

Exercise 9.1.12 *Let $\mathbf{P}_3$ be the set of all polynomials of degree 3 or less. That is, these are of the form ax^3+bx^2+cx+d. Addition and scalar multiplication of polynomials are defined as usual. Show that $\mathbf{P}_3$ is a vector space.*

Exercise 9.1.13 *Let $X=\{1,2,\ldots,n\}$, and consider the space $\mathbf{Func}_{X,\mathbb{R}}$ of real-valued functions defined on X. Explain how $\mathbf{Func}_{X,\mathbb{R}}$ can be considered as $\mathbb{R}^n$.*

Exercise 9.1.14 *Prove the cancellation law from Proposition 9.9.*

9.2 Linear combinations, span, and linear independence

Outcomes

A. *Determine if a vector is within a given span.*

B. *Determine if a set is spanning.*

C. *Determine if a set is linearly independent.*

In this section, we will again explore concepts introduced earlier in terms of $\mathbb{R}^n$ and extend them to apply to abstract vector spaces.

We can now revisit many of the concepts first introduced in Chapter 2 in the context of general vector spaces. We will look at linear combinations, span, and linear independence in this section, and at subspaces, bases, and dimension in the next section.

Definition 9.10: Linear combination

*Let V be a vector space over a field K. Let $\mathbf{u}_1,\ldots,\mathbf{u}_n\in V$. A vector $\mathbf{v}\in V$ is called a **linear combination** of $\mathbf{u}_1,\ldots,\mathbf{u}_n$ if there exist scalars $a_1,\ldots,a_n\in K$ such that*

$$\mathbf{v}=a_1\mathbf{u}_1+\ldots+a_n\mathbf{u}_n.$$

Example 9.11: Linear combination of matrices

Write the matrix $A=\begin{bmatrix}1&3\\-1&2\end{bmatrix}$ as a linear combination of

$$\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad\text{and}\quad\begin{bmatrix}0&-1\\1&0\end{bmatrix}.$$

Solution. We must find coefficients a, b, c, d such that

$$\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a+b & c-d \\ c+d & a-b \end{bmatrix}.$$

This yields a system of four equations in four variables:

$$\begin{aligned} a+b &= 1, \\ c+d &= -1, \\ c-d &= 3, \\ a-b &= 2. \end{aligned}$$

We can easily solve the system of equations to find the unique solution $a = \frac{3}{2}$, $b = -\frac{1}{2}$, $c = 1$, $d = -2$. Therefore

$$\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

♠

Example 9.12: Linear combination of polynomials

Write the polynomial $p(x) = 7x^2 + 4x - 3$ *as a linear combination of*

$$q_1(x) = x^2, \quad q_2(x) = (x+1)^2, \quad \text{and} \quad q_3(x) = (x+2)^2.$$

Solution. Note that $q_2(x) = (x+1)^2 = x^2 + 2x + 1$ and $q_3(x) = (x+2)^2 = x^2 + 4x + 4$. We must find coefficients a, b, c such that $p(x) = aq_1(x) + bq_2(x) + cq_3(x)$, or equivalently,

$$7x^2 + 4x - 3 = ax^2 + b(x^2 + 2x + 1) + c(x^2 + 4x + 4).$$

Collecting equal powers of x, we can rewrite this as

$$7x^2 + 4x - 3 = (a+b+c)x^2 + (2b+4c)x + (b+4c).$$

Since two polynomials are equal if and only if each corresponding coefficient is equal, this yields a system of three equations in three variables

$$\begin{aligned} a+b+c &= 7, \\ 2b+4c &= 4, \\ b+4c &= -3. \end{aligned}$$

We can easily solve this system of equations and find that the unique solution is $a = \frac{5}{2}$, $b = 7$, $c = -\frac{5}{2}$. Therefore

$$p(x) = \frac{5}{2}q_1(x) + 7q_2(x) - \frac{5}{2}q_3(x).$$

♠

As in Chapter 2, the span of a set of vectors is defined as the set of all of its linear combinations. We generalize the concept of span to consider spans of arbitrary (possibly finite, possibly infinite) sets of vectors.

Definition 9.13: Span of a set of vectors

*Let V be a vector space over some field K, and let S be a set of vectors (i.e., a subset of V). The **span** of S is the set of all linear combinations of elements of S. In symbols, we have*

$$\operatorname{span} S \;=\; \{a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \mid \mathbf{u}_1,\ldots,\mathbf{u}_k \in S \text{ and } a_1,\ldots,a_k \in K\}.$$

It is important not to misunderstand this definition. Even when the set S is infinite, each *individual* element $\mathbf{v} \in \operatorname{span} S$ is a linear combination of only *finitely many* elements $\mathbf{u}_1,\ldots,\mathbf{u}_k$ of S. The definition does not talk about infinite linear combinations

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \ldots$$

Indeed, such infinite sums do not typically exist. However, different elements $\mathbf{v},\mathbf{w} \in \operatorname{span} S$ can be linear combinations of a different (finite) number of vectors of S. For example, it is possible that $\mathbf{v}$ is a linear combination of 10 elements of S, and $\mathbf{w}$ is a linear combination of 100 elements of S.

Example 9.14: Spans of sequences

Consider the vector space $\mathbf{Seq}_K$ of infinite sequences. For every $k \in \mathbb{N}$, let e^k be the sequence whose k^{th} element is 1 and that is 0 everywhere else, i.e.,

$$\begin{aligned} e^0 &= (1,0,0,0,0,\ldots),\\ e^1 &= (0,1,0,0,0,\ldots),\\ e^2 &= (0,0,1,0,0,\ldots), \end{aligned}$$

and so on. Let $S = \{e^k \mid k \in \mathbb{N}\}$. Which of the following sequences are in $\operatorname{span} S$?

(a) $f = (1,1,1,0,0,0,0,0,\ldots)$ (followed by infinitely many zeros),

(b) $g = (1,2,0,5,0,0,0,0,\ldots)$ (followed by infinitely many zeros),

(c) $h = (1,1,1,1,1,1,1,1,\ldots)$ (followed by infinitely many ones),

(d) $k = (1,0,1,0,1,0,1,0,\ldots)$ (forever alternating between 1 and 0).

Solution.

(a) We have $f \in \operatorname{span} S$, because $f = e^0 + e^1 + e^2$.

(b) We have $g \in \operatorname{span} S$, because $g = 1e^0 + 2e^1 + 5e^3$.

(c) The sequence h is not in $\operatorname{span} S$, because each element of $\operatorname{span} S$ is, by definition, a linear combinations of *finitely many* elements of S. No linear combinations of finitely many e^k can end in infinitely many ones. Note that we are not permitted to write an infinite sum such as $e^0 + e^1 + e^2 + \ldots$. Such infinite sums are not defined in vector spaces.

(d) The sequence k is not in $\operatorname{span} S$, for the same reason. We would need to add infinitely many sequences of the form e^k to get a sequence that contains infinitely many non-zero elements. However, this is not permitted by the definition of span. ♠

Example 9.15: Span of polynomials

Let $p(x) = 7x^2 + 4x - 3$. Is $p(x) \in \operatorname{span}\left\{x^2,\ (x+1)^2,\ (x+2)^2\right\}$?

Solution. The answer is yes, because we found in Example 9.12 that $p(x) = \frac{5}{2}x^2 + 7(x+1)^2 - \frac{5}{2}(x+2)^2$. ♠

We say that a set of vectors S is a **spanning set** for V if $V = \operatorname{span} S$.

Example 9.16: Spanning set

Let $S = \left\{x^2,\ (x+1)^2,\ (x+2)^2\right\}$. Show that S is a spanning set for $\mathbf{P}_2$, the vector space of all polynomials of degree at most 2.

Solution. This is analogous to Example 9.12. Consider an arbitrary element $p(x) = p_2x^2 + p_1x + p_0$ of $\mathbf{P}_2$. We must show that $p(x) \in \operatorname{span} S$, i.e., that there exists $a, b, c \in K$ such that

$$p(x) = ax^2 + b(x+1)^2 + c(x+2)^2.$$

We can equivalently rewrite this equation as

$$p_2x^2 + p_1x + p_0 \ = \ (a+b+c)x^2 \ + \ (2b+4c)x \ + \ (b+4c),$$

which yields the system of equations

$$\begin{aligned} a+b+c &= p_2 \\ 2b+4c &= p_1 \\ b+4c &= p_0 \end{aligned} \quad \simeq \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & p_2 \\ 0 & 2 & 4 & p_1 \\ 0 & 1 & 4 & p_0 \end{array}\right] \quad \simeq \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & p_2 \\ 0 & 2 & 4 & p_1 \\ 0 & 0 & 4 & 2p_0 - p_1 \end{array}\right].$$

Since the system has rank 3, it has a solution. Therefore, $p(x) \in \operatorname{span} S$. Since $p(x)$ was an arbitrary element of $\mathbf{P}_2$, it follows that S is a spanning set for $\mathbf{P}_2$. ♠

To define the concept of linear independence in a general vector space, it will be convenient to base our definition on the "alternative" characterization of Theorem 5.12. Here too, we generalize the definition to an arbitrary (finite or infinite) set of vectors.

Definition 9.17: Linear independence

Let V be a vector space over some field K. A finite set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is called ***linearly independent*** *if the equation*

$$a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k = \mathbf{0}$$

has only the trivial solution $a_1, \ldots, a_k = 0$. An infinite set S of vectors is called linearly independent if every finite subset of S is linearly independent. A set of vectors is called ***linearly dependent*** *if it is not linearly independent.*

Example 9.18: Linearly independent polynomials

Determine whether the polynomials x^2, x^2+2x-1, and $2x^2-x+3$ are linearly independent.

Solution. According to the definition of linear independence, we must solve the equation

$$ax^2 + b(x^2+2x-1) + c(2x^2-x+3) \;=\; 0.$$

If there is a non-trivial solution, the polynomials are linearly dependent. If there is only the trivial solution, they are linearly independent. We first rearrange the left-hand side to collect equal powers of x:

$$(a+b+2c)x^2 + (2b-c)x + (3c-b) \;=\; 0.$$

This turns into a system of 3 equations in 3 variables:

$$\begin{array}{rcl} a+b+2c &=& 0 \\ 2b-c &=& 0 \\ 3c-b &=& 0 \end{array} \quad\simeq\quad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \quad\simeq\quad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since the system has rank 3, there are no free variables. The only solution is $a=b=c=0$, and the polynomials are linearly independent. ♠

Example 9.19: Linearly independent sequences

Let K be a field, and consider again the sequences from Example 9.14,

$$\begin{aligned} e^0 &= (1,0,0,0,0,\ldots), \\ e^1 &= (0,1,0,0,0,\ldots), \\ e^2 &= (0,0,1,0,0,\ldots), \end{aligned}$$

and so on. Let $S = \{e^0, e^1, e^2, \ldots\}$. This is an infinite subset of $\mathbf{Seq}_K$. Show that S is linearly independent.

Solution. Since S is an infinite set, we have to show that every finite subset of S is linearly independent. So consider a finite subset

$$\left\{e^{k_1}, e^{k_2}, \ldots, e^{k_n}\right\} \subseteq S$$

and assume that

$$a_1e^{k_1} + a_2e^{k_2} + \ldots + a_ne^{k_n} = 0. \tag{9.1}$$

We have to show that $a_1, \ldots, a_n = 0$. Consider some index $i \in \{1, \ldots, n\}$. Then the k_i^{th} element of $a_1e^{k_1} + \ldots + a_ne^{k_n}$ is equal to a_i by the left-hand side of (9.1), but it is also equal to 0 by the right-hand side of (9.1). It follows that $a_i = 0$ for all $i \in \{1, \ldots, n\}$, and therefore $\left\{e^{k_1}, e^{k_2}, \ldots, e^{k_n}\right\}$ is linearly independent. Since $\left\{e^{k_1}, e^{k_2}, \ldots, e^{k_n}\right\}$ was an arbitrary finite subset of S, it follows, by definition, that S is linearly independent. ♠

Example 9.20: Linearly dependent matrices

Determine whether the following elements of $\mathbf{M}_{2,2}$ *are linearly independent:*

$$M_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}.$$

Solution. To determine whether $\{M_1, M_2, M_3\}$ is linearly independent, we look for solutions to

$$aM_1 + bM_2 + cM_3 = 0.$$

Notice that this equation has non-trivial solutions, for example $a = 2$, $b = 3$ and $c = -1$. Therefore the matrices are linearly dependent. ♠

Example 9.21: Linearly independent functions

In the vector space $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$ *of real-valued functions on the real numbers, show that the functions* $f(x) = \sin x$ *and* $g(x) = \cos x$ *are linearly independent.*

Solution. Assume $A\sin x + B\cos x = 0$. Note that this is an equality of functions, which means that it is true for all x. In particular, substituting $x = 0$ into the equation, and using the fact that $\sin 0 = 0$ and $\cos 0 = 1$, we have

$$0 = A\sin 0 + B\cos 0 = A\cdot 0 + B\cdot 1 = B,$$

and therefore $B = 0$. On the other hand, substituting $x = \frac{\pi}{2}$ into the equation, and using the fact that $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$, we have

$$0 = A\sin\frac{\pi}{2} + B\cos\frac{\pi}{2} = A\cdot 1 + B\cdot 0 = A,$$

and therefore $A = 0$. Therefore, the equation $A\sin x + B\cos x = 0$ only has the trivial solution $A = B = 0$, and it follows that $\sin x$ and $\cos x$ are linearly independent. ♠

The properties of linear independence that were discussed in Chapter 2 remain true in the general setting of vector spaces. For example, the first two parts of Proposition 5.16 apply without change. (The third part specifically mentions $\mathbb{R}^n$, but can be generalized to any vector space of dimension n). We also have the usual characterization of linear dependence in terms of redundant vectors:

Proposition 9.22: Linear dependence and redundant vectors

Let V *be a vector space, and let* $\mathbf{u}_1, \mathbf{u}_2, \ldots$ *be a (finite or infinite) sequence of vectors in* V*. If* $\mathbf{u}_1, \mathbf{u}_2, \ldots$ *are linearly dependent, then at least one of the vectors can be written as a linear combination of earlier vectors in the sequence:*

$$\mathbf{u}_j = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_{j-1}\mathbf{u}_{j-1},$$

for some j*. As in Section 5.2.1, we say that the vector* $\mathbf{u}_j$ *is* ***redundant****.*

Proof. Suppose that the vectors are linearly dependent. Then the equation $b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k = \mathbf{0}$ has a non-trivial solution for some k. In other words, there exist scalars $b_1, \ldots, b_k$, not all equal to zero, such that $b_1\mathbf{u}_1 + \ldots + b_k\mathbf{u}_k = \mathbf{0}$. Let j be the largest index such that $b_j \neq 0$. Then $b_1\mathbf{u}_1 + \ldots + b_j\mathbf{u}_j = \mathbf{0}$. Dividing by b_j and solving for $\mathbf{u}_j$, we have $\mathbf{u}_j = -\frac{b_1}{b_j}\mathbf{u}_1 - \ldots - \frac{b_{j-1}}{b_j}\mathbf{u}_{j-1}$, so $\mathbf{u}_j$ can be written as a linear combination of earlier vectors as claimed. ♠

Example 9.23: Polynomials of increasing degree

Consider a sequence of non-zero polynomials $p_1(x), \ldots, p_k(x)$ of increasing degree, i.e., such that the degree of each $p_i(x)$ is strictly larger than that of $p_{i-1}(x)$. Show that $p_1(x), \ldots, p_k(x)$ are linearly independent in the vector space $\mathbf{P}$.

Solution. A polynomial of degree n cannot be a linear combination of polynomials of degree less than n. Therefore, none of the polynomials $p_1(x), \ldots, p_k(x)$ can be written as a linear combination of earlier polynomials. By Proposition 9.22, $p_1(x), \ldots, p_k(x)$ are linearly independent. ♠

Theorems 5.18 and 5.19 also remain true in the setting of general vector spaces. The original proofs can be used without change. Thus, if $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, then every vector $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ can be uniquely written as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Also, given any finite set of vectors, we can find a subset of the vectors that is linearly independent and has the same span.

We finish this section with a useful observation about linear independence. Namely, given a linearly independent set of vectors and one more vector that is not in their span, then we can add the vector to the set and it will remain linearly independent.

Proposition 9.24: Adding to a linearly independent set

Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent and $\mathbf{v} \notin \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Then the set

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}\}$$

is also linearly independent.

Proof. Assume, on the contrary, that the set were linearly dependent. Then by Proposition 9.22, one of the vectors can be written as a linear combination of earlier vectors. This vector cannot be one of the $\mathbf{u}_i$, because $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent. It also cannot be $\mathbf{v}$, because $\mathbf{v} \notin \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Therefore, our assumption cannot be true, and the set is linearly independent. ♠

Exercises

Exercise 9.2.1 *Let V be a vector space and suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a set of vectors in V. Show that $\mathbf{0}$ is in* $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

Exercise 9.2.2 *Determine whether* $p(x) = 4x^2 - x$ *is in* $\operatorname{span}\{x^2+x,\ x^2-1,\ -x+2\}$.

Exercise 9.2.3 *Determine whether* $p(x) = -x^2+x+2$ *is in* $\operatorname{span}\{x^2+x+1,\ 2x^2+x\}$.

Exercise 9.2.4

(a) *Write* $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ *as a linear combination of*

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

(b) *Show that the above set of four matrices is a spanning set for* $\mathbf{M}_{2,2}$, *the vector space of all* 2×2-*matrices.*

Exercise 9.2.5 *Let* K *be a field, and consider the vector space* $\mathbf{Seq}_K$ *of infinite sequences of scalars. A sequence* $a = (a_i)_{i\in\mathbb{N}}$ *is called* ***finitely supported*** *if all but finitely many elements of the sequence are zero. In other words,* a *is finitely supported if there exists some* $N \in \mathbb{N}$ *such that* $a_k = 0$ *for all* $k \geq N$. *Let* $e^0, e^1, e^2, \ldots$ *be the sequences from Example 9.14. Show that* $a \in \operatorname{span}\{e^0, e^1, e^2, \ldots\}$ *if and only if* a *is finitely supported.*

Exercise 9.2.6 *For each of the following sets of polynomials, determine whether the set is linearly independent. If it is linearly dependent, write one polynomial as a linear combination of the other polynomials in the set.*

(a) $\{x+1,\ x^2+2,\ x^2-x-3\}$.

(b) $\{x^2+x,\ -2x^2-4x-6,\ 2x-2\}$.

Exercise 9.2.7 *Determine whether each of the following sets of matrices is linearly independent. If it is linearly dependent, write one matrix as a linear combination of the other matrices in the set.*

(a) $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -7 & 2 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

Exercise 9.2.8 *Consider polynomials*

$$\left\{ a_i x^3 + b_i x^2 + c_i x + d_i \mid i = 1,2,3,4 \right\}.$$

Show that this collection of polynomials is linearly independent if and only if

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$$

is an invertible matrix.

Exercise 9.2.9 *Assume* $\mathbf{u},\mathbf{v},\mathbf{w}$ *are linearly independent elements of some vector space* V. *Consider the set of vectors*

$$R = \left\{2\mathbf{u}-\mathbf{w},\ \mathbf{w}+\mathbf{v},\ 3\mathbf{v}+\tfrac{1}{2}\mathbf{u}\right\}.$$

Determine whether R *is linearly independent.*

9.3 Subspaces

Outcomes

A. *Determine whether a set of vectors is a subspace of a given vector space.*

B. *Determine whether two sets of vectors span the same subspace.*

In this section we will consider subspaces of general vector spaces.

Definition 9.25: Subspace

Let V *be a vector space over a field* K. *A subset* $W \subseteq V$ *is said to be a* ***subspace*** *of* V *if the following conditions hold:*

1. $\mathbf{0} \in W$, *where* $\mathbf{0}$ *is the additive unit of* V.
2. W *is* ***closed under addition****: Whenever* $\mathbf{u},\mathbf{v} \in W$, *then* $\mathbf{u}+\mathbf{v} \in W$.
3. W *is* ***closed under scalar multiplication****: Whenever* $k \in K$ *and* $\mathbf{u} \in W$, *then* $k\mathbf{u} \in W$.

Example 9.26: Subspaces of $\mathbb{R}^3$

As we have seen in Section 5.3, the subspaces of $\mathbb{R}^3$ *are:*

- *the zero subspace* $\{\mathbf{0}\}$;
- *lines through the origin;*
- *planes through the origin;*
- $\mathbb{R}^3$ *itself.*

Example 9.27: Space of continuous functions

Let $V = \mathbf{Func}_{\mathbb{R},\mathbb{R}}$, the vector space of functions from real numbers to real numbers. Let $W \subseteq V$ be the subset of continuous *functions. Then W is a subspace of V.*

Proof. We know from calculus that:

1. the zero function, defined by $f(x) = 0$ for all x, is continuous;
2. if f, g are continuous functions, then $f + g$ is continuous;
3. if f is a continuous function and k a constant, then kf is continuous.

It follows that W contains 0, and is closed under addition and scalar multiplication. Therefore, W is a subspace of V. ♠

Example 9.28: Space of differentiable functions

Let $V = \mathbf{Func}_{\mathbb{R},\mathbb{R}}$. Recall from calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is called ***differentiable*** *if the derivative $f'(x)$ exists for all $x \in \mathbb{R}$. Let $W \subseteq V$ be the subset of differentiable functions. Then W is a subspace of V.*

Proof. We know from calculus that:

1. The zero function, defined by $f(x) = 0$ for all x, is differentiable. In fact, its derivative is $f'(x) = 0$.
2. If f, g are differentiable functions, then $h = f + g$ is differentiable. In fact, $h'(x) = f'(x) + g'(x)$.
3. if f is a differentiable function and k a constant, then $h = kf$ is differentiable. In fact, $h' = kf'$.

It follows that W contains 0, and is closed under addition and scalar multiplication. Therefore, W is a subspace of V. ♠

Example 9.29: Space of sequences satisfying a linear recurrence

Let $V = \mathbf{Seq}_{\mathbb{R}}$, the vector space of sequences of real numbers. Let

$$W = \{a \in \mathbf{Seq}_{\mathbb{R}} \mid \text{for all } n \geq 0,\ a_{n+2} = a_n + a_{n+1}\}.$$

In other words, W is the set of all sequences satisfying the recurrence relation $a_{n+2} = a_n + a_{n+1}$. Then W is a subspace of V.

Proof. Before we prove that W is a subspace, let us first consider an example. The following sequences are elements of W, because they both satisfy the recurrence:

$$\begin{aligned} a &= (1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots), \\ b &= (1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ \ldots). \end{aligned}$$

Note that if we add these sequences, we get

$$a+b = (2, \ 4, \ 6, 10, 16, 26, 42, 68, \ldots),$$

which again satisfies the recurrence. Therefore, the set W is closed under the addition of these particular sequences a and b. We now prove the properties in general.

1. Let z be the zero sequence, defined by $z_n = 0$ for all n. Then z satisfies the recurrence relation, since for all $n \geq 0$, $z_{n+2} = 0 = z_n + z_{n+1}$. Therefore $z \in W$.

2. To show that W is closed under addition, consider any two sequences $a, b \in W$, and let $c = a+b$. Then for all $n \geq 0$,

$$c_{n+2} = a_{n+2} + b_{n+2} = (a_n + a_{n+1}) + (b_n + b_{n+1}) = (a_n + b_n) + (a_{n+1} + b_{n+1}) = c_n + c_{n+1},$$

so c satisfies the recurrence. It follows that $c \in W$, and therefore W is closed under addition.

3. To show that W is closed under scalar multiplication, consider any $k \in \mathbb{R}$ and $a \in W$, and let $c = ka$. Then for all $n \geq 0$,

$$c_{n+2} = ka_{n+2} = k(a_n + a_{n+1}) = ka_n + ka_{n+1} = c_n + c_{n+1},$$

so c satisfies the recurrence. It follows that $c \in W$, and therefore W is closed under scalar multiplication.

♠

Example 9.30: Solution space of a linear differential equation

Let $V = \mathbf{Func}_{\mathbb{R},\mathbb{R}}$. Recall from calculus that a ***differential equation*** *is an equation about an unknown function and its derivatives. For example*

$$f'' = -f$$

is a differential equation. The functions $f(x) = \sin x$, $f(x) = \cos x$, and $f(x) = 0$ are examples of solutions of this differential equation. Let W be the set of all functions that are solutions of the differential equation $f'' = -f$. Then W is a subspace of V.

Proof.

1. The zero function $f(x) = 0$ is a solution of the differential equation, and therefore an element of W.

2. To show that W is closed under addition, let $f, g \in W$ and consider $h = f + g$. Then $f'' = -f$ and $g'' = -g$, and therefore $h'' = f'' + g'' = -f + (-g) = -h$. Therefore, $h \in W$, and W is closed under addition.

3. To show that W is closed under scalar multiplication, let $k \in \mathbb{R}$ and $f \in W$, and consider $h = kf$. Then $f'' = -f$, and therefore $h'' = kf'' = k(-f) = -h$. It follows that $h \in W$, and therefore W is closed under scalar multiplication.

♠

Example 9.31: Subspace of polynomials

Consider $\mathbf{P}_2$, the vector space of polynomials of degree at most 2, with coefficients in a field K. Fix some element $r \in K$, and let $W \subseteq \mathbf{P}_2$ be the subset of polynomials that have r as a root. Then W is a subspace of $\mathbf{P}_2$.

Proof. We can express W as follows:

$$W = \{p \in \mathbf{P}_2 \mid p(r) = 0\}.$$

We need to show that W is a subspace.

1. The zero polynomial, given by $0(x) = 0$, satisfies $0(r) = 0$, so $0 \in W$.
2. To show that W is closed under addition, assume $p, q \in W$, and let $s = p + q$. Then $p(r) = 0$ and $q(r) = 0$, therefore $s(r) = p(r) + q(r) = 0$. It follows that $s \in W$.
3. To show that W is closed under scalar multiplication, assume $p(x) \in W$ and k be a scalar. Then $(kp)(r) = k(p(r)) = k0 = 0$, and therefore $kp \in W$.

♠

Example 9.32: Trivial subspaces of V

Let V be an arbitrary vector space over a field K. Then $\{\mathbf{0}\}$ is a subspace of V, called the ***zero subspace****. Also, V is a subspace of itself.*

Proof. Clearly $\{\mathbf{0}\}$ contains $\mathbf{0}$, and is closed under addition and scalar multiplication because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for all k. Similarly, V contains $\mathbf{0}$ and is closed under addition and scalar multiplication, because addition and scalar multiplication are operations on V. Therefore, both $\{\mathbf{0}\}$ and V are subspaces of V. ♠

The interest of subspaces lies in the fact that they are vector spaces in their own right, as stated in the following proposition.

Proposition 9.33: Subspaces are vector spaces

Let W be a subspace of a vector space V. Then W satisfies the vector space axioms (A1)–(A4) and (SM1)–(SM4), with respect to the same operations (addition and scalar multiplication) as those defined on V.

Proof. Since W is a subspace, it is closed under addition and scalar multiplication. This ensures that addition and scalar multiplication are well-defined operations on W. The axioms (A1), (A2), (A4), and (SM1)–(SM4) all obviously hold in W, because they hold in V (two elements of W are equal in W if and only if they are equal in V). The axiom (A3) holds because $\mathbf{0} \in W$. ♠

We end this section with an observation about spans.

Proposition 9.34: Span is smallest subspace containing given vectors

Let V be a vector space over some field K, and consider a set of vectors $S \subseteq V$. Then $\operatorname{span} S$ is the smallest subspace of V containing S. More explicitly, we have:

(a) The set $\operatorname{span} S$ is a subspace of V, and $S \subseteq \operatorname{span} S$.

(b) If W is any other subspace of V such that $S \subseteq W$, then $\operatorname{span} S \subseteq W$.

Proof.

(a) To show that $\operatorname{span} S$ is a subspace, first note that $\mathbf{0} \in \operatorname{span} S$, because $\mathbf{0}$ is the empty linear combination. Also, if $\mathbf{v}, \mathbf{u} \in \operatorname{span} S$, then by definition of span, there exist $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_\ell \in S$ and $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in K$ such that

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k, \\ \mathbf{u} &= b_1\mathbf{u}_1 + \ldots + b_\ell\mathbf{u}_\ell. \end{aligned}$$

Then

$$\mathbf{v} + \mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k + b_1\mathbf{u}_1 + \ldots + b_\ell\mathbf{u},$$

and therefore $\mathbf{v} + \mathbf{u} \in \operatorname{span} S$. It follows that $\operatorname{span} S$ is closed under addition. The proof for scalar multiplication is similar. Finally, every $\mathbf{v} \in S$ is trivially a linear combination of itself, $\mathbf{v} = 1\mathbf{v}$, and therefore $S \subseteq \operatorname{span} S$.

(b) Consider any other subspace W of V such that $S \subseteq W$. To show that $\operatorname{span} S \subseteq W$, consider an arbitrary element $\mathbf{v} \in \operatorname{span} S$. By definition of span, there exist $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$ and $a_1, \ldots, a_k \in K$ such that $\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k$. By assumption, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$. Since W is closed under addition and scalar multiplication, it follows that $\mathbf{v} \in W$. Since $\mathbf{v}$ was an arbitrary element of $\operatorname{span} S$, it follows that $\operatorname{span} S \subseteq W$.

♠

While the last proposition looks technical, it can actually be useful for proving that two sets of vectors span the same subspace. The following in an example of this.

Example 9.35: Equal spans

Show that the sets $S = \{x^2 - 2x + 1,\ x - 1\}$ and $T = \{x^2 - 1,\ x^2 - x\}$ span the same subspace of $\mathbf{P}_2$. In other words, show that

$$\operatorname{span} S = \operatorname{span} T.$$

Solution. To show that two sets are equal, we must show that each is a subset of the other. So we will show $\operatorname{span} S \subseteq \operatorname{span} T$ and $\operatorname{span} T \subseteq \operatorname{span} S$. By Proposition 9.34, it is sufficient to show that $S \subseteq \operatorname{span} T$ and $T \subseteq \operatorname{span} S$, i.e., we must show that every element of S is a linear combination of elements of T and vice versa.

1. $S \subseteq \operatorname{span} T$. We have

$$\begin{array}{rcll} x^2-2x+1 & = & (-1)(x^2-1)+2(x^2-x) & \in \operatorname{span} T, \\ x-1 & = & 1(x^2-1)-1(x^2-x) & \in \operatorname{span} T. \end{array}$$

   Since each element of S is an element of $\operatorname{span} T$, it follows that $S \subseteq \operatorname{span} T$. By Proposition 9.34, this implies that $\operatorname{span} S \subseteq \operatorname{span} T$.

2. $T \subseteq \operatorname{span} S$. We have

$$\begin{array}{rcll} x^2-1 & = & 1(x^2-2x+1)-2(x-1) & \in \operatorname{span} S, \\ x^2-x & = & 1(x^2-2x+1)-1(x-1) & \in \operatorname{span} S. \end{array}$$

   Since each element of T is an element of $\operatorname{span} S$, it follows that $T \subseteq \operatorname{span} S$. By Proposition 9.34, this implies that $\operatorname{span} T \subseteq \operatorname{span} S$.

♠

Exercises

Exercise 9.3.1 *Consider the set of symmetric $n \times n$-matrices, i.e., matrices satisfying $A = A^T$. Show that this set of symmetric matrices is a subspace of $\mathbf{M}_{n,n}$, the vector space of $n \times n$-matrices.*

Exercise 9.3.2 *Consider the set of all vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right] \in \mathbb{R}^2$ such that $x+y \geq 0$. Is this a subspace of $\mathbb{R}^2$?*

Exercise 9.3.3 *Consider the set of all vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right] \in \mathbb{R}^2$ such that $xy = 0$. Is this a subspace of $\mathbb{R}^2$?*

Exercise 9.3.4 *Let V be the set of those polynomials $ax^2+bx+c \in \mathbf{P}_2$ such that $a+b+c=0$. Is V a subspace of $\mathbf{P}_2$? Explain.*

Exercise 9.3.5 *Let U, W be subspaces of a vector space V and consider $U+W$ defined as the set of all vectors that can be written of the form $\mathbf{u}+\mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Show that $U+W$ is a subspace of V.*

Exercise 9.3.6 *Let U, W be subspaces of a vector space V. Then $U \cap W$ consists of all vectors which are in both U and W. Show that $U \cap W$ is a subspace of V.*

Exercise 9.3.7 *Let U, W be subspaces of a vector space V. Then $U \cup W$ consists of all vectors which are in either U or W. Show that $U \cup W$ is not necessarily a subspace of V by giving an example where $U \cup W$ fails to be a subspace.*

Exercise 9.3.8 *Let $U = \left\{ [x,y,z]^T \in \mathbb{R}^3 \;\middle|\; |x| \leq 4 \right\}$. Is U a subspace of $\mathbb{R}^3$?*

Exercise 9.3.9 *Let W be the subset of $\mathbf{M}_{2,2}$ given by*

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a,b,c,d \in K,\; a+b = c+d \right\}$$

Is W a subspace of $\mathbf{M}_{2,2}$?

Exercise 9.3.10 *Let $U = \left\{ A \in \mathbf{M}_{2,2} \;\middle|\; A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A \right\}$. Show that U is a subspace of $\mathbf{M}_{2,2}$.*

Exercise 9.3.11 *Let W be the subset of $\mathbf{P}_3$ given by*

$$W = \left\{ ax^3 + bx^2 + cx + d \mid a,b,c,d \in K, d = 0 \right\}$$

Is W a subspace of $\mathbf{P}_3$?

Exercise 9.3.12 *Let W be the subset of $\mathbf{P}_3$ given by*

$$W = \{p(x) \mid p(2) = 1\}$$

Is W a subspace of $\mathbf{P}_3$?

Exercise 9.3.13 *Let W be the subset of $\mathbf{Seq}_{\mathbb{R}}$ consisting of all sequences that are alternating, i.e., where $a_i \geq 0$ for even i and $a_i \leq 0$ for odd i, or vice versa. Is W a subspace of $\mathbf{Seq}_{\mathbb{R}}$?*

Exercise 9.3.14 *Let W be the subset of $\mathbf{Seq}_K$ consisting of all sequences that satisfy the recurrence relation $a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$. Is W a subspace of $\mathbf{Seq}_K$?*

Exercise 9.3.15 *A sequence $(a_i)_{i\in\mathbb{N}}$ is called **periodic** if there exists some $k > 0$ such that for all i, $a_{i+k} = a_i$. The number k is called a **period** of the sequence. For example, the following is a periodic sequence with period 3:*

$$(1,5,-7,1,5,-7,1,5,-7,1,5,-7,1,\ldots).$$

(a) Show that the set of all periodic sequences of a fixed period k forms a subspace of $\mathbf{Seq}_K$.

(b) More difficult: show that the set of all periodic sequences of all periods forms a subspace of $\mathbf{Seq}_K$.

Exercise 9.3.16 *A function $f : \mathbb{R} \to \mathbb{R}$ is called **symmetric** if $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Let W be the subset of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$ consisting of all symmetric functions. Is W a subspace of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$?*

Exercise 9.3.17 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to **vanish at infinity** if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. Let W be the subset of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$ consisting of all functions that vanish at infinity. Prove that W is a subspace of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$.*

Exercise 9.3.18 *Show that the sets $S = \left\{x+2, (x+2)^2\right\}$ and $T = \left\{x^2-4, x^2+x-2\right\}$ span the same subspace of $\mathbf{P}_2$.*

Exercise 9.3.19 *Show that the sets $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$ span the same subspace of $\mathbf{M}_{2,2}$.*

9.4 Basis and dimension

Outcomes

A. *Find a basis of a given vector space.*

B. *Determine the dimension of a vector space.*

C. *Extend a linearly independent set of vectors to a basis.*

D. *Shrink a spanning set of vectors to a basis.*

Definition 9.36: Basis

Let V be a vector space. A set B of vectors is called a ***basis*** *of V if*

1. *B is a spanning set for V, and*

2. *B is linearly independent.*

Example 9.37: Bases of $\mathbf{P}_2$

Consider the vector space $\mathbf{P}_2$ of polynomials of degree at most 2 with coefficients in a field K.

- $\{1, x, x^2\}$ *is a basis of $\mathbf{P}_2$.*
- $\{x^2, (x+1)^2, (x+2)^2\}$ *is a basis of $\mathbf{P}_2$.*
- $\{1, x-1, (x-1)^2\}$ *is a basis of $\mathbf{P}_2$.*

Unlike $\mathbb{R}^n$, a vector space like $\mathbf{P}_2$ does not necessarily have a "standard" basis. One basis might be useful for one application, and another basis for a different application.

Proof. It is easy to verify that each set of vectors is linearly independent and spanning. See Examples 9.16, 9.18, and 9.23 for similar calculations. ♠

Example 9.38: An infinite basis

Consider the vector space $\mathbf{P}$ of all polynomials with coefficients in a field K. The following is a basis for $\mathbf{P}$:

$$\{1, x, x^2, x^3, x^4, \ldots\}.$$

Note that this basis is infinite.

Proof. The polynomials $1, x, x^2, x^3, x^4, \ldots$ are linearly independent by Proposition 9.22. Namely, if they were linearly dependent, then one of the polynomials could be written as a linear combination of earlier ones. However, this is not possible because a polynomial of degree n cannot be a linear combination of polynomials of degree less than n.

To show that the polynomials $1, x, x^2, x^3, x^4, \ldots$ are a spanning set, consider an arbitrary element $p(x)$ of $\mathbf{P}$. Then by definition, $p(x)$ is of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

for some $n \geq 0$ and $a_0, \ldots, a_n \in K$. But then $p(x)$ is a linear combination of $1, \ldots, x^n$, i.e., it is in the span of $\{1,\ x,\ x^2,\ x^3,\ x^4,\ \ldots\}$. ♠

Example 9.39: Not a basis

Consider the vector space $\mathbf{Seq}_K$ *of infinite sequences. As before, let* e^k *be the sequence whose* k^{th} *element is* 1 *and that is* 0 *everywhere else, i.e.,*

$$\begin{aligned} e^0 &= (1,0,0,0,0,\ldots), \\ e^1 &= (0,1,0,0,0,\ldots), \\ e^2 &= (0,0,1,0,0,\ldots), \end{aligned}$$

and so on. Then the set

$$\{e^0,\ e^1,\ e^2,\ \ldots\}$$

is not a basis of $\mathbf{Seq}_K$. *Indeed, although we saw in Example 9.19 that the sequences* $e^0, e^1, e^2, \ldots$ *are linearly independent, Example 9.14 shows that they are not spanning. Indeed,*

$$W = \operatorname{span}\{e^0,\ e^1,\ e^2,\ \ldots\}$$

is a subspace of $\mathbf{Seq}_K$*, consisting exactly of the* ***finitely supported*** *sequences, i.e., those sequences that have only finitely many non-zero components. Thus,* $\{e^0,\ e^1,\ e^2,\ \ldots\}$ *is a basis of* W.

The following theorem ensures that every vector space has a basis. We will not prove this theorem, because when the spaces are infinite-dimensional, the proof uses mathematics that is beyond the scope of this book. The proof uses a reasoning principle called the **axiom of choice**, which allows us to prove the existence of a basis even in cases where we cannot find an actual concrete example of a basis. For example, it is not possible to give a specific example of a basis for the space $\mathbf{Seq}_K$, even though the following theorem guarantees that such a basis exists.

Theorem 9.40: Existence of bases

Every vector space has a basis.

The Exchange Lemma, which we proved in the context of $\mathbb{R}^n$ in Section 5.4, is true in general vector spaces.

Lemma 9.41: Exchange Lemma

Let V *be a vector space over a field* K. *Suppose* $\mathbf{u}_1, \ldots, \mathbf{u}_r$ *are linearly independent elements of* $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$. *Then* $r \leq s$.

The proof is exactly the same as that of Lemma 5.37, so we do not repeat it here. As in Section 5.4, an important consequence of the Exchange Lemma is that any two bases of a vector space have the same size.

Theorem 9.42: Bases are of the same size

Let V be a vector space over some field K, and let B_1 and B_2 be bases of V. Then either B_1 and B_2 are both finite and have the same number of elements, or else B_1 and B_2 are both infinite.

Proof. We first show that B_1 and B_2 are either both finite or both infinite. Assume one of them, say B_1, is finite and contains s vectors. Since B_1 is spanning and B_2 is linearly independent, it follows from the Exchange Lemma that B_2 cannot contain more than s vectors, and in particular, B_2 must be finite. So the sets are either both finite or both infinite. If they are both finite, say of size s and r, then by the Exchange Lemma, we have $s \leq r$ and $r \leq s$, hence $r = s$. ♠

This allows us to define the dimension of a vector space.

Definition 9.43: Dimension

Let V be a vector space over a field K. If V has a basis consisting of n vectors, we say that V has ***dimension*** *n, and we write $\dim(V) = n$. In this case we also say that V is* ***finite-dimensional****. If V has an infinite basis, we say that V is* ***infinite-dimensional****, and we write $\dim(V) = \infty$.*

Note that the dimension is well-defined by Theorems 9.40 and 9.42, since these theorems ensure that every vector space has a basis (and therefore a dimension), and that any two bases are of the same size (and therefore a vector space cannot have more than one dimension).

We now calculate the dimensions of some vector spaces we encountered in Sections 9.1 and 9.3.

- The space $\mathbb{R}^n$ has dimension n.
- The space $\mathbf{P}_2$ has dimension 3. We found several bases for this space in Example 9.37.
- The space $\mathbf{M}_{m,n}$ has dimension mn. A possible basis consists of all the matrices that contain a single 1 and zeros everywhere else.
- The space $\mathbf{Func}_{X,K}$ is infinite-dimensional if X is an infinite set. If X is a finite set of n elements, then this space is n-dimensional. In that case, a basis is given by the set of functions whose value is 1 for one input and 0 for all other inputs.
- The space $\mathbf{Seq}_K$ is infinite-dimensional. We found an infinite linearly independent set in Example 9.19, showing that the space cannot be finite-dimensional.
- The space $\mathbf{P}$ is infinite-dimensional. We found a basis for this space in Example 9.38.
- The subspace of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$ consisting of the continuous functions is infinite-dimensional. For example, the functions $\{1, x, x^2, x^3, \ldots\}$ form an infinite, linearly independent set of continuous functions.
- The subspace of $\mathbf{Func}_{\mathbb{R},\mathbb{R}}$ consisting of the differentiable functions is infinite-dimensional. Again, the set $\{1, x, x^2, x^3, \ldots\}$ is an infinite linearly independent set in this space.

Example 9.44: Space of sequences satisfying a linear recurrence

In Example 9.29, we considered the space W of sequences of real numbers that satisfy the recurrence $a_{n+2} = a_n + a_{n+1}$. What is the dimension of this space?

Solution. The space is 2-dimensional. The easiest way to see this is to observe that a sequence $a \in W$ is determined by its first two elements. We can say that the first two elements of the sequence are parameters, and all the other elements are then computed by the recurrence relation. Specifically, suppose $a_0 = x$ and $a_1 = y$. Using the recurrence relation to compute the remaining elements, we have

$$\begin{aligned} a &= (x,\ y,\ x+y,\ x+2y,\ 2x+3y,\ 3x+5y,\ \ldots) \\ &= x(1,0,1,1,2,3,\ldots) + y(0,1,1,2,3,5,\ldots). \end{aligned}$$

Since this is the general form of the elements of W, and since the two sequences starting with 1,0 and 0,1 are clearly linearly independent, it follows that

$$\{(1,0,1,1,2,3,\ldots),\ (0,1,1,2,3,5,\ldots)\}$$

is a basis of W. ♠

Example 9.45: Solution space of a linear differential equation

In Example 9.30, we considered the space of solutions of the differential equation $f'' = -f$. What is the dimension of this space?

Solution. From calculus, we know that the general solution of the differential equation $f'' = -f$ is

$$f(x) = A\sin x + B\cos x,$$

where A, B are constants. We also know, from Example 9.21, that $\sin x$ and $\cos x$ are linearly independent. It follows that $\{\sin x, \cos x\}$ is a basis for the solution space. The solution space is therefore 2-dimensional. ♠

We conclude this section by stating two properties of bases that generalize Theorem 5.19 and Lemma 5.44: every linearly independent set can be extended to a basis by adding 0 or more vectors, and every spanning set can be reduced to a basis by removing 0 or more vectors.

Proposition 9.46: Extending a linearly independent set to a basis

Let V be a vector space, and let $S \subseteq V$ be a linearly independent set of vectors. Then S can be extended to a basis of V, i.e., there exists a basis B of V such that $S \subseteq B$.

Proposition 9.47: Shrinking a spanning set to a basis

Let V be a vector space, and let $S \subseteq V$ be a spanning set of V. Then S can be shrunk to a basis, i.e., there exists a basis B of V such that $B \subseteq S$.

Example 9.48: Extending a linearly independent set to a basis

Let $S \subseteq \mathbf{M}_{2,2}$ be the linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Enlarge S to a basis of $\mathbf{M}_{2,2}$.

Solution. We can obtain a basis of $\mathbf{M}_{2,2}$ by adding two more linearly independent matrices

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The resulting basis is

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

♠

Example 9.49: Shrinking a spanning set to a basis

Consider the spanning set $S \subseteq \mathbf{P}_2$ given by

$$S = \left\{ 1,\ x,\ 2x+1,\ x^2+1,\ x^2+2 \right\}$$

Shrink S to a basis of $\mathbf{P}_2$.

Solution. We use a version of the casting-out method. We examine each element S from left to right and cast out the elements that are linear combinations of previous elements. Clearly the first two elements, 1 and x, are linearly independent. The next element, $2x+1$, is redundant because it is a linear combination of 1 and x. The next element x^2+1 is linearly independent of 1 and x. The final element x^2+2 is redundant because it is a linear combination of 1 and x^2+1. Therefore, the following subset of S is a basis of $\mathbf{P}_2$:

$$B = \left\{ 1,\ x,\ x^2+1 \right\}.$$

♠

Exercises

Exercise 9.4.1 *Let $\mathbf{P}_3$ be the vector space of polynomials of degree at most 3. Determine which of the following are bases for this vector space.*

(a) $\left\{ x^3+1,\ x^2+x,\ 2x^3+x^2,\ 2x^3-x^2-3x+1 \right\}$.

(b) $\{x+1,\ x^3+x^2+2x,\ x^2+x,\ x^3+x^2+x\}$.

Exercise 9.4.2 *Determine whether the following is a basis for* $\mathbf{P}_2$*, the vector space of polynomials of degree at most* 2*.*

$$\{x^2+x+1,\ 2x^2+2x+1,\ x+1\}.$$

Exercise 9.4.3 *Find a basis for the following subspace of* $\mathbf{P}_2$*:*

$$W = \operatorname{span}\{1+x+x^2,\ 1+2x,\ 1+5x-3x^2\}.$$

Exercise 9.4.4 *Find a basis for the following subspace of* $\mathbf{P}_3$*:*

$$W = \operatorname{span}\{1+x-x^2+x^3,\ 1+2x+3x^3,\ -1+3x+5x^2+7x^3,\ 1+6x+4x^2+11x^3\}.$$

Exercise 9.4.5 *Extend the following linearly independent set of polynomials to a basis of* $\mathbf{P}_3$*:*

$$\{x^3+x^2-x-1,\ 3x^3+2x^2+2x-1\}.$$

Exercise 9.4.6 *Let* V *be a* 5*-dimensional vector space. If you have* 5 *linearly independent vectors in* V*, can you conclude that the vectors span* V*?*

Exercise 9.4.7 *Let* V *be a* 5*-dimensional vector space. If you have* 6 *vectors in* V*, is it possible that they are linearly independent? Explain.*

Exercise 9.4.8 *Find a basis for the vector space of symmetric* 3×3*-matrices, i.e., matrices satisfying* $A = A^T$*. What is the dimension of this space?*

Exercise 9.4.9 *Let* W *be the subspace of* $\mathbf{P}_3$ *(over the field* $\mathbb{R}$*) consisting of all polynomials* $p(x)$ *that satisfy* $p(3) = 0$*. Find a basis for* W*. What is the dimension of* W*?*

Exercise 9.4.10 *Find a basis for* $U = \left\{A \in \mathbf{M}_{2,2} \;\middle|\; A\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A\right\}$*. What is the dimension of* U*?*

Exercise 9.4.11

(a) *Let* k *be a positive integer, let* $W_k \subseteq \mathbf{Seq}_K$ *be the subspace consisting of all sequences that are periodic with period* k *(see Exercise 9.3.15). Find a basis for* W_k*. What is its dimension?*

(b) *More difficult: find a basis for the infinite-dimensional vector space consisting of all periodic sequences of all periods.*

Exercise 9.4.12 *Let* $K = \mathbb{Q}$*, the field of rational numbers. Consider vectors of the form* $a + b\sqrt{2}$ *where* a, b *are rational numbers. Show that this collection of vectors is a vector space over* $\mathbb{Q}$ *and give a basis for this vector space. What is its dimension?*

9.5 Application: Error correcting codes

Outcomes

A. *Determine the block length, message length, Hamming distance, and rate of a code.*

B. *Determine whether a code is m-error detecting and m-error correcting.*

C. *Find generator and check matrices for a linear code.*

D. *Use the syndrome method to correct errors in code blocks.*

E. *Construct and use Hamming codes.*

Binary codes

When transmitting or storing information on digital media, the information is usually encoded as a sequence of bits, i.e., 0s and 1s. For example, in the ASCII code, each symbol is encoded as a sequence of 8 bits. The letter “A” is encoded as 01000001, the letter “B” is encoded as 01000010, and so on. We can think of a sequence of n bits as an n-dimensional column vector over the field $\mathbb{Z}_2$, i.e., as an element of $\mathbb{Z}_2^n$. For our purposes, it is convenient to continue writing bit sequences horizontally, but we will consider this to be merely an alternate notation for a column vector.

One issue with digital data is that the data can sometimes be corrupted. DVDs may be scratched, magnetic storage may depolarize, and data sent by radio transmission may be subject to interference. This can result in errors in the data, such as some 0s being changed to 1s or vice versa. One of the ways to deal with such errors is to introduce **redundancy** in the way the data is encoded.

A very simple example of redundant encoding is the so-called **3-repetition code**. This simply means to repeat each bit 3 times. Thus, the bit 0 is encoded as 000 and the bit 1 as 111. For example, the bit string 100101 is encoded as 111000000111000111. With this encoding, single bit errors are easy to detect and correct. Assuming that at most one error occurs within each 3-bit block, the blocks can be decoded by “majority decision”. Namely, if the bits in a block are not the same, we assume that the error occurred in the bit that is in the minority. The following table shows the decoding scheme:

Received	Likely error	Corrected	Decoded
000	none	000	0
100	bit 1	000	0
010	bit 2	000	0
001	bit 3	000	0
011	bit 1	111	1
101	bit 2	111	1
110	bit 3	111	1
111	none	111	1

Example 9.50: Decoding the 3-repetition code

A message was encoded with the 3-repetition code. The receiver receives the following data: 001011000100111110. *What was the likely message?*

Solution. We start by dividing the received data into blocks of 3 bits:

$$001\ 011\ 000\ 100\ 111\ 110.$$

We then decode each block separately, using the majority rule. The first block 001 has a majority of zeros, so the likely error is in the third bit and the likely decoding is 0. The second block 011 has a majority of ones, so the likely error is in the first bit and the likely decoding is 1. Continuing this way, we find that the likely original message was 01 00 11. ♠

Since the 3-repetition code can be used to correct some errors, it is called an **error correcting code**. This particular code only works if there are not too many errors; namely, it can correct at most one error per code block. The main drawback of the 3-repetition code is that it increases the message length by a factor of 3. In this section, we will use linear algebra over the field $\mathbb{Z}_2$ to construct better error correcting codes.

Most error correcting codes do not work by encoding individual bits. Rather, the codes work by dividing the message into *message blocks* of length k, and encoding each such message block by a *code block* of length n. This leads us to the following definition:

Definition 9.51: Code

Let n and k be positive integers with $k \leq n$. A **code** *with* **message length** *k and* **block length** *n is a set C of 2^k different vectors in $\mathbb{Z}_2^n$. We call the elements of C the* **code blocks**. *Since there are 2^k different code blocks, they can be used to encode* **message blocks** *of length k. We say that the* **rate** *of the code is $\frac{k}{n}$, because for every n bits of encoded data transmitted, k bits of decoded data are obtained.*

Example 9.52: A simple code

Consider the following code with message length $k = 2$ and block length $n = 5$:

Message block	*Code block*
00	00000
01	00111
10	11100
11	11011

(a) *Encode the message* 011110.

(b) *Decode the message* 111000011111011.

(c) *What is the rate of this code? Is it better or worse than the rate of the 3-repetition code?*

Solution. (a) We divide the message into blocks of length 2: 01 11 10. Then we encode each block separately: 00111 11011 11100. (b) We divide the message into blocks of length 5: 11100 00111 11011.

Then we decode each block separately: 10 01 11. (c) The rate is $\frac{2}{5} = 0.4$. It is slightly higher, and therefore better, than the rate of the 3-repetition code, which is $\frac{1}{3} \approx 0.33$. ♠

Hamming distance and error correction

The error correction capabilities of a code depend on a property called the *Hamming distance* of the code, which we now define.

Definition 9.53: Hamming weight and Hamming distance

- *Let* $\mathbf{v}$ *be a vector in* $\mathbb{Z}_2^n$*. The* ***Hamming weight*** *of* $\mathbf{v}$*, denoted* $W(\mathbf{v})$*, is the number of components of* $\mathbf{v}$ *that are equal to 1.*
- *Let* $\mathbf{v}, \mathbf{w}$ *be two vectors in* $\mathbb{Z}_2^n$*. The* ***Hamming distance*** *between* $\mathbf{v}$ *and* $\mathbf{w}$*, denoted* $D(\mathbf{v}, \mathbf{w})$*, is the number of components where* $\mathbf{v}$ *and* $\mathbf{w}$ *differ. We can also express this as the Hamming weight of* $\mathbf{v} - \mathbf{w}$*, i.e.,* $D(\mathbf{v}, \mathbf{w}) = W(\mathbf{v} - \mathbf{w})$*.*
- *Finally, we say that the* ***Hamming distance of a code*** *is equal to the smallest Hamming distance between any two code blocks.*

A code with block length n*, message length* k*, and Hamming distance* d *is also called an* (n, k, d)***-code****.*

Example 9.54: Hamming distance

Calculate the Hamming distance between 00111 *and* 11100*. What is the Hamming distance of the 3-repetition code? What is the Hamming distance of the code in Example 9.52?*

Solution. The vectors 00111 and 11100 differ in 4 places, so their Hamming distance is $D(00111, 11100) = 4$. This is also equal to the Hamming weight of $00111 - 11100 = 11011$.

The 3-repetition code has only two code blocks: 000 and 111. Since their Hamming distance is 3, the Hamming distance of the code is also 3.

To calculate the Hamming distance of the code from Example 9.52, we calculate the Hamming distance between all pairs of code blocks:

$$\begin{aligned}
D(00000, 00111) &= 3, \\
D(00000, 11100) &= 3, \\
D(00000, 11011) &= 4, \\
D(00111, 11100) &= 4, \\
D(00111, 11011) &= 3, \\
D(11100, 11011) &= 3.
\end{aligned}$$

Since the smallest distance between any two code blocks is 3, the Hamming distance of the code is 3. ♠

The significance of a code's Hamming distance is explained by the following definition and proposition.

Definition 9.55: Error detection and error correction

- *We say that a code C is m-**error detecting** if for all valid code blocks $\mathbf{v}$, whenever $\mathbf{w}$ is obtained from $\mathbf{v}$ by introducing up to m bit errors, then $\mathbf{w}$ is not a valid code block.*
- *We say that a code C is m-**error correcting** if for all valid code blocks $\mathbf{v}$, whenever $\mathbf{w}$ is obtained from $\mathbf{v}$ by introducing up to m bit errors, then $\mathbf{v}$ is the only valid code block within Hamming distance m of $\mathbf{w}$.*

Proposition 9.56: Error detection and error correction

Consider a code with Hamming distance d. Then the code is:

- *m-error detecting if $m \leq d-1$;*
- *m-error correcting if $2m \leq d-1$.*

Proof. Assume up to m errors have happened. In other words, let $\mathbf{v}$ be a valid code block, and let $\mathbf{w}$ be the code block obtained from $\mathbf{v}$ by introducing up to m errors. To prove the first claim, assume $m \leq d-1$. Then $D(\mathbf{v},\mathbf{w}) \leq m \leq d-1$. Since d is the minimum distance between any two valid code blocks, $\mathbf{w}$ cannot be a valid code block. Hence, the errors can be detected. To prove the second claim, assume $2m \leq d-1$. Then $D(\mathbf{v},\mathbf{w}) \leq m$. Assume that there exists another valid code block $\mathbf{u}$ within Hamming distance m of $\mathbf{w}$, i.e., assume $D(\mathbf{w},\mathbf{u}) \leq m$. Then

$$D(\mathbf{v},\mathbf{u}) \leq D(\mathbf{v},\mathbf{w}) + D(\mathbf{w},\mathbf{u}) \leq m+m \leq d-1.$$

Therefore, the Hamming distance of $\mathbf{v}$ and $\mathbf{u}$ is at most $d-1$, contradicting the assumption that the code has Hamming distance d. Hence, there is no such code block $\mathbf{u}$, and the errors can be corrected. ♠

Example 9.57: Error detection and error correction

Consider a code with Hamming distance 3. How many errors per code block can the code detect? How many can it correct? Also answer this question for Hamming distance 2, 4 and 5.

Solution. By Proposition 9.56, a code with Hamming distance 3 can detect up to 2 errors and correct up to 1 error. The answers for other Hamming distances are summarized in the following table:

Hamming distance	Errors detected	Errors corrected
2	1	0
3	2	1
4	3	1
5	4	2

♠

Example 9.58: Decoding

The following message has been encoded using the code of Example 9.52. It contains some errors, but no more than 1 error per code block. Can you decode the message?

11101 01000 11011 11111 00011

Solution. Since the code of Example 9.52 has Hamming distance 3, it can correct up to 1 error per code block, so we are able to decode the message uniquely. For each code block, we must find the unique valid code block that is within Hamming distance 1 or less.

Received	Error	Corrected	Decoded
11101	bit 5	11100	10
01000	bit 2	00000	00
11011	none	11011	11
11111	bit 3	11011	11
00011	bit 3	00111	01

The decoded message is 10 00 11 11 01. ♠

We can say that a "good" error correcting code is one that has a high rate (of message length divided by block length) and a large Hamming distance.

Linear codes

To construct a code of message length n, we need to specify a set of 2^n code blocks. If n is large, it is not really feasible to write down a list of all the code blocks, and to check all their Hamming distances by hand. For example, a code of message length $n = 10$ requires $2^{10} = 1024$ code blocks, and we need to check more than half a million Hamming distances. Instead, we will focus on a particular class of codes that is much easier to describe. These are the linear codes.

Definition 9.59: Linear code

*A **linear code** is a subspace of $\mathbb{Z}_2^n$. If the subspace is k-dimensional, then the code has message length k and block length n.*

The advantage of a linear code is that to specify the code blocks, we only need to list k basis elements, rather than all 2^k elements of the code.

Example 9.60: Linear code

Show that the code from Example 9.52 is linear. What is a basis for the code?

Solution. Consider all linear combinations of 11100 and 00111 (with scalars in $\mathbb{Z}_2$):

$$\begin{aligned}
0(11100)+0(00111) &= 00000,\\
0(11100)+1(00111) &= 00111,\\
1(11100)+0(00111) &= 11100,\\
1(11100)+1(00111) &= 11011.
\end{aligned}$$

This shows that the code of Example 9.52 is span$\{11100, 00111\}$, and hence a subspace of $\mathbb{Z}_2^5$. A basis for the code is $\{11100, 00111\}$ ♠

From Section 5.5, we know that every subspace of $\mathbb{Z}_2^n$, and therefore every linear code, is the column space of some matrix G, and also the null space of some matrix H. Such matrices are called a *generator matrix* and a *check matrix* for the code, respectively.

Definition 9.61: Generator matrix and check matrix

Consider a linear code C with block length n and message length k, i.e., a k-dimensional subspace of $\mathbb{Z}_2^n$.

- *A **generator matrix** for the code is an $n \times k$-matrix G such that C is the column space of G.*
- *A **check matrix** for the code is an $(n-k) \times n$-matrix H such that C is the null space of H.*

Example 9.62: Generator matrix and check matrix

Find a generator matrix and check matrix for the linear code from Example 9.52.

Solution. We already found in Example 9.60 that $\{11100, 00111\}$ is a basis for this code. We can obtain a generator matrix by using the basis vectors as columns:

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

To find a check matrix, assume that $[a,b,c,d,e]$ is a row of the check matrix. Then for every code block $\mathbf{v} \in C$, we must have $[a,b,c,d,e]\,\mathbf{v} = 0$. Since the code blocks are spanned by 11100 and 00111, it suffices to consider the two equations

$$\begin{bmatrix} a & b & c & d & e \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1a + 1b + 1c + 0d + 0e = 0$$

and

$$\begin{bmatrix} a & b & c & d & e \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0a + 0b + 1c + 1d + 1e = 0.$$

Solving this system of equations, we find that the following is a basis for the solution space:

$$\left\{ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix} \right\}$$

We can use these basic solutions as the rows of the check matrix:

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

♠

Encoding and the generator matrix

Let C be a linear code with generator matrix G. Since C is the column space of the generator matrix, there is a simple encoding method: we can simply multiply the generator matrix G by a message block to obtain the corresponding code block. In other words, if $\mathbf{u} \in \mathbb{Z}_2^k$ is a message block, we can use $\mathbf{v} = G\mathbf{u} \in \mathbb{Z}_2^n$ as the code block.

Example 9.63: Using the generator matrix for encoding

Use the generator matrix

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

to encode the message 01 11 10.

Solution. The message blocks are $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We obtain the corresponding code blocks by multiplication with the generator matrix:

$$\mathbf{v}_1 = G\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = G\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = G\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the encoded message is 00111 11011 11100. Note that this is the same answer as in Example 9.52(a). ♠

Decoding and the check matrix

Let C be a linear code with check matrix H. By definition, C is the null space of the check matrix. This means that a vector $\mathbf{v} \in \mathbb{Z}_2^n$ is a valid code block (i.e., a code block without errors) if and only if $H\mathbf{v} = \mathbf{0}$. So we can easily use the check matrix to determine whether a code block contains errors or not.

However, something even better is true. The value of $H\mathbf{v}$, when it is not zero, tell us not only *that* an error has occurred, but also *which* error has occurred! To see why, consider a code block $\mathbf{v}'$ possibly containing errors. Then $\mathbf{v}' = \mathbf{v} + \mathbf{e}$, where $\mathbf{v}$ is a valid code block and $\mathbf{e}$ is an **error pattern**. The error pattern is the vector $\mathbf{e}$ that has a 1 in every component in which an error occurred, and a 0 everywhere else. Then we have

$$H\mathbf{v}' = H(\mathbf{v} + \mathbf{e}) = H\mathbf{v} + H\mathbf{e} = \mathbf{0} + H\mathbf{e} = H\mathbf{e}.$$

Therefore, the value of $H\mathbf{v}'$ only depends on the error pattern, and not on $\mathbf{v}$. The value $\mathbf{s} = H\mathbf{v}' = H\mathbf{e}$ is called the **syndrome** of the error. A code is error correcting, for a class of error patterns, if and only if each such error pattern has a different syndrome. In that case, we can simply make a table of all the syndromes and corresponding error patterns, as an efficient method for correcting errors. Such a table is called a **syndrome table** and the corresponding decoding method is called **syndrome decoding**.

Example 9.64: Syndrome decoding

Consider the linear code of Example 9.52 with check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Make a syndrome table for all single-bit errors. Then use your syndrome table to decode the message 11101 01000 11011 11111 00011.

Solution. Since we are only interested in single-bit errors, there are six error patterns to consider: 00000 (no error), 10000, 01000, 00100, 00010, and 00001. For each of these error patterns $\mathbf{e}$, we compute the corresponding syndrome $H\mathbf{e}$. For example,

$$H\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad H\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad H\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and so on. We put this information in a table (writing the vectors horizontally as usual in this section). This is the syndrome table:

Error pattern $\mathbf{e}$	Syndrome $H\mathbf{e}$
00000	000
10000	111
01000	100
00100	011
00010	010
00001	001

To decode the message 11101 01000 11011 11111 00011, we first calculate the syndrome of each code block by multiplying it by the check matrix. The syndrome table then tells us the corresponding error pattern, which we can use to correct the code block.

For example, consider the first code block 11101. Multiplying by H, we get the syndrome 001:

$$H\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The syndrome table then tells us that the corresponding error pattern is 00001. Therefore, the corrected code block is $11101+00001=11100$. The decoded block is 10. We proceed in the same way for all the code blocks:

Received code block $\mathbf{v}'$	Syndrome $H\mathbf{v}'$	Error pattern $\mathbf{e}$	Corrected code block $\mathbf{v}=\mathbf{v}'+\mathbf{e}$	Decoded
11101	001	00001	11100	10
01000	100	01000	00000	00
11011	000	00000	11011	11
11111	011	00100	11011	11
00011	011	00100	00111	01

Therefore, the decoded message is 10 00 11 11 01. Note that this is the same answer we got in Example 9.58. But the syndrome table gives a more systematic method of finding the error patterns, which we previously had to do by guessing, or by comparing to all possible code blocks. ♠

Example 9.65: An invalid code block

Using the code of the previous example, suppose you have received the code block 10010. *Can you decode it?*

Solution. The syndrome of this code block is

$$H\begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Since the syndrome 101 is not in our syndrome table, it is not the syndrome of any single-bit error. Therefore, the code block 10010 must contain more than one error. Since our code is only 1-error correcting, this error cannot be corrected. (In fact, there are two valid code words within Hamming distance 2, namely 00000 and 11011). ♠

Hamming codes

The syndrome decoding method immediately gives us the following theorem:

Theorem 9.66: Check matrices of 1-error correcting codes

Consider a linear code with check matrix H. Then the code is 1-error correcting if and only if all columns of the check matrix are non-zero and distinct.

Proof. The syndrome of the zero error pattern is always $H\mathbf{0}=\mathbf{0}$. Let $\mathbf{e}_i$ be the error pattern containing a single bit error in the i^{th} component. Then its syndrome is $H\mathbf{e}_i$, which is the i^{th} column of H.

If H has distinct, non-zero columns, then each possible single bit error has a different syndrome. Therefore, the errors can be corrected.

Conversely, if H has a column that is zero, or two columns that are equal, then the corresponding error patterns have the same syndrome, and therefore cannot be corrected. ♠

This theorem was discovered by Richard Hamming in 1950. Hamming then realized that the best single-error correcting binary codes can be constructed by letting the check matrix have *all* possible non-zero columns. The resulting codes are called Hamming codes.

Definition 9.67: Hamming code

Let $r \geq 2$ The r^{th} ***Hamming code*** *is a linear code whose check matrix H is an $r \times (2^r - 1)$-matrix that has all possible non-zero column vectors as its columns.*

From the size of the check matrix, we know that the code length is $n = 2^r - 1$. Since the check matrix has rank r, its null space has dimension $k = n - r$. Thus, the r^{th} Hamming code is an $(n, k, 3)$-code, where $n = 2^r - 1$ and $k = 2^r - 1 - r$.

It is customary to order the columns of the check matrix so that the last r columns are the standard basis vectors. In other words, the check matrix is usually taken to be of the form

$$H = \left[\begin{array}{c|c} A & I \end{array}\right]$$

where I is the $r \times r$-identity matrix and A is the $r \times k$-matrix consisting of the remaining columns of H. In that case, we can take the $n \times k$-matrix

$$G = \left[\begin{array}{c} I \\ \hline A \end{array}\right]$$

as the generator matrix, where I is the $k \times k$-identity matrix. Specifically, for this choice of G, we have $HG = 0$, ensuring that the column space of G is contained in the null space of H. Moreover, since $\operatorname{rank} G = k$, the column space of G is k-dimensional, and therefore equal to the null space of H, so that G is indeed a correct generator matrix for the code.

Example 9.68: Hamming code for $r = 3$

Construct a check matrix and generator matrix for the Hamming code with $r = 3$. Then encode the message 0101 1110 0111.

Solution. The check matrix must be a 3×7-matrix whose columns are all the possible non-zero vectors of length 3. Moreover, we will follow the convention of using the standard basis vectors as the last 3 columns. Such a matrix is

$$H = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

The corresponding generator matrix G is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

To encode the message 0101 1110 0111, we multiply the generator matrix by each code block:

$$G\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad G\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the encoded message is 0101010 1110000 0111001. Note that the fact that the top part of the generator matrix G is the identity matrix has the pleasant effect that the k first bits of each code block are the corresponding message block. This makes decoding especially convenient (after any errors have been corrected first). ♠

Example 9.69: Decoding a Hamming code

Using the Hamming $(7,4,3)$-code of the previous example, correct the errors in the message 1011101 1101001 0001101 1110000 *and decode it.*

Solution. We do not need to make a syndrome table, because the syndromes are exactly the columns of the check matrix H. More precisely, the i^{th} column of the check matrix is the syndrome of the error pattern containing a single-bit error in the i^{th} bit. We use the check matrix

$$H = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

to calculate the syndrome of each code block:

$$H\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad H\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad H\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad H\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From these syndromes, we can read off the error locations and correct the errors:

Received code block	Syndrome	Error position	Error pattern	Corrected code block	Decoded
1011101	111	bit 4	0001000	1010101	1010
1101001	101	bit 2	0100000	1001001	1001
0001101	010	bit 6	0000010	0001111	0001
1110000	000	none	0000000	1110000	1110

Thus, the decoded message is 1010 1001 0001 1110. ♠

Example 9.70: Hamming code for $r = 2$

What is the Hamming code for $r = 2$? Have you seen this code before?

Solution. The check matrix and generator matrix for the Hamming code with $r = 2$ are

$$H = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \quad \text{and} \quad G = \left[\begin{array}{c} 1 \\ \hline 1 \\ 1 \end{array}\right].$$

The code has block size $n = 3$ and message length $k = 1$. The encoding function is

$$G\left[\ 0\ \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad G\left[\ 1\ \right] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This is exactly the 3-repetition code. ♠

We end this section by remarking that the Hamming codes are very efficient. The rate of the r^{th} Hamming code is

$$\frac{k}{n} = \frac{2^r - 1 - r}{2^r - 1} = 1 - \frac{r}{2^r - 1},$$

which is very close to 1 when r is large. The following table lists the block sizes, message sizes, and rates of all Hamming codes up to $r = 8$.

r	Block size n	Message size k	Rate
2	3	1	0.333
3	7	4	0.571
4	15	11	0.733
5	31	26	0.839
6	63	57	0.905
7	127	120	0.945
8	255	247	0.969

Of course, since the codes are only 1-error correcting, one cannot increase the block size indefinitely, or else the probability of having two or more errors in a block becomes too large. There exist more sophisticated error correcting codes with larger Hamming distances, which can correct many errors per code block. You might learn more about such codes in a course on applied modern algebra.

Exercises

Exercise 9.5.1 *Encode the message* 0110 *using the 3-repetition code. Decode the message* 100 011 000 110 101 110 *after correcting single-bit errors.*

Exercise 9.5.2 *Use the code of Example 9.52 to encode the message* 01 11 10 00 01. *Use the syndrome method to correct all single-bit errors in the message* 10100 00001 11001. *What is the decoded message?*

Exercise 9.5.3 *Consider the code*

$$C = \{000000, 011100, 111010, 101001, 100110, 110101, 010011, 001111\}.$$

Is this a linear code? What are the message length and block length of this code? What is its Hamming distance? How many errors per code block can this code detect? How many can it correct?

Exercise 9.5.4 *Consider a linear code with check matrix*

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Find a generator matrix for this code (hint: the columns of G should form a basis for the null space of H). List all possible code blocks. What is the Hamming distance of the code? Make a syndrome table for this code.

Exercise 9.5.5 *Use the generator matrix*

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to encode the message 111 110 101 001. *Find a check matrix for this code and make a syndrome table. Is the code 1-error correcting? Use the syndrome method to correct and decode the message* 111000 101111 100111 110100.

Exercise 9.5.6 *Construct check and generator matrices of a Hamming code for $r = 4$. What is the block length and message length of this code? Encode the message* 00110011000 10100000001. *Decode the message* 001000010010000 010000001000001 *after correcting single-bit errors.*

10. Linear transformation of vector spaces

10.1 Definition and examples

Outcomes

A. *Understand the definition of a linear transformation in the context of vector spaces.*

B. *Determine whether a function is linear or not.*

In Chapter 6, we defined a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to be a function that preserves addition and scalar multiplication. We now revisit this concept in the more general setting of vector spaces V and W.

Definition 10.1: Linear transformation

Let V and W be vector spaces over some field K. A function $T : V \to W$ is called a ***linear transformation*** *from V to W if it satisfies the following two conditions:*

1. *T preserves addition, i.e., for all $\mathbf{v}, \mathbf{w} \in V$, we have $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$;*
2. *T preserves scalar multiplication, i.e, for all $\mathbf{v} \in V$ and $k \in K$, we have $T(k\mathbf{v}) = kT(\mathbf{v})$.*

A linear transformation is also sometimes called a ***linear function*** *or a* ***linear map****. A linear transformation $T : V \to V$ (i.e., when $V = W$) is also sometimes called an* ***operator****.*

Our first example of a linear transformation is a matrix transformation. We have already seen this in Section 6.2.

Example 10.2: Matrix transformation

Let A be an $m \times n$-matrix. Then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation, called a ***matrix transformation****. This was proved in Proposition 6.4.*

There are many interesting examples of linear transformations on vector spaces other than $\mathbb{R}^n$. We will consider a few such examples.

Example 10.3: Derivative operator

Let $\mathbf{P}_n$ *be the vector space of real polynomials of degree at most* n. *The function* $D : \mathbf{P}_n \to \mathbf{P}_{n-1}$ *is defined by*

$$D(p(x)) = p'(x),$$

where $p'(x)$ *denotes the derivative of the polynomial* $p(x)$. *The function* D *is called the* ***derivative operator***.

(a) Compute $D(x^3)$, $D(2x^2+x)$, *and* $D(ax^3+bx^2+cx+d)$.

(b) Show that D *is a linear transformation.*

Solution.

(a) In each case, we simply take the derivative:

$$\begin{aligned} D(x^3) &= 3x^2, \\ D(2x^2+x) &= 4x+1, \\ D(ax^3+bx^2+cx+d) &= 3ax^2+2bx+c. \end{aligned}$$

(b) First, we note that if $p(x)$ is a polynomial of degree at most n, then its derivative $p'(x)$ is a polynomial of degree at most $n-1$. Therefore, the derivative operator D is a well-defined function from $\mathbf{P}_n$ to $\mathbf{P}_{n-1}$. To show that it preserves addition, consider any two polynomials $p(x), q(x) \in \mathbf{P}_n$. From calculus, we know that the derivative of $p(x)+q(x)$ is $p'(x)+q'(x)$. Therefore,

$$D(p(x)+q(x)) = (p(x)+q(x))' = p'(x)+q'(x) = D(p(x))+D(q(x)),$$

and so D preserves addition. To show that it preserves scalar multiplication, consider $p(x) \in \mathbf{P}_n$ and $k \in \mathbb{R}$. From calculus, we know that the derivative of $kp(x)$ is $kp'(x)$, and therefore

$$D(kp(x)) = (kp(x))' = kp'(x) = kD(p(x)).$$

Hence, D preserves scalar multiplication. It follows that D is a linear transformation.

♠

It is important to understand that we are not claiming that the derivative $p'(x)$ of a polynomial $p(x)$ is a linear function. It is of course a polynomial. Rather, what the above example shows is that the act of *taking* the derivative is a linear operation, i.e., the derivative of a sum is the sum of the derivatives, and the derivative of a constant times a function is a constant times the derivative.

Example 10.4: A differential equation

Solve the equation $p(x) = x^3 + D(p(x))$, *where* $D : \mathbf{P}_3 \to \mathbf{P}_2$ *is the derivative operator of Example 10.3.*

Solution. Every element of P_3 is of the form $p(x) = ax^3+bx^2+cx+d$. We can write the equation $p(x) = x^3 + D(p(x))$ as

$$(ax^3+bx^2+cx+d) = x^3 + (3ax^2+2bx+c).$$

For the left-hand side and right-hand side to be equal, we must have $a = 1$, $b = 3a$, $c = 2b$, and $d = c$. This yields the unique solution $(a,b,c,d) = (1,3,6,6)$, or $p(x) = x^3 + 3x^2 + 6x + 6$. ♠

Example 10.5: The shift and unshift operators

Consider the vector space $\mathbf{Seq}_K$ *of infinite sequences of elements of* K. *The function* $\operatorname{shift} : \mathbf{Seq}_K \to \mathbf{Seq}_K$ *is defined by shifting the entire sequence to the left and dropping the first element:*

$$\operatorname{shift}(a_0, a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3, a_4, \ldots).$$

The function $\operatorname{unshift} : \mathbf{Seq}_K \to \mathbf{Seq}_K$ *is defined by shifting the entire sequence to the right and adding* 0 *as the new first element:*

$$\operatorname{unshift}(a_0, a_1, a_2, a_3, \ldots) = (0, a_0, a_1, a_2, \ldots).$$

(a) *Compute* $\operatorname{shift}(1,2,3,\ldots)$, $\operatorname{unshift}(\operatorname{shift}(1,1,1,\ldots))$, *and* $\operatorname{shift}(\operatorname{unshift}(1,1,1,\ldots))$.

(b) *Show that* shift *and* unshift *are linear transformations.*

Solution.

(a) We have

$$\begin{aligned}
\operatorname{shift}(1,2,3,4,\ldots) &= (2,3,4,5,\ldots), \\
\operatorname{unshift}(\operatorname{shift}(1,1,1,1,\ldots)) &= \operatorname{unshift}(1,1,1,1,\ldots) = (0,1,1,1,\ldots), \\
\operatorname{shift}(\operatorname{unshift}(1,1,1,1,\ldots)) &= \operatorname{shift}(0,1,1,1,\ldots) = (1,1,1,1,\ldots).
\end{aligned}$$

(b) To show that shift is a linear transformation, we show that it preserves addition and scalar multiplication. Let $a = (a_0, a_1, a_2, \ldots)$ and $b = (b_0, b_1, b_2, \ldots)$. Then

$$\begin{aligned}
\operatorname{shift}(a+b) &= \operatorname{shift}(a_0+b_0,\ a_1+b_1,\ a_2+b_2,\ \ldots) \\
&= (a_1+b_1,\ a_2+b_2,\ a_3+b_3,\ \ldots) \\
&= (a_1,\ a_2,\ a_3,\ \ldots) + (b_1,\ b_2,\ b_3,\ \ldots) \\
&= \operatorname{shift}(a) + \operatorname{shift}(b), \\
\operatorname{shift}(ka) &= \operatorname{shift}(ka_0,\ ka_1,\ ka_2,\ \ldots) \\
&= (ka_1,\ ka_2,\ ka_3,\ \ldots) \\
&= k(a_1,\ a_2,\ a_3,\ \ldots) \\
&= k \operatorname{shift}(a).
\end{aligned}$$

Therefore, shift is a linear transformation. The proof for unshift is similar.

♠

Example 10.6: Properties of shift and unshift

Show that for all sequences $a \in \mathbf{Seq}_K$, *we have* $\operatorname{shift}(\operatorname{unshift}(a)) = a$. *On the other hand, show that in general,* $\operatorname{unshift}(\operatorname{shift}(a)) \neq a$.

Solution. For $a = (a_0, a_1, a_2, \ldots)$, we have

$$\text{shift}(\text{unshift}(a)) = \text{shift}(\text{unshift}(a_0, a_1, a_2, \ldots)) = \text{shift}(0, a_0, a_1, \ldots) = (a_0, a_1, a_2, \ldots) = a.$$

On the other hand, we have

$$\text{unshift}(\text{shift}(a)) = \text{unshift}(\text{shift}(a_0, a_1, a_2, \ldots)) = \text{unshift}(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots).$$

The latter is not equal to a unless $a_0 = 0$. ♠

Example 10.7: Recurrence as a linear equation

Find the general solution to the following equation, where $a \in \mathbf{Seq}_K$:

$$\text{shift}(\text{shift}(a)) = \text{shift}(a) + a.$$

Where have you seen this equation before?

Solution. For $a = (a_0, a_1, a_2, \ldots)$, we have

$$\begin{aligned} \text{shift}(\text{shift}(a)) &= (a_2, a_3, a_4, \ldots), \\ \text{shift}(a) &= (a_1, a_2, a_3, \ldots), \\ a &= (a_0, a_1, a_2, \ldots). \end{aligned}$$

Therefore a is a solution of the equation $\text{shift}(\text{shift}(a)) = \text{shift}(a) + a$ if and only if

$$\begin{aligned} a_2 &= a_1 + a_0, \\ a_3 &= a_2 + a_1, \\ a_4 &= a_3 + a_2, \end{aligned}$$

and so on. In other words, a is a solution if and only if $a_{n+2} = a_{n+1} + a_n$ holds for all $n \geq 0$. This is nothing but the recurrence relation of Example 9.29. We already calculated the general solution in Example 9.44. The general solution is

$$a = (x,\ y,\ x+y,\ x+2y,\ 2x+3y,\ 3x+5y,\ \ldots),$$

and a basis for the solution space is

$$\{(1,0,1,1,2,3,\ldots),\ (0,1,1,2,3,5,\ldots)\}.$$

♠

We conclude this section by stating some elementary properties of linear transformations. "Elementary" means that these properties follow directly from the definition, i.e., from the fact that linear transformations preserve addition and scalar multiplication.

Proposition 10.8: Properties of linear transformations

Let V and W be vector spaces over a field K, and let $T : V \to W$ be a linear transformation. Then

- *T preserves the zero vector: $T(\mathbf{0}) = \mathbf{0}$.*
- *T preserves additive inverses: $T(-\mathbf{v}) = -T(\mathbf{v})$.*
- *T preserves linear combinations:*

$$T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k) \;=\; a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_kT(\mathbf{v}_k).$$

Proof. To prove the first property, let $k = 0$ in the equation $T(k\mathbf{v}) = kT(\mathbf{v})$. Since $0\mathbf{v} = \mathbf{0}$ and $0T(\mathbf{v}) = \mathbf{0}$ by Proposition 9.9, we therefore have $T(\mathbf{0}) = \mathbf{0}$. Similarly, to prove the second property, let $k = -1$ in the equation $T(k\mathbf{v}) = kT(\mathbf{v})$. Finally, the third property is a direct consequence of the fact that T preserves addition and scalar multiplication:

$$\begin{aligned} T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k) &= T(a_1\mathbf{v}_1) + T(a_2\mathbf{v}_2) + \ldots + T(a_k\mathbf{v}_k) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_kT(\mathbf{v}_k). \end{aligned}$$

♠

Exercises

Exercise 10.1.1 *Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. For each of these functions T, show that it is not linear either by showing that T does not preserve addition, or by showing that it does not preserve scalar multiplication, or by showing that it does not preserve the zero vector.*

(a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z + 1 \\ 2y - 3x + z \end{bmatrix}$.

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y^2 + 3z \\ 2y + 3x + z \end{bmatrix}$.

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin x + 2y + 3z \\ 2y + 3x + z \end{bmatrix}$.

Exercise 10.1.2 *Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. For each function T, show that T is a linear transformation. Do this by showing that T is a matrix transformation, i.e., find a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$.*

(a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y-3x+z \end{bmatrix}.$

(b) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x+2y+z \\ 3x-11y+2z \end{bmatrix}.$

(c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y+z \\ x+2y+6z \end{bmatrix}.$

(d) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y-5x+z \\ x+y+z \end{bmatrix}.$

Exercise 10.1.3 *Which of the following functions are linear transformations?*

(a) *The function* $f : \mathbf{Seq}_K \to \mathbf{Seq}_K$ *defined by* $f((a_0, a_1, a_2, \ldots)) = (-a_1, -a_3, -a_5, \ldots)$. *In words, the function* f *removes the even-numbered elements, and negates the odd-numbered elements.*

(b) *The function* $f : \mathbf{M}_{2,2} \to \mathbf{M}_{2,2}$ *defined by* $f(A) = AB$. *Here,* $\mathbf{M}_{2,2}$ *is the vector space of* 2×2*-matrices with real entries, and* B *is a fixed matrix.*

(c) *The function* $f : \mathbf{M}_{2,2} \to \mathbf{M}_{2,2}$ *defined by* $f(A) = A+B$, *where* B *is a fixed matrix.*

(d) *The function* $f : \mathbf{M}_{2,2} \to \mathbf{M}_{2,2}$ *defined by* $f(A) = A^T$.

Exercise 10.1.4 *Recall the vector space* $\mathbf{P}$ *of polynomials with coefficients in a field* K. *Consider the function* $M : \mathbf{P} \to \mathbf{P}$ *defined by* $M(p(x)) = xp(x)$.

(a) *Compute* $M(x^3)$, $M(2x^2+x)$, *and* $M(ax^2+bx+c)$.

(b) *Show that* M *is a linear transformation.*

Exercise 10.1.5 *Recall the vector space* $\mathbf{P}$ *of polynomials with coefficients in a field* K. *Consider the function* $S : \mathbf{P} \to \mathbf{P}$ *defined by* $S(p(x)) = p(x+1)$.

(a) *Compute* $S(x^3)$, $S(2x^2+x)$, *and* $S(ax^2+bx+c)$.

(b) *Show that* S *is a linear transformation.*

Exercise 10.1.6 *Consider the shift function* $\operatorname{shift} : \mathbf{Seq}_K \to \mathbf{Seq}_K$ *from Example 10.5. Find a basis for the solution space of each of the following equations:*

(a) $\operatorname{shift}(a) = a$.

(b) $\operatorname{shift}(a) = -a$.

(c) $\operatorname{shift}(\operatorname{shift}(a)) = \operatorname{shift}(a) + 2a$.

10.2 The algebra of linear transformations

Outcomes

A. *Use algebraic properties of linear transformations to manipulate expressions.*

Two linear transformations are considered to be equal if they act in the same way on all vectors. This is the content of the following definition.

Definition 10.9: Equal transformations

Let S and T be linear transformations from V to W. We say that S and T are **equal**, *and we write $S = T$, if for all $\mathbf{v} \in V$,*

$$S(\mathbf{v}) = T(\mathbf{v}).$$

We now consider several operations on linear transformations. These include addition and scalar multiplication of linear transformations, as well as the zero transformation.

Definition 10.10: Addition and scalar multiplication of linear transformations

Let V and W be vector spaces over a field K.

(a) *The* **zero transformation** $0 : V \to W$ *is defined by* $0(\mathbf{v}) = \mathbf{0}$ *for all* $\mathbf{v} \in V$.

(b) *If $T, S : V \to W$ are linear transformations, then their* **sum** $T + S : V \to W$ *is the linear transformation defined by*

$$(T + S)(\mathbf{v}) = T(\mathbf{v}) + S(\mathbf{v})$$

for all $\mathbf{v} \in V$.

(c) *If $T : V \to W$ is a linear transformation and $k \in K$, the linear transformation $kT : V \to W$ is defined by*

$$(kT)(\mathbf{v}) = k(T(\mathbf{v}))$$

for all $\mathbf{v} \in V$.

Proposition 10.11: Addition and scalar multiplication of linear transformations

If $T, S : V \to W$ are linear transformations, then so are $T + S$ and kT.

Proof. To show that $T + S$ is linear, we must verify that it preserves addition and scalar multiplication. Let $\mathbf{v}, \mathbf{u} \in V$. Then we have

$$\begin{aligned}
(T+S)(\mathbf{v}+\mathbf{u}) &= T(\mathbf{v}+\mathbf{u}) + S(\mathbf{v}+\mathbf{u}) && \text{by definition of } T+S, \\
&= (T(\mathbf{v}) + T(\mathbf{u})) + (S(\mathbf{v}) + S(\mathbf{u})) && \text{by linearity of } T \text{ and } S, \\
&= (T(\mathbf{v}) + S(\mathbf{v})) + (T(\mathbf{u}) + S(\mathbf{u})) && \text{by the associative and commutative laws of vectors,} \\
&= (T+S)(\mathbf{v}) + (T+S)(\mathbf{u}) && \text{by definition of } T+S.
\end{aligned}$$

Therefore, $T+S$ preserves addition. Also, for $\mathbf{v}\in V$ and $\ell\in K$, we have

$$\begin{aligned}(T+S)(\ell\mathbf{v}) &= T(\ell\mathbf{v})+S(\ell\mathbf{v}) && \text{by definition of } T+S\\ &= \ell T(\mathbf{v})+\ell S(\mathbf{v}) && \text{by linearity of } T \text{ and } S\\ &= \ell(T(\mathbf{v})+S(\mathbf{v})) && \text{by the distributive law over vector addition}\\ &= \ell((T+S)(\mathbf{v})) && \text{by definition of } T+S.\end{aligned}$$

Therefore, $T+S$ preserves scalar multiplication, and hence $T+S$ is a linear transformation. The proof that kT is a linear transformation is left as an exercise. ♠

There operations satisfy the following properties:

Proposition 10.12: Properties of addition and scalar multiplication of linear transformations

Let V,W be vector spaces over a field K, let $T,S,R : V\to W$ be linear transformations, and let $k,\ell\in K$ be scalars. Then the following hold:

- *Commutative law of addition: $T+S=S+T$.*
- *Associative law of addition: $(T+S)+R=T+(S+R)$.*
- *The existence of an additive unit: there exists an element $\mathbf{0}\in V$ such that for all T, $T+0=T$.*
- *The law of additive inverses: $T+(-T)=0$.*
- *The distributive law over vector addition: $k(T+S)=kT+kS$.*
- *The distributive law over scalar addition: $(k+\ell)T=kT+\ell T$.*
- *The associative law for scalar multiplication: $k(\ell T)=(k\ell)T$.*
- *The rule for multiplication by one: $1T=T$.*

But these 8 properties are just the vector space laws (A1)–(A4) and (SM1)–(SM4)! Therefore, the set of linear transformations from V to W, with the above operations of addition and scalar multiplication, forms a vector space.

Definition 10.13: Vector space of linear transformations

Let V,W be vector spaces over a field K. We define $\mathbf{Lin}_{V,W}$ to be the vector space of all linear transformations from V to W, with the above operations of addition and scalar multiplication.

Another important operation is the composition of linear transformations. We have already encountered this in Definition 6.16 for the case of $\mathbb{R}^n$. Here, we generalize it to linear transformations of arbitrary vector spaces. We also consider the identity transformation on a vector space, which forms the unit for composition.

Definition 10.14: Composition of linear transformations

Let V, U, W be vector spaces over a field K, and let $S : V \to U$ and $T : U \to W$ be linear transformations.

(a) *The **composition** $T \circ S : V \to W$ is the linear transformation defined by*

$$(T \circ S)(\mathbf{v}) = T(S(\mathbf{v}))$$

for all $\mathbf{v} \in V$.

(b) *The **identity transformation** $1_V : V \to V$ is defined by $1_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. We often omit the subscript and just write $1 : V \to V$ when V is clear from the context.*

Composition of linear transformations satisfies the following properties:

Proposition 10.15: Properties of composition of linear transformations

Composition of linear transformations satisfies the following properties, for all R, S, T as appropriate for each law:

- *Associative law of composition: $(T \circ S) \circ R = T \circ (S \circ R)$.*
- *Unit laws of composition: $T \circ 1_V = T$ and $1_W \circ T = T$, where $T : V \to W$.*
- *Distributive laws: $(T + S) \circ R = T \circ R + S \circ R$ and $P \circ (T + S) = P \circ T + P \circ S$.*
- *Zero laws: $0 \circ R = 0$ and $R \circ 0 = 0$.*
- *Compatibility with scalar multiplication: $(kT) \circ S = k(T \circ S) = T \circ (kS)$, where $k \in K$.*

Finally, we consider the notion of the inverse of a linear transformation.

Definition 10.16: Inverse of a linear transformation

*Let V, U be vector spaces over a field K, and let $S : V \to U$ and $T : U \to V$ be linear transformations. We say that S and T are **inverses** if*

$$T \circ S = 1_V \quad \text{and} \quad S \circ T = 1_U.$$

In this case, we also write $S = T^{-1}$ and $T = S^{-1}$.

Proposition 10.17: Uniqueness of inverses

Inverses are unique. In other words, if S and S' are two inverses of T, then $S = S'$.

Proof. Suppose both S and S' are inverses of T. Consider $S \circ T \circ S'$. Since S and T are inverses, this is equal to S', but since T and S' are inverses, it is also equal to S. Therefore, $S = S'$. ♠

Proposition 10.18: Properties of inverses of linear transformations

- *If $S : V \to U$ and $T : U \to W$ are both invertible, then so is $T \circ S$, and we have*
$$(T \circ S)^{-1} = S^{-1} \circ T^{-1}.$$
- $1^{-1} = 1$.
- *If $S : V \to U$ is invertible and k is a non-zero scalar, then*
$$(kS)^{-1} = k^{-1} S^{-1}.$$

Exercises

Exercise 10.2.1 *Suppose V and W are vector spaces, $T, S : V \to W$ and $R, Q : W \to W$ are linear transformations, and k is a scalar. Which of the following equalities are valid?*

(a) $R \circ (T + kS) = R \circ T + kR \circ S$.

(b) $(R + Q) \circ (R + Q) = R \circ R + 2R \circ Q + Q \circ Q$.

(c) $(R + Q) \circ (T + S) = R \circ T + R \circ S + Q \circ T + Q \circ S$.

(d) *If T, S, and $T + S$ are invertible, then* $(T + S)^{-1} = T^{-1} + S^{-1}$.

Exercise 10.2.2 *Finish the proof of Proposition 10.11, i.e., prove that if $T : V \to W$ is a linear transformation and k a scalar, then $kT : V \to W$ is a linear transformation.*

10.3 Linear transformations defined on a basis

Outcomes

A. *Check whether two linear transformations are equal by considering their action on a spanning set.*

B. *Specify a linear transformation by considering its action on a basis.*

Recall that, by definition, two linear transformations $S, T : V \to W$ are equal if and only if for all $\mathbf{v} \in V$, we have $S(\mathbf{v}) = T(\mathbf{v})$. However, this is not a very practical way of checking whether $S = T$, as it theoretically requires checking $S(\mathbf{v}) = T(\mathbf{v})$ for each one of infinitely many vectors $\mathbf{v}$. The following proposition states that it is sufficient to check the actions of S and T on a spanning set of vectors.

Proposition 10.19: Equality of linear transformations

Let V and W be vector spaces over a field K, and let $S, T : V \to W$ be linear transformations. Moreover, let $X \subseteq V$ be a spanning set of V, i.e., such that $V = \operatorname{span} X$. If $S(\mathbf{v}) = T(\mathbf{v})$ for all $\mathbf{v} \in X$, then $S = T$.

Proof. Assume that $S(\mathbf{v}) = T(\mathbf{v})$ holds for all $\mathbf{v} \in X$. To show that $S = T$, let $\mathbf{u} \in V$ be an arbitrary vector. Since X is a spanning set, we can write $\mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$, for some $\mathbf{v}_1, \ldots, \mathbf{v}_n \in X$ and $a_1, \ldots, a_n \in K$. By assumption, $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ for all i, because $\mathbf{v}_i \in X$. Then we have

$$\begin{aligned} S(\mathbf{u}) &= S(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) \\ &= a_1S(\mathbf{v}_1) + \ldots + a_nS(\mathbf{v}_n) \\ &= a_1T(\mathbf{v}_1) + \ldots + a_nT(\mathbf{v}_n) \\ &= T(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) \\ &= T(\mathbf{u}). \end{aligned}$$

Since $\mathbf{u} \in V$ was arbitrary, it follows that $S = T$. ♠

Therefore, if we know how a linear transformation acts on a spanning set (and in particular, on a basis), then we know how it acts on the entire space. There is also a kind of converse to this: given a basis of V, we can map the basis vectors to any elements of W we like, and this will always determine a unique linear transformations. This is the content of the following theorem.

Theorem 10.20: Linear transformation defined on a basis

Suppose V and W are vector spaces over a field K. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of V, and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ are any vectors in W (which may or may not be distinct). Then there exists a unique linear transformation $T : V \to W$ such that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \quad T(\mathbf{v}_2) = \mathbf{w}_2, \quad \ldots \quad T(\mathbf{v}_n) = \mathbf{w}_n.$$

Proof. To show that such a linear transformation T exists, we first define a function $T : V \to W$ as follows. Given any $\mathbf{v} \in V$, there exists a unique set of coordinates $a_1, \ldots, a_n \in K$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n.$$

Then define

$$T(\mathbf{v}) = a_1\mathbf{w}_1 + \ldots + a_n\mathbf{w}_n.$$

This defines a function $T : V \to W$. Next, we must check that T is linear. To show that T preserves addition, consider $\mathbf{v}, \mathbf{v}' \in V$, with $\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$ and $\mathbf{v}' = b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n$. Then

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}') &= T((a_1 + b_1)\mathbf{v}_1 + \ldots + (a_n + b_n)\mathbf{v}_n) \\ &= (a_1 + b_1)\mathbf{w}_1 + \ldots + (a_n + b_n)\mathbf{w}_n \end{aligned}$$

$$\begin{aligned} &= (a_1\mathbf{w}_1 + \ldots + a_n\mathbf{w}_n) + (b_1\mathbf{w}_1 + \ldots + b_n\mathbf{w}_n) \\ &= T(\mathbf{v}) + T(\mathbf{v}'). \end{aligned}$$

Therefore, T preserves addition. To show that T preserves scalar multiplication, consider $\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$ and $k \in K$. Then

$$\begin{aligned} T(k\mathbf{v}) &= T(ka_1\mathbf{v}_1 + \ldots + ka_n\mathbf{v}_n) \\ &= ka_1\mathbf{w}_1 + \ldots + ka_n\mathbf{w}_n \\ &= k(a_1\mathbf{w}_1 + \ldots + a_n\mathbf{w}_n) \\ &= kT(\mathbf{v}). \end{aligned}$$

Therefore, T preserves scalar multiplication. It follows that T is linear. Next, we must show that T satisfies the condition of the theorem, i.e., that $T(\mathbf{v}_i) = \mathbf{w}_i$ for each i. But this is clearly the case, because in this case, $a_i = 1$ and $a_j = 0$ for all $j \neq i$. We have shown that there exists a linear function T satisfying all of the conditions required by the theorem.

Finally, the only thing left to show is uniqueness. But this follows from Proposition 10.19. Namely, if T' is another linear transformation such that $T'(\mathbf{v}_i) = \mathbf{w}_i$ for all i, then T and T' agree on $\mathbf{v}_1, \ldots, \mathbf{v}_n$, which is a basis and hence a spanning set. By Proposition 10.19, $T = T'$. ♠

Example 10.21: Linear transformation defined on a basis

Recall that $\{x^2, (x+1)^2, (x+2)^2\}$ *is a basis of* $\mathbf{P}_2$. *Consider the linear function* $T : \mathbf{P}_2 \to \mathbf{M}_{22}$ *defined by*

$$T(x^2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad T((x+1)^2) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad T((x+2)^2) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Find $T(4x)$.

Solution. Let $\mathbf{v}_1 = x^2$, $\mathbf{v}_2 = (x+1)^2$, $\mathbf{v}_3 = (x+2)^2$, $\mathbf{w}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. We must first find a, b, c such that $4x = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$. We do this by solving a system of equations, using the same method as in Example 9.12. We find that $a = -3$, $b = 4$, and $c = -1$. Therefore

$$T(4x) = T(-3\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3) = -3\mathbf{w}_1 + 4\mathbf{w}_2 - \mathbf{w}_3 = \begin{bmatrix} -3 & 1 \\ -1 & 3 \end{bmatrix}.$$

♠

Exercises

Exercise 10.3.1 *Let* $T : \mathbf{P}_2 \to \mathbb{R}$ *be a linear transformation such that*

$$T(x^2) = 1, \quad T(x^2 + x) = 5, \quad \text{and} \quad T(x^2 + x + 1) = -1.$$

Find $T(ax^2+bx+c)$.

Exercise 10.3.2 *Let vectors* $\mathbf{v}_1,\ldots,\mathbf{v}_n \in \mathbb{R}^n$ *and* $\mathbf{w}_1,\ldots,\mathbf{w}_n \in \mathbb{R}^m$ *be given. Let* A *be the matrix whose columns are* $\mathbf{v}_1,\ldots,\mathbf{v}_n$, *and assume that* A^{-1} *exists. Show that there exists a linear transformation* T *such that* $T(\mathbf{v}_i) = \mathbf{w}_i$ *for* $i = 1,\ldots,n$.

10.4 The matrix of a linear transformation

Outcomes

A. *Find the matrix of a linear transformation with respect to general bases in vector spaces.*

In Section 6.2, we saw that linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ are in one-to-one correspondence with $m\times n$-matrices. Here, we will generalize this correspondence to arbitrary finite-dimensional vector spaces. There is an important difference, however. While $\mathbb{R}^n$ comes with a natural coordinate system (i.e., every vector in $\mathbb{R}^n$ has a first component, second component, and so on), there is no distinguished coordinate system on an arbitrary vector space. To define the matrix of a linear transformation $T : V \to W$, we must first choose a basis, or equivalently a coordinate system, for V and for W. Different choices of basis will give rise to different matrices.

Let V be a vector space with basis $B = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$. Recall from Section 5.4.3 that the **coordinates** of a vector $\mathbf{v}$ with respect to the basis B are the unique scalars $a_1,\ldots,a_n$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n.$$

As before, we write

$$[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

to denote the coordinates of $\mathbf{v}$ with respect to the basis B. We will now see how to use bases and coordinates to encode any linear map between finite-dimensional vector spaces as a matrix.

Proposition 10.22: The matrix of a linear transformation

Let V *and* W *be finite-dimensional vector spaces over a field* K. *Assume* B *is a basis of* V *and* C *is a basis of* W. *Let* $T : V \to W$ *be a linear transformation. Then there exists a unique* $m\times n$*-matrix* A *such that for all* $\mathbf{v} \in V$,

$$A[\mathbf{v}]_B = [T\mathbf{v}]_C.$$

Moreover, A *can be computed as follows: the* i^{th} *column of* A *holds the coordinates of* $T(\mathbf{v}_i)$, *where* $\mathbf{v}_i$ *is the* i^{th} *vector of the basis* B, *and the coordinates are computed with respect to the basis* C.

Proof. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. By Theorem 10.20, the linear transformation T is completely determined by the images of the basis vectors, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n) \in W$. Since $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a basis of W, we can write each $T(\mathbf{v}_i)$ as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$:

$$\begin{array}{rcl} T(\mathbf{v}_1) & = & a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \ldots + a_{m1}\mathbf{w}_m, \\ T(\mathbf{v}_2) & = & a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \ldots + a_{m2}\mathbf{w}_m, \\ & \ldots & \\ T(\mathbf{v}_n) & = & a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \ldots + a_{mn}\mathbf{w}_m. \end{array}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Then A is an $m \times n$-matrix. We must prove that it has the desired property, i.e., that $A[\mathbf{v}]_B = [T\mathbf{v}]_C$, for all $\mathbf{v} \in V$. Since both the left-hand side and the right-hand side are linear functions of $\mathbf{v}$, it suffices to check that this property holds for basis vectors. Consider, therefore, one of the basis vectors $\mathbf{v}_i$. Note that $\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 1\mathbf{v}_i + \ldots + 0\mathbf{v}_n$. Therefore, the coordinates of $\mathbf{v}_i$ with respect to the basis B are

$$[\mathbf{v}_i]_B = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the usual i^{th} standard basis vector. On the other hand, since $T(\mathbf{v}_i) = a_{1i}\mathbf{w}_1 + \ldots + a_{mi}\mathbf{w}_m$, we have

$$[T(\mathbf{v}_i)]_C = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = A\mathbf{e}_i.$$

Here, in the last equation, we have used the fact that $A\mathbf{e}_i$ is the same thing as the i^{th} column of A. We therefore have $[T(\mathbf{v}_i)]_C = A\mathbf{e}_i = A[\mathbf{v}_i]_B$, as desired. ♠

Definition 10.23: The matrix of a linear transformation

The matrix A of Proposition 10.22 is called the ***matrix of the linear transformation*** T ***with respect to the bases*** B ***and*** C*. We also write*

$$A = [T]_{C,B}.$$

Therefore,

$$[T]_{C,B}[\mathbf{v}]_B = [T\mathbf{v}]_C$$

for all $\mathbf{v} \in V$.

Example 10.24: Finding the matrix of a linear transformation

Find the matrix of the derivative operator $D : \mathbf{P}_3 \to \mathbf{P}_2$ with respect to the basis $B = \{1, x, x^2, x^3\}$ of $\mathbf{P}_3$ and the basis $C = \{1, x, x^2\}$ of $\mathbf{P}_2$.

Solution. We first find the images of each basis vector of the basis B, and we write each of them as a linear combination of basis vectors from the basis C. Let us denote the basis vectors of B as $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$, and $\mathbf{v}_4 = x^3$, and the basis vectors of C as $\mathbf{w}_1 = 1$, $\mathbf{w}_2 = x$, and $\mathbf{w}_3 = x^2$. We have

$$\begin{array}{lllllllll} D(\mathbf{v}_1) & = & D(1) & = & 0 & = & 0+0x+0x^2 & = & 0\mathbf{w}_1+0\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_2) & = & D(x) & = & 1 & = & 1+0x+0x^2 & = & 1\mathbf{w}_1+0\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_3) & = & D(x^2) & = & 2x & = & 0+2x+0x^2 & = & 0\mathbf{w}_1+2\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_4) & = & D(x^3) & = & 3x^2 & = & 0+0x+3x^2 & = & 0\mathbf{w}_1+0\mathbf{w}_2+3\mathbf{w}_3. \end{array}$$

Therefore, we have

$$A = [D]_{C,B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

♠

Example 10.25: Same linear transformation, different bases

Find the matrix of the derivative operator $D : \mathbf{P}_3 \to \mathbf{P}_2$ *with respect to the basis* $B' = \{1, x+1, x^2+x+1, x^3+x^2+x+1\}$ *of* $\mathbf{P}_3$ *and the basis* $C' = \{1, x-1, x^2-1\}$ *of* $\mathbf{P}_2$.

Solution. This is the same linear transformation as in the previous example, but we are given different bases. Let us denote the basis vectors of B' as $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x+1$, $\mathbf{v}_3 = x^2+x+1$, and $\mathbf{v}_4 = x^3+x^2+x+1$, and the basis vectors of C' as $\mathbf{w}_1 = 1$, $\mathbf{w}_2 = x-1$, and $\mathbf{w}_3 = x^2-1$. We must write each $D(\mathbf{v}_i)$ as a linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, which requires solving a system of linear equations for each of them. We have:

$$\begin{array}{lllllll} D(\mathbf{v}_1) & = & D(1) & = & 0 & = & 0\mathbf{w}_1+0\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_2) & = & D(x+1) & = & 1 & = & 1\mathbf{w}_1+0\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_3) & = & D(x^2+x+1) & = & 2x+1 & = & 3\mathbf{w}_1+2\mathbf{w}_2+0\mathbf{w}_3, \\ D(\mathbf{v}_4) & = & D(x^3+x^2+x+1) & = & 3x^2+2x+1 & = & 6\mathbf{w}_1+2\mathbf{w}_2+3\mathbf{w}_3. \end{array}$$

Therefore, the matrix is

$$[D]_{C',B'} = \begin{bmatrix} 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

♠

The last two examples illustrate that a linear transformation can have many different matrices, because the matrix depends not only on the linear transformation, but also on the given bases. The art of linear algebra often lies in choosing "convenient" bases for a given application. Often, a "convenient" basis is one that gives rise to simple matrices, for example, matrices containing many zeros, or matrices that are diagonal.

Example 10.26: Finding a convenient basis

Let $T : \mathbf{P}_3 \to \mathbf{M}_{2,2}$ be the linear transformation defined by

$$T(ax^3+bx^2+cx+d) = \begin{bmatrix} a+d & b-c \\ b+c & a-d \end{bmatrix}$$

for all $ax^3+bx^2+cx+d \in \mathbf{P}_3$. Let $B = \{x^3, x^2, x, 1\}$ and

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be bases of $\mathbf{P}_3$ and $\mathbf{M}_{2,2}$, respectively.

(a) Find $[T]_{C,B}$.

(b) Find a basis C' of $\mathbf{M}_{2,2}$ such that $[T]_{C',B}$ is the identity matrix.

Solution. (a) We have

$$\begin{aligned}
T(x^3) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
T(x^2) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
T(x) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
T(1) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Therefore,

$$[T]_{C,B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

(b) Since the matrices $T(x^3)$, $T(x^2)$, $T(x)$, and $T(1)$ form a basis of $\mathbf{M}_{2,2}$, we can take C' to consist of these four matrices, i.e.,

$$C' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Then

$$[T]_{C',B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the identity matrix. ♠

We end this section with some properties of matrices of linear transformations.

Proposition 10.27: Properties of matrices of linear transformations

Let V, W, and U be finite-dimensional vector spaces with respective bases B, C, and D. Suppose $T, T' : V \to W$ and $S : W \to U$ are linear transformations. The following hold:

(a) $[S \circ T]_{D,B} = [S]_{D,C} [T]_{C,B}$.

(b) $[1_V]_{B,B} = I$.

(c) T is invertible if and only if $[T]_{C,B}$ is invertible, and in that case, $\left[T^{-1}\right]_{B,C} = ([T]_{C,B})^{-1}$.

(d) $[0]_{C,B} = 0$.

(e) $[T + T']_{C,B} = [T]_{C,B} + [T']_{C,B}$.

(f) $[kT]_{C,B} = k\,[T]_{C,B}$.

Exercises

Exercise 10.4.1 *Let* $B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ *be a basis of* $\mathbb{R}^2$ *and let* $\mathbf{x} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ *be a vector in* $\mathbb{R}^2$. *Find* $[\mathbf{x}]_B$.

Exercise 10.4.2 *Let* $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ *be a basis of* $\mathbb{R}^3$ *and let* $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ *be a vector in* $\mathbb{R}^2$. *Find* $[\mathbf{x}]_B$.

Exercise 10.4.3 *Suppose* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *is a linear transformation such that*

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be the standard basis of* $\mathbb{R}^3$, *and let*

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be another basis.

(a) *Find the matrix of* T *with respect to* E, *i.e., find* $[T]_{E,E}$.

(b) *Find* $[T]_{B,B}$.

Exercise 10.4.4 *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a linear transformation defined by*

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}.$$

Consider the two bases

$$B_1 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

and

$$B_2 = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

Find the matrix M_{B_2,B_1} *of* T *with respect to the bases* B_1 *and* B_2.

Exercise 10.4.5 *Let* $M = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, *and consider the linear transformation* $T : \mathbf{M}_{2,2} \to \mathbf{M}_{2,2}$ *given by* $T(A) = MAM$. *Find the matrix of* T *with respect to the basis*

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.\right\}$$

Exercise 10.4.6 *Consider the linear transformation* $T : \mathbf{P}_3 \to \mathbf{P}_3$ *given by* $T(p(x)) = p(x+1)$. *Find* $[T]_{B,B}$, *where* $B = \{1, x, x^2, x^3\}$.

Exercise 10.4.7 *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ *and consider the linear function* $T(\mathbf{w}) = \mathrm{proj}_{\mathbf{v}}(\mathbf{w})$. *Find the matrix of* T *with respect to the standard basis of* $\mathbb{R}^3$.

Exercise 10.4.8 *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ *and consider the linear function* $T(\mathbf{w}) = \mathrm{proj}_{\mathbf{v}}(\mathbf{w})$. *Find the matrix of* T *with respect to the basis*

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Exercise 10.4.9 *Suppose that* V *and* W *are finite-dimensional vector spaces with bases* B *and* C, *respectively. Let* $T : V \to W$ *be a linear transformation such that*

$$T(\mathbf{v}_i) = \mathbf{w}_i$$

for $i = 1, \ldots, n$. *Let* M *be the matrix whose columns are* $[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_n]_B$, *and let* N *be the matrix whose columns are* $[\mathbf{w}_1]_C, \ldots, [\mathbf{w}_n]_C$. *Suppose that* M *is invertible. Show that* $[T]_{C,B} = NM^{-1}$.

11. Inner product spaces

One structure that we often use in $\mathbb{R}^n$, but which is missing from abstract vector spaces, is the dot product. In Chapter 2, we saw how to use dot products to compute the length of a vector, the angle between two vectors, and to decide when two vectors are orthogonal. We also used dot products to define the projection of one vector onto another, and to find shortest distances between various objects (such as points and planes, two lines, etc).

In this chapter, we will consider inner product spaces. An inner product space is essentially an abstract vector spaces that has been equipped with an operation that works "like" the dot product.

11.1 Real inner product spaces

Outcomes

A. *Check whether an operation is an inner product.*

B. *Give an example of a vector space on which more than one inner product can be defined.*

C. *Calculate the inner product of vectors in various examples of inner product spaces.*

D. *Calculate the norm of a vector and the angle between two vectors.*

E. *Use the Cauchy-Schwarz inequality and the triangle inequality to reason about the size of inner products and norms of vectors.*

In this section, we fix K to be the field of real numbers.

Definition 11.1: Real inner product space

A ***(real) inner product space*** *is a real vector space V equipped with an operation that assigns to any pair of vectors $\mathbf{u},\mathbf{v}\in V$ a real number $\langle \mathbf{u},\mathbf{v}\rangle$, called the* ***inner product*** *of $\mathbf{u}$ and $\mathbf{v}$. This operation must satisfy the following properties:*

1. *Symmetry:* $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle$.
2. *Linearity:* $\langle \mathbf{u},k\mathbf{v}+\ell\mathbf{w}\rangle = k\langle \mathbf{u},\mathbf{v}\rangle + \ell\langle \mathbf{u},\mathbf{w}\rangle$.
3. *The positive definite property:* $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$, *and moreover,* $\langle \mathbf{u},\mathbf{u}\rangle = 0$ *if and only if* $\mathbf{u}=\mathbf{0}$.

Example 11.2: $\mathbb{R}^n$ with the usual dot product

$V = \mathbb{R}^n$, with the usual dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n$$

is an inner product space.

Proof. The relevant properties were shown in Proposition 2.25. ♠

There exist other inner product operations on $\mathbb{R}^n$ besides the dot product. The following is an example.

Example 11.3: $\mathbb{R}^2$ with a non-standard inner product

Let $V = \mathbb{R}^2$, and consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Define an operation $\langle -, - \rangle$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_1 v_2 + u_2 v_1 + 2 u_2 v_2.$$

Then $\mathbb{R}^2$, with this operation, is an inner product space.

Proof. We must show the three properties are satisfied. For symmetry, note that a scalar is equal to its own transpose. Therefore

$$\langle \mathbf{u}, \mathbf{v} \rangle \;=\; \mathbf{u}^T A \mathbf{v} \;=\; (\mathbf{u}^T A \mathbf{v})^T \;=\; \mathbf{v}^T A^T \mathbf{u}.$$

But this is equal to $\langle \mathbf{v}, \mathbf{u} \rangle$ because A is a symmetric matrix, i.e., $A^T = A$. For linearity, note that, by properties of matrix multiplication,

$$\langle \mathbf{u}, k\mathbf{v} + \ell\mathbf{w} \rangle \;=\; \mathbf{u}^T A (k\mathbf{v} + \ell\mathbf{w}) \;=\; k\mathbf{u}^T A \mathbf{v} + \ell \mathbf{u}^T A \mathbf{w} \;=\; k\langle \mathbf{u}, \mathbf{v} \rangle + \ell \langle \mathbf{u}, \mathbf{w} \rangle.$$

For the positive definite property, we have

$$\langle \mathbf{u}, \mathbf{u} \rangle \;=\; u_1^2 + 2u_1 u_2 + 2u_2^2 \;=\; (u_1^2 + 2u_1 u_2 + u_2^2) + u_2^2 \;=\; (u_1 + u_2)^2 + u_2^2 \;\geq\; 0,$$

because the sum of two squares is always non-negative. Moreover, if equality holds in the last equation, then we must have $u_1 + u_2 = 0$ and $u_2 = 0$, and this is only possible if $\mathbf{u} = \mathbf{0}$. ♠

Example 11.4: Continuous functions on an interval

Let $a < b$ be real numbers, and let

$$[a, b] = \{x \mid a \leq x \leq b\}$$

be the closed interval. Let $V = C[a, b]$ be the vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. Given two functions $f, g \in C[a, b]$, define

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx.$$

This is an inner product.

Proof. This follows from well-known properties of integrals. For symmetry, we have

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g,f\rangle.$$

For linearity, we have

$$\langle f,kg+\ell h\rangle = \int_a^b f(x)(kg(x)+\ell h(x))\,dx = k\int_a^b f(x)g(x)\,dx+\ell\int_a^b f(x)h(x)\,dx = k\langle f,g\rangle+\ell\langle f,h\rangle.$$

For the positive definite property, we have

$$\langle f,f\rangle = \int_a^b f(x)^2\,dx \geq 0.$$

This is ≥ 0 because $a<b$ and the integral over a non-negative function is non-negative. Moreover, since f is continuous, we know from calculus that the last integral can only be equal to 0 if f is the constant zero function. ♠

Example 11.5: Polynomials

Let $\mathbf{P}$ *be the vector space of polynomials (of any degree) with real coefficients. Let* $a<b$ *be real numbers, and consider the inner product defined by*

$$\langle p,q\rangle = \int_a^b p(x)q(x)\,dx.$$

This is an inner product space.

Proof. The proof is the same as for $C[a,b]$. ♠

Example 11.6: Real Hilbert space

Let $\mathbf{Hilb}_{\mathbb{R}}$ *be the vector space of all infinite sequences of real numbers* $a=(a_0,a_1,a_2,\ldots)$ *satisfying*

$$a_0^2+a_1^2+a_2^2+\ldots<\infty.$$

These are called the ***square summable*** *sequences. (One needs to do a little bit of work to show that it is indeed a vector space; in particular, to show that the sum of two square summable sequences is square summable.) On this space, we can define an inner product as follows:*

$$\langle a,b\rangle = a_0b_0+a_1b_1+a_2b_2+\ldots.$$

The details will be worked out in Exercise 11.1.6. This inner product space is called ***(real) Hilbert space****.*

We now look at some properties of inner products. The first thing to note is that while the axioms require linearity in the right component, symmetry ensures that linearity in the left component also holds.

Proposition 11.7: Left linearity

Let V be an inner product space. Then $\langle k\mathbf{u}+\ell\mathbf{v},\mathbf{w}\rangle = k\langle\mathbf{u},\mathbf{w}\rangle+\ell\langle\mathbf{v},\mathbf{w}\rangle$.

Proof. This follows directly from symmetry. We have

$$\langle k\mathbf{u}+\ell\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{w},k\mathbf{u}+\ell\mathbf{v}\rangle = k\langle\mathbf{w},\mathbf{u}\rangle+\ell\langle\mathbf{w},\mathbf{v}\rangle = k\langle\mathbf{u},\mathbf{w}\rangle+\ell\langle\mathbf{v},\mathbf{w}\rangle.$$

♠

We also note that as a consequence of right linearity, we have $\langle\mathbf{u},\mathbf{0}\rangle = 0$. This can be seen by letting $k=\ell=0$ in the equation for right linearity. Similarly, $\langle\mathbf{0},\mathbf{w}\rangle = 0$ follows from left linearity.

Another important property of inner products is the Cauchy-Schwarz inequality. We have already encountered this in the context of the dot product in Section 2.6.

Theorem 11.8: Cauchy-Schwarz inequality

Let $\mathbf{u},\mathbf{v}$ be vectors in an inner product space. Then

$$\langle\mathbf{u},\mathbf{v}\rangle^2 \leq \langle\mathbf{u},\mathbf{u}\rangle\cdot\langle\mathbf{v},\mathbf{v}\rangle. \tag{11.1}$$

Proof. The proof is almost identical to that of Proposition 2.27. First note that if $\mathbf{u}=\mathbf{0}$, then both sides of (11.1) are equal to zero, and so there is nothing to show. Therefore, we will assume in what follows that $\mathbf{u}\neq\mathbf{0}$. Define a function of $t\in\mathbb{R}$ by

$$f(t) = \langle t\mathbf{u}+\mathbf{v},t\mathbf{u}+\mathbf{v}\rangle.$$

Then by the positive definite property, we know that $f(t)\geq 0$ for all $t\in\mathbb{R}$. Also from linearity and symmetry, we have

$$\begin{aligned} f(t) &= \langle t\mathbf{u},t\mathbf{u}+\mathbf{v}\rangle+\langle\mathbf{v},t\mathbf{u}+\mathbf{v}\rangle \\ &= t^2\langle\mathbf{u},\mathbf{u}\rangle+t\langle\mathbf{u},\mathbf{v}\rangle+t\langle\mathbf{v},\mathbf{u}\rangle+\langle\mathbf{v},\mathbf{v}\rangle \\ &= t^2\langle\mathbf{u},\mathbf{u}\rangle+2t\langle\mathbf{u},\mathbf{v}\rangle+\langle\mathbf{v},\mathbf{v}\rangle. \end{aligned}$$

This means the graph of $y=f(t)$ is a parabola which opens upwards and is never negative. It follows that this function has at most one root. From the quadratic formula, we know that a quadratic function at^2+bt+c has one or zero roots if and only if $b^2-4ac\leq 0$. Applying this reasoning to the function $f(t)$, we obtain

$$(2\langle\mathbf{u},\mathbf{v}\rangle)^2-4\langle\mathbf{u},\mathbf{u}\rangle\langle\mathbf{v},\mathbf{v}\rangle < 0,$$

which is equivalent to $\langle\mathbf{u},\mathbf{v}\rangle^2\leq\langle\mathbf{u},\mathbf{u}\rangle\cdot\langle\mathbf{v},\mathbf{v}\rangle$. ♠

We can use the inner product to define the norm of a vector.

Definition 11.9: Norm

*Let $\mathbf{u}$ be a vector in an inner product space. Then the **norm** of $\mathbf{u}$ is defined as*

$$\|\mathbf{u}\| = \sqrt{\langle\mathbf{u},\mathbf{u}\rangle}.$$

Example 11.10: Norm in $C[-1,1]$

Calculate the norm of $f(x) = x^2$ in $C[-1,1]$.

Solution. In the vector space $C[-1,1]$, the inner product is defined as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

We therefore have

$$\langle f, f\rangle = \int_{-1}^{1} f(x)^2\,dx = \int_{-1}^{1} x^4\,dx = \left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{2}{5}.$$

Therefore, $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{2}{5}}$. ♠

A vector $\mathbf{u}$ in an inner product space is called **normalized** or a **unit vector** if $\|\mathbf{u}\| = 1$. We note that if $\mathbf{v}$ is any non-zero vector in an inner product space, then

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is normalized.

By the Cauchy-Schwarz inequality, we have the following properties:

Proposition 11.11: Inner product and norm

Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space. Then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

Proof. The Cauchy-Schwarz inequality states that $\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$. The claim follows by taking the square root of both sides of the equation. ♠

Proposition 11.12: Triangle inequality

Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space. Then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

*This is called the **triangle inequality**.*

Proof. The proof is essentially the same as that of Proposition 2.28. We have

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2
\end{aligned}$$

$$\begin{aligned} &\leq \|\mathbf{u}\|^2 + 2\,|\langle \mathbf{u},\mathbf{v}\rangle| + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

The triangle inequality follows by taking square roots of both sides. ♠

Generalizing the situation in $\mathbb{R}^n$, we can define the angle between any two vectors in an inner product space. Note that by the Cauchy-Schwarz inequality,

$$\frac{|\langle \mathbf{u},\mathbf{v}\rangle|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \leq 1,$$

and therefore,

$$-1 \leq \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \leq 1.$$

This ensures that the following definition is well-defined.

Definition 11.13: Angle between two vectors

*Let $\mathbf{u},\mathbf{v}$ be vectors in an inner product space. The **angle** between $\mathbf{u}$ and $\mathbf{v}$ is, by definition, the unique θ such that $0 \leq \theta \leq \pi$ and*

$$\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Example 11.14: Find the angle between two vectors

Find the angle between 1 and x^2 in $C[-1,1]$.

Solution. We have

$$\begin{aligned} \langle 1,1\rangle &= \int_{-1}^{1} 1\cdot 1\,dx \;=\; 2, \\ \langle x^2,x^2\rangle &= \int_{-1}^{1} x^2\cdot x^2\,dx \;=\; \frac{2}{5}, \\ \langle 1,x^2\rangle &= \int_{-1}^{1} 1\cdot x^2\,dx \;=\; \frac{2}{3}. \end{aligned}$$

Therefore

$$\cos\theta \;=\; \frac{\langle 1,x^2\rangle}{\|1\|\,\|x^2\|} \;=\; \frac{\frac{2}{3}}{\sqrt{2}\sqrt{\frac{2}{5}}} \;=\; \frac{\sqrt{5}}{3}.$$

The angle θ is $\cos^{-1}(\frac{\sqrt{5}}{3})$, which is approximately 0.7297 radians or 41.81 degrees.

♠

Exercises

Exercise 11.1.1 *For each matrix A, determine whether the formula $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ determines an inner product on $\mathbb{R}^2$.*

$$(a)\; A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad (b)\; A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad (c)\; A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad (d)\; A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (e)\; A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Exercise 11.1.2 *Consider the inner product space $C[0,1]$ as in Example 11.4. Compute the following inner products:*

$$(a)\; \langle 1, x \rangle, \quad (b)\; \langle x, x^2 \rangle, \quad (c)\; \langle 1+x, 2+x^2 \rangle.$$

Exercise 11.1.3 *Consider the inner product space $C[0,1]$ as in Example 11.4. Compute the following norms:*

$$(a)\; \|1\|, \quad (b)\; \|x\|, \quad (c)\; \left\|x^2+1\right\|.$$

Exercise 11.1.4 *For $\mathbf{u}, \mathbf{v}$ vectors in $\mathbb{R}^3$, define the product $\mathbf{u} * \mathbf{v} = u_1 v_1 + 2 u_2 v_2 + 3 u_3 v_3$. Show that*

$$|\mathbf{u} * \mathbf{v}| \leq (\mathbf{u} * \mathbf{u})^{1/2} (\mathbf{v} * \mathbf{v})^{1/2}.$$

Hint: first show that the operation $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} * \mathbf{v}$ is an inner product on $\mathbb{R}^3$, then use Proposition 11.11.

Exercise 11.1.5 *In $C[-1,1]$, find (a) the angle between x and x^2, (b) the angle between x and x^3.*

Exercise 11.1.6 . *In this exercise, we will work out the details of Example 11.6. We must show that $\mathbf{Hilb}_{\mathbb{R}}$ is a vector space. We will do this by showing that it is a subspace of $\mathbf{Seq}_{\mathbb{R}}$. Further, we must show that the inner product is well-defined. This requires some knowledge of convergent series from calculus.*

(a) *Assume $a = (a_0, a_1, \ldots)$ and $b = (b_0, b_1, \ldots)$ are square summable sequences. Show that the series $a_0 b_0 + a_1 b_1 + a_2 b_2 + \ldots$ converges to a real number. Hint: consider the series $|a_0 b_0| + |a_1 b_2| + \ldots$ and use the Cauchy-Schwarz inequality and the absolute convergence test.*

(b) *Using the result of part (a), show that $\mathbf{Hilb}_{\mathbb{R}}$ is a subspace of $\mathbf{Seq}_{\mathbb{R}}$.*

(c) *Show that the operation $\langle a, b \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + \ldots$ is an inner product on $\mathbf{Hilb}_{\mathbb{R}}$.*

11.2 Orthogonality

Outcomes

A. *Determine whether two vectors in an inner product space are orthogonal.*

B. *Find the orthogonal complement of a set of vectors.*

C. *Determine whether a set of vectors is orthogonal and/or orthonormal.*

D. *Check whether a basis is orthogonal and/or orthonormal.*

E. *Calculate the Fourier coefficients of a vector with respect to an orthogonal basis.*

Definition 11.15: Orthogonality

*Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space. We say that $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

We also write $\mathbf{u} \perp \mathbf{v}$ to indicate that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

We note that the zero vector $\mathbf{0}$ is orthogonal to all vectors, because $\langle \mathbf{u}, \mathbf{0} \rangle = 0$ follows from the linearity of the inner product. We also note that $\mathbf{u} \perp \mathbf{v}$ if and only if $\mathbf{v} \perp \mathbf{u}$; this follows from the symmetry of the inner product.

Definition 11.16: Orthogonal complement

*Let S be a subset of an inner product space V. The **orthogonal complement** of S is the set*

$$S^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in S\}.$$

Proposition 11.17: Orthogonal complement

If S is any subset of an inner product space V, then $S^{\perp}$ is a subspace.

Proof. We clearly have $\mathbf{0} \in S^{\perp}$, because $\mathbf{0}$ is orthogonal to all vectors. To show that $S^{\perp}$ is closed under addition, assume $\mathbf{v}, \mathbf{v}' \in S^{\perp}$. We have to show $\mathbf{v} + \mathbf{v}' \in S^{\perp}$. So take an arbitrary $\mathbf{w} \in S$. Then we have

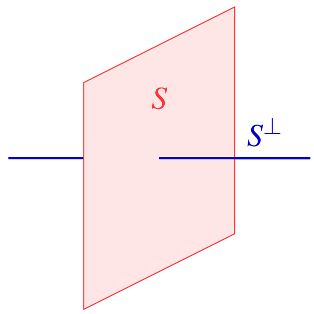
$$\langle \mathbf{v} + \mathbf{v}', \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}', \mathbf{w} \rangle = 0 + 0 = 0.$$

Therefore, $\mathbf{v} + \mathbf{v}' \in S^{\perp}$. Finally, to show that $S^{\perp}$ is closed under scalar multiplication, assume $\mathbf{v} \in S^{\perp}$ and $k \in \mathbb{R}$. We have to show $k\mathbf{v} \in S^{\perp}$. So take an arbitrary $\mathbf{w} \in S$. Then we have

$$\langle k\mathbf{v}, \mathbf{w} \rangle = k \langle \mathbf{v}, \mathbf{w} \rangle = k0 = 0.$$

Therefore, $k\mathbf{v} \in S^{\perp}$. It follows that $S^{\perp}$ is a subspace of V. ♠

Here is an illustration of a subspace S of $\mathbb{R}^3$ and its orthogonal complement $S^{\perp}$:



Example 11.18: Orthogonal complement

Consider the inner product space $\mathbf{P}_3$ *of polynomials of degree at most* 3*, with the inner product defined by*

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find the orthogonal complement of $\{x^2\}$.

Solution. We have to compute the set of all polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$ that are orthogonal to x^2. So let us compute the inner product:

$$\begin{aligned}\langle p(x), x^2\rangle &= \int_{-1}^{1} (ax^3 + bx^2 + cx + d)x^2\,dx \\ &= \int_{-1}^{1} ax^5 + bx^4 + cx^3 + dx^2\,dx \\ &= 0a + \frac{2}{5}b + 0c + \frac{2}{3}d.\end{aligned}$$

Setting this equal to 0, we see that $\langle p(x), x^2\rangle = 0$ if and only if $\frac{2}{5}b + \frac{2}{3}d = 0$, or equivalently, $3b + 5d = 0$. The basic solutions are

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

giving the following basis for the space of polynomials orthogonal to x^2:

$$\{x^3, \quad 5x^2 - 3, \quad x\}.$$

♠

We now consider orthogonal sets and bases.

Definition 11.19: Orthogonal and orthonormal sets of vectors

A set of vectors $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ *in an inner product space is called an* ***orthogonal set*** *if the vectors are non-zero and pairwise orthogonal, i.e., for all* i, $\mathbf{u}_i \neq \mathbf{0}$ *and for all* $i \neq j$, $\mathbf{u}_i \perp \mathbf{u}_j$.

Moreover, the set of vectors $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ *is called* ***orthonormal*** *if it is orthogonal and each vector is normalized, i.e.,* $\|\mathbf{u}_i\| = 1$.

The interest of orthogonal and orthonormal sets of vectors lies, among other things, in the fact that they are automatically linearly independent.

Proposition 11.20: Orthogonal set is linearly independent

If $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ *is an orthogonal set of vectors, then* $\mathbf{u}_1,\dots,\mathbf{u}_k$ *are linearly independent.*

Proof. Assume $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ is an orthogonal set. To show that $\mathbf{u}_1,\dots,\mathbf{u}_k$ are linearly independent, assume

$$a_1\mathbf{u}_1 + \dots + a_k\mathbf{u}_k = \mathbf{0}.$$

We must show that $a_1,\dots,a_k = 0$. So pick some $i \in \{1,\dots,k\}$. We compute

$$\begin{aligned}\langle \mathbf{u}_i, a_1\mathbf{u}_1 + \dots + a_k\mathbf{u}_k\rangle &= a_1\langle \mathbf{u}_i,\mathbf{u}_1\rangle + \dots + a_i\langle \mathbf{u}_i,\mathbf{u}_i\rangle + \dots + a_k\langle \mathbf{u}_i,\mathbf{u}_k\rangle \\ &= 0 + \dots + a_i\langle \mathbf{u}_i,\mathbf{u}_i\rangle + \dots + 0 \\ &= a_i\langle \mathbf{u}_i,\mathbf{u}_i\rangle.\end{aligned}$$

On the other hand,

$$\langle \mathbf{u}_i, a_1\mathbf{u}_1 + \dots + a_k\mathbf{u}_k\rangle = \langle \mathbf{u}_i, \mathbf{0}\rangle = 0.$$

It follows that $a_i\langle \mathbf{u}_i,\mathbf{u}_i\rangle = 0$. Since $\langle \mathbf{u}_i,\mathbf{u}_i\rangle \neq 0$, it follows that $a_i = 0$. Since the choice of i was arbitrary, it follows that $a_1,\dots,a_k = 0$, and $\mathbf{u}_1,\dots,\mathbf{u}_k$ are linearly independent. ♠

Example 11.21: An orthogonal set of functions

Consider the inner product space $C[0,2\pi]$. *The following functions form an (infinite) orthogonal set:*

$$\begin{aligned} g_0(x) &= 1 \\ f_1(x) &= \sin x \\ g_1(x) &= \cos x \\ f_2(x) &= \sin 2x \\ g_2(x) &= \cos 2x \\ &\vdots \\ f_k(x) &= \sin kx \\ g_k(x) &= \cos kx \\ &\vdots \end{aligned}$$

Proof. We have to check that any two functions are orthogonal to each other. This is true because of trigonometric identities. We have the following trigonometric formulas:

$$\begin{aligned}\sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),\\ \cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)),\\ \sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)).\end{aligned}$$

Using these, we can compute the relevant inner products. Assume $i \neq j$. Then

$$\begin{aligned}\langle g_i, g_j\rangle &= \int_0^{2\pi}\cos(ix)\cos(jx)\,dx\\ &= \frac{1}{2}\int_0^{2\pi}\cos((i-j)x)+\cos((i+j)x)\,dx\\ &= \frac{1}{2}\left[\frac{1}{(i-j)}\sin((i-j)x)+\frac{1}{(i+j)}\sin((i+j)x)\right]_0^{2\pi}\\ &= 0,\end{aligned}$$

and therefore $g_i \perp g_j$. The proofs of $f_i \perp f_j$ and $f_i \perp g_j$ are similar. ♠

Definition 11.22: Orthogonal and orthonormal bases

*Let V be an inner product space, and let W be a subspace of V. We say that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an **orthogonal basis** for W if it is an orthogonal set and spans W.*

*If, moreover, each $\mathbf{u}_i$ is normalized, we say that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an **orthonormal basis** for W.*

We note that, by Proposition 11.20, every orthogonal (or orthonormal) basis is automatically linearly independent, and therefore an actual basis of W.

Example 11.23: Orthogonal, orthonormal, and non-orthogonal bases

Consider $\mathbb{R}^2$ as an inner product space with the usual dot product.

- $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ *is an orthonormal basis of $\mathbb{R}^2$.*
- $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ *is an orthonormal basis of $\mathbb{R}^2$.*
- $\left\{\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}0\\3\end{bmatrix}\right\}$ *is an orthogonal basis of $\mathbb{R}^2$, but not orthonormal.*
- $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ *is an orthogonal basis of $\mathbb{R}^2$, but not orthonormal.*
- $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ *is a basis of $\mathbb{R}^2$, but neither orthogonal nor orthonormal.*

So why are we interested in orthogonal and orthonormal bases? The main reason is that finding coordinates is *much easier* when the bases are orthogonal (and even better when they are orthonormal). We have the following property:

Proposition 11.24: Fourier coefficients

Let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthogonal basis of some space W, and suppose $\mathbf{v} \in W$. Then

$$\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n,$$

where

$$a_i = \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}.$$

In this situation, the coordinates $a_1, \ldots, a_n$ are also called the ***Fourier coefficients*** *of $\mathbf{v}$ (with respect to the orthogonal basis B).*

In case B is an orthonormal basis, the formula is even simpler. In that case:

$$a_i = \langle \mathbf{u}_i, \mathbf{v} \rangle.$$

Proof. Since B is a basis of W, we know that there exist coefficients $a_1, \ldots, a_n$ such that $\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n$. It remains to verify that the coefficients satisfy the required formula. This is a simple calculation. We have

$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{v} \rangle &= \langle \mathbf{u}_i, a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n \rangle \\
&= a_1\langle \mathbf{u}_i, \mathbf{u}_1 \rangle + \ldots + a_i\langle \mathbf{u}_i, \mathbf{u}_i \rangle + \ldots + a_n\langle \mathbf{u}_i, \mathbf{u}_n \rangle \\
&= 0 + \ldots + a_i\langle \mathbf{u}_i, \mathbf{u}_i \rangle + \ldots + 0 \\
&= a_i\langle \mathbf{u}_i, \mathbf{u}_i \rangle.
\end{aligned}$$

Since $\mathbf{u}_i \neq \mathbf{0}$, we have $\langle \mathbf{u}_i, \mathbf{u}_i \rangle \neq 0$ by the positive definite property. We can therefore divide both sides of the equation by $\langle \mathbf{u}_i, \mathbf{u}_i \rangle$ to obtain

$$a_i = \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle},$$

as desired. Finally, if the basis is orthonormal, then $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = 1$, and the denominator disappears, so that $a_i = \langle \mathbf{u}_i, \mathbf{v} \rangle$. ♠

Example 11.25: Fourier coefficients

Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for an inner product space V, such that $\|\mathbf{u}_1\| = 1$, $\|\mathbf{u}_2\| = \sqrt{2}$, and $\|\mathbf{u}_3\| = 2$. Moreover, suppose that $\mathbf{v} \in V$ is a vector such that $\langle \mathbf{u}_1, \mathbf{v} \rangle = 3$, $\langle \mathbf{u}_2, \mathbf{v} \rangle = -1$, and $\langle \mathbf{u}_3, \mathbf{v} \rangle = 2$. Find the coordinates of $\mathbf{v}$ with respect to B.

Solution. We have to find a_1, a_2, a_3 such that

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3.$$

We have sufficient information to compute a_1, a_2, a_3. Namely,

$$\begin{aligned} a_1 &= \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\|\mathbf{u}_1\|^2} = \frac{3}{1} = 3, \\ a_2 &= \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} = \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\|\mathbf{u}_2\|^2} = \frac{-1}{2} = -0.5, \\ a_3 &= \frac{\langle \mathbf{u}_3, \mathbf{v} \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} = \frac{\langle \mathbf{u}_3, \mathbf{v} \rangle}{\|\mathbf{u}_3\|^2} = \frac{2}{4} = 0.5. \end{aligned}$$

♠

Proposition 11.24 shows that if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis, then we can solve $\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n$ without having to solve a system of equations. While this is a useful thing to be able to do, perhaps it is merely a convenience (solving a system of equations would also be fine). However, there is another very useful property of Fourier coefficients. The coefficient a_i only depends on the basis vector $\mathbf{u}_i$, and not on any of the other basis vectors. This is useful in situations where only *part* of an orthogonal basis is known. We can calculate the corresponding coefficients without having to know the rest of the basis.

Example 11.26: Finding partial coordinates

Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal basis of $\mathbb{R}^4$. We have been told that

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

but it is not known what $\mathbf{u}_3$ and $\mathbf{u}_4$ are. Find the first two coordinates of the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

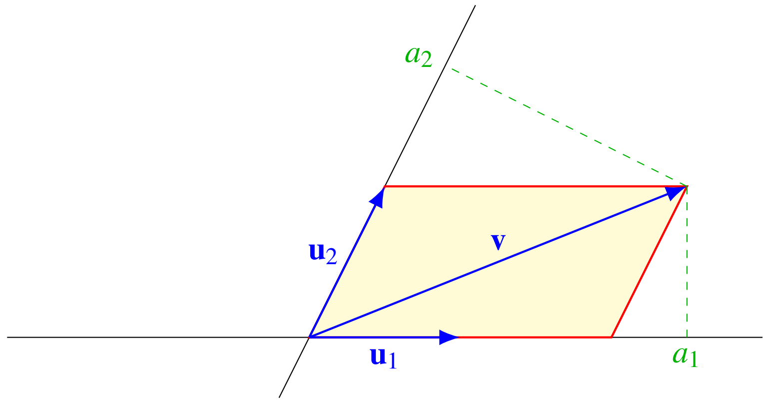
with respect to the basis B.

Solution. We have $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$, where

$$a_1 = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} = \frac{-1}{6} \quad \text{and} \quad a_2 = \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} = \frac{4}{14}.$$

So the first two coordinates are $-\frac{1}{6}$ and $\frac{2}{7}$. ♠

A word of warning is in order: Fourier coefficients do not work when the basis is not orthogonal. Consider the following picture, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal.



The *coordinates* of $\mathbf{v}$ with respect to $\mathbf{u}_1, \mathbf{u}_2$ are $(2,1)$, because $\mathbf{v} = 2\mathbf{u}_1 + 1\mathbf{u}_2$, as indicated by the shaded parallelogram. On the other hand, the formula for the *Fourier coefficients* is concerned with the orthogonal projections of $\mathbf{v}$ onto $\mathbf{u}_1$ and $\mathbf{u}_2$, as indicated by the dashed lines. It yields the coefficients

$$\begin{aligned} a_1 &= \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \approx 2.5, \\ a_2 &= \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \approx 1.8. \end{aligned}$$

These are not the same as the coordinates of $\mathbf{v}$, because the dashed lines are orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, instead of parallel to them. In summary, the Fourier coefficients of a vector are equal to its coordinates *only if the basis is orthogonal*.

When $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are orthogonal, we have a convenient formula for the norm of $a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k$:

Proposition 11.27: Norm of orthogonal linear combination

Let V be an inner product space, and suppose that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are orthogonal. Then

$$\|a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k\|^2 = a_1^2\,\|\mathbf{u}_1\|^2 + \ldots + a_k^2\,\|\mathbf{u}_k\|^2.$$

Proof. We have

$$\begin{aligned} \|a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k\|^2 &= \langle a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k, a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k \rangle \\ &= \begin{array}{llll} \quad a_1^2\langle \mathbf{u}_1, \mathbf{u}_1 \rangle & + a_1a_2\langle \mathbf{u}_1, \mathbf{u}_2 \rangle & + \ldots & + a_1a_n\langle \mathbf{u}_1, \mathbf{u}_n \rangle \\ + a_2a_1\langle \mathbf{u}_2, \mathbf{u}_1 \rangle & + \quad a_2^2\langle \mathbf{u}_2, \mathbf{u}_2 \rangle & + \ldots & + a_2a_n\langle \mathbf{u}_2, \mathbf{u}_n \rangle \\ + \qquad \vdots & \qquad \vdots & & \qquad \vdots \\ + a_na_1\langle \mathbf{u}_n, \mathbf{u}_1 \rangle & + a_na_2\langle \mathbf{u}_n, \mathbf{u}_2 \rangle & + \ldots & + \quad a_n^2\langle \mathbf{u}_n, \mathbf{u}_n \rangle \end{array} \\ &= a_1^2\,\|\mathbf{u}_1\|^2 + \ldots + a_k^2\,\|\mathbf{u}_k\|^2. \end{aligned}$$

Here we have used the fact that $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ when $i \neq j$. ♠

We conclude this section by making a connection between arbitrary inner products and dot products. Once an orthonormal basis has been chosen on an inner product space, computing inner products is essentially the same as computing dot products. The following proposition makes this more precise.

Proposition 11.28: Inner product and dot product

Let V be an inner product space, and suppose that $B = \{\mathbf{u}_1, \dots \mathbf{u}_n\}$ is an orthonormal basis. Then for any pair of vectors $\mathbf{v}, \mathbf{w} \in V$, their inner product is equal to the dot product of their coordinate vectors with respect to the basis B, i.e.

$$\langle \mathbf{v}, \mathbf{w} \rangle = [\mathbf{v}]_B \cdot [\mathbf{w}]_B .$$

Proof. By definition of coordinate vectors, we have

$$[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad [\mathbf{w}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

where $\mathbf{v} = a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n$ and $\mathbf{w} = b_1\mathbf{u}_1 + \dots + b_n\mathbf{u}_n$. We calculate

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \langle a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n, b_1\mathbf{u}_1 + \dots + b_n\mathbf{u}_n \rangle \\ &= \quad a_1b_1\langle \mathbf{u}_1, \mathbf{u}_1 \rangle + a_1b_2\langle \mathbf{u}_1, \mathbf{u}_2 \rangle + \dots + a_1b_n\langle \mathbf{u}_1, \mathbf{u}_n \rangle \\ &\quad + a_2b_1\langle \mathbf{u}_2, \mathbf{u}_1 \rangle + a_2b_2\langle \mathbf{u}_2, \mathbf{u}_2 \rangle + \dots + a_2b_n\langle \mathbf{u}_2, \mathbf{u}_n \rangle \\ &\quad + \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &\quad + a_nb_1\langle \mathbf{u}_n, \mathbf{u}_1 \rangle + a_nb_2\langle \mathbf{u}_n, \mathbf{u}_2 \rangle + \dots + a_nb_n\langle \mathbf{u}_n, \mathbf{u}_n \rangle \\ &= a_1b_1 + a_2b_2 + \dots + a_nb_n \\ &= [\mathbf{v}]_B \cdot [\mathbf{w}]_B . \end{aligned}$$

Here we have used the fact that $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 1$ when $i = j$ and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ otherwise, which holds because B is orthonormal. ♠

Exercises

Exercise 11.2.1 *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, *and consider* $\mathbb{R}^3$ *with the inner product given by* $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$. *Which of the following vectors are orthogonal to each other?*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 7 \\ -5 \\ -2 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 10 \\ -2 \\ -7 \end{bmatrix}.$$

Exercise 11.2.2 *On* $C[-1,1]$, *which of the following functions are orthogonal to each other?*

$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = x^3 - x, \quad f_4(x) = 1 - x^4.$$

Exercise 11.2.3 *Consider the inner product space* $\mathbf{P}_3$ *of polynomials of degree at most* 3*, with the inner product defined by*

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Find the orthogonal complement of $\{x^2,x\}$.

(b) Find the orthogonal complement of $\{x+1\}$.

Exercise 11.2.4 *Consider* $\mathbb{R}^3$ *as an inner product space with the usual dot product. For each of the following bases of* $\mathbb{R}^3$*, state whether it is orthonormal, orthogonal, or neither.*

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) $\left\{ \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{bmatrix} \right\}$.

Exercise 11.2.5 *Suppose* $B = \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ *is an orthogonal basis for an inner product space* V*, such that* $\|\mathbf{u}_1\| = 2$*,* $\|\mathbf{u}_2\| = \sqrt{3}$*, and* $\|\mathbf{u}_3\| = \sqrt{5}$*. Moreover, suppose that* $\mathbf{v} \in V$ *is a vector such that* $\langle \mathbf{v},\mathbf{u}_1\rangle = 1$*,* $\langle \mathbf{v},\mathbf{u}_2\rangle = 2$*, and* $\langle \mathbf{v},\mathbf{u}_3\rangle = -4$*. Find the coordinates of* $\mathbf{v}$ *with respect to* B.

Exercise 11.2.6 *Suppose* $B = \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ *is an orthogonal basis of* $\mathbb{R}^3$*. We have been told that*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

but it is not known what $\mathbf{u}_2$ *and* $\mathbf{u}_3$ *are. Find the first coordinate of the vector*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

with respect to the basis B.

11.3 The Gram-Schmidt orthogonalization procedure

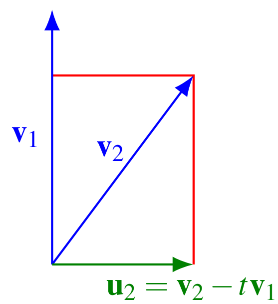
Outcomes

A. *Use the Gram-Schmidt procedure to find an orthogonal basis of a subspace of an inner product space.*

B. *Find an orthonormal basis of a subspace.*

Although we have already seen some potential uses for orthogonal bases, we have not yet seen very many examples of such bases. In this section, we will look at the Gram-Schmidt orthogonalization procedure, a method for turning any basis into an orthogonal one.

The basic idea is very simple: if two vectors $\mathbf{v}_1,\mathbf{v}_2$ are not orthogonal, then we can make them orthogonal by replacing $\mathbf{v}_2$ by a vector of the form $\mathbf{u}_2=\mathbf{v}_2-t\mathbf{v}_1$, for a suitable parameter t.



But what is the correct value of t? It turns out that this value is uniquely determined by the requirement that $\mathbf{v}_1$ and $\mathbf{u}_2$ must be orthogonal. We calculate

$$\langle \mathbf{v}_1,\mathbf{u}_2\rangle \;=\; \langle \mathbf{v}_1,\mathbf{v}_2-t\mathbf{v}_1\rangle \;=\; \langle \mathbf{v}_1,\mathbf{v}_2\rangle - t\langle \mathbf{v}_1,\mathbf{v}_1\rangle.$$

Setting this equal to 0 yields the unique solution

$$t=\frac{\langle \mathbf{v}_1,\mathbf{v}_2\rangle}{\langle \mathbf{v}_1,\mathbf{v}_1\rangle}$$

Note that this is exactly the same thing as the Fourier coefficient of $\mathbf{v}_2$ in the direction of $\mathbf{v}_1$. The following proposition summarizes what we have found so far. For consistency with our later notation, we also rename the first basis vector $\mathbf{v}_1$ to $\mathbf{u}_1$.

Proposition 11.29: Gram-Schmidt orthogonalization procedure for 2 vectors

Let $\{\mathbf{v}_1,\mathbf{v}_2\}$ be a basis for some subspace W of an inner product space V. Define vectors $\mathbf{u}_1,\mathbf{u}_2$ as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1,\\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1.
\end{aligned}$$

Then $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal basis of W.

Example 11.30: Gram-Schmidt orthogonalization procedure for 2 vectors

In $\mathbb{R}^3$ with the usual dot product, find an orthogonal basis for

$$\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$$

Solution. Let $\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$. We calculate

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix},\\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}.\end{aligned}$$

Therefore the desired orthogonal basis is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}\right\}$. ♠

The procedure for finding an orthogonal basis of a k-dimensional space is very similar. We adjust each basis vector $\mathbf{v}_i$ by subtracting a suitable linear combination of previous orthogonal basis vectors.

Proposition 11.31: Gram-Schmidt orthogonalization procedure for k vectors

Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ be a basis for some subspace W of an inner product space V. Define vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k$ as follows:

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{v}_1,\\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1,\\ \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{u}_1,\mathbf{v}_3\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\mathbf{v}_3\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2,\\ &\vdots\\ \mathbf{u}_k &= \mathbf{v}_k - \frac{\langle \mathbf{u}_1,\mathbf{v}_k\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\mathbf{v}_k\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2 - \ldots - \frac{\langle \mathbf{u}_{k-1},\mathbf{v}_k\rangle}{\langle \mathbf{u}_{k-1},\mathbf{u}_{k-1}\rangle}\mathbf{u}_{k-1}.\end{aligned}$$

Then $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthogonal basis of W.

Proof. First, it is clear that $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ and $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ span the same subspace, as each $\mathbf{v}_i$ is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_i$ and conversely, each $\mathbf{u}_i$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_i$. So the only thing we must check is that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthogonal set. In other words, we must show that $\langle \mathbf{u}_j,\mathbf{u}_i\rangle = 0$ for all $j < i$. We prove this by induction on i, i.e., we assume it is already true for all pairs of indices smaller than i. To show $\langle \mathbf{u}_j, \mathbf{u}_i \rangle = 0$, we calculate:

$$\begin{aligned}
\langle \mathbf{u}_j, \mathbf{u}_i \rangle &= \langle \mathbf{u}_j, \mathbf{v}_i - \tfrac{\langle \mathbf{u}_1, \mathbf{v}_i \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \ldots - \tfrac{\langle \mathbf{u}_j, \mathbf{v}_i \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}\mathbf{u}_j - \ldots - \tfrac{\langle \mathbf{u}_{i-1}, \mathbf{v}_i \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle}\mathbf{u}_{i-1} \rangle \\
&= \langle \mathbf{u}_j, \mathbf{v}_i \rangle - \tfrac{\langle \mathbf{u}_1, \mathbf{v}_i \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\langle \mathbf{u}_j, \mathbf{u}_1 \rangle - \ldots - \tfrac{\langle \mathbf{u}_j, \mathbf{v}_i \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}\langle \mathbf{u}_j, \mathbf{u}_j \rangle - \ldots - \tfrac{\langle \mathbf{u}_{i-1}, \mathbf{v}_i \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle}\langle \mathbf{u}_j, \mathbf{u}_{i-1} \rangle \\
&= \langle \mathbf{u}_j, \mathbf{v}_i \rangle - 0 - \ldots - \tfrac{\langle \mathbf{u}_j, \mathbf{v}_i \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}\langle \mathbf{u}_j, \mathbf{u}_j \rangle - \ldots - 0 \\
&= \langle \mathbf{u}_j, \mathbf{v}_i \rangle - \langle \mathbf{u}_j, \mathbf{v}_i \rangle \\
&= 0.
\end{aligned}$$

It follows that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is orthogonal, as desired. ♠

Example 11.32: Gram-Schmidt orthogonalization procedure

In $\mathbb{R}^4$, find an orthogonal basis for

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Solution. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We calculate

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ -3/4 \end{bmatrix}, \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{u}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1/2}{3/4}\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ -3/4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \\ 0 \end{bmatrix}.
\end{aligned}$$

Therefore the orthogonal basis is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ -3/4 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \\ 0 \end{bmatrix} \right\}$. ♠

The Gram-Schmidt procedure is sensitive to reordering the vectors. For example, if we order the original basis vectors in Example 11.32 in the opposite order, we end up with a different orthogonal basis at the end. Sometimes this can simplify the calculations, as the following example shows.

Example 11.33: Gram-Schmidt orthogonalization procedure: reordering the vectors

In $\mathbb{R}^4$, find an orthogonal basis for

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Solution. Note that these are the same basis vectors as in Example 11.32, but listed in a different order. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We calculate

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \\ \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{u}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

This time, we end up with the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. ♠

In the next example, we will consider $\mathbb{R}^n$, but with a non-standard inner product.

Example 11.34: Gram-Schmidt orthogonalization procedure, non-standard inner product

Let

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 6 & -1 \\ -2 & -1 & 9 \end{bmatrix},$$

and consider the vector space $\mathbb{R}^3$ *with the inner product given by* $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$. *Let*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \textit{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Apply the Gram-Schmidt procedure to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *to find an orthogonal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *for* $\mathbb{R}^3$ *with respect to the above inner product.*

Solution. As usual, we start with

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next, we calculate

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{v}_2 \rangle &= \mathbf{u}_1^T A \mathbf{v}_2 = 2, \\ \langle \mathbf{u}_1, \mathbf{u}_1 \rangle &= \mathbf{u}_1^T A \mathbf{u}_1 = 1. \end{aligned}$$

Therefore,

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{u}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, we calculate

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{v}_3 \rangle &= \mathbf{u}_1^T A \mathbf{v}_3 = -2, \\ \langle \mathbf{u}_2, \mathbf{v}_3 \rangle &= \mathbf{u}_2^T A \mathbf{v}_3 = 3, \\ \langle \mathbf{u}_2, \mathbf{u}_2 \rangle &= \mathbf{u}_2^T A \mathbf{u}_2 = 2. \end{aligned}$$

Therefore,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{u}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-2}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3/2 \\ 1 \end{bmatrix}.$$

So the desired orthogonal basis is

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3/2 \\ 1 \end{bmatrix} \right\}.$$

Note that it is not orthogonal with respect to the dot product, but with respect to the inner product defined above. ♠

Example 11.35: Legendre polynomials

Consider the vector space $\mathbf{P}$ *of polynomials, with the inner product*

$$\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Use the Gram-Schmidt procedure to find an orthogonal basis for $\operatorname{span}\{1,x,x^2,x^3\}$.

Solution. Let $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$, and $\mathbf{v}_4 = x^3$. We follow the Gram-Schmidt procedure:

$$\mathbf{u}_1 \;=\; \mathbf{v}_1 \;=\; 1.$$

Before we calculate $\mathbf{u}_2$, we have to evaluate two integrals:

$$\begin{aligned}
\langle \mathbf{u}_1,\mathbf{v}_2\rangle &= \int_{-1}^{1} 1\cdot x\,dx \;=\; \left[\tfrac{1}{2}x^2\right]_{-1}^{1} \;=\; 0,\\
\langle \mathbf{u}_1,\mathbf{u}_1\rangle &= \int_{-1}^{1} 1\cdot 1\,dx \;=\; [x]_{-1}^{1} \;=\; 2.
\end{aligned}$$

Then

$$\mathbf{u}_2 \;=\; \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 \;=\; x - \frac{0}{2}\cdot 1 \;=\; x.$$

To calculate $\mathbf{u}_3$, we first evaluate three integrals:

$$\begin{aligned}
\langle \mathbf{u}_1,\mathbf{v}_3\rangle &= \int_{-1}^{1} 1\cdot x^2\,dx \;=\; \left[\tfrac{1}{3}x^3\right]_{-1}^{1} \;=\; \frac{2}{3},\\
\langle \mathbf{u}_2,\mathbf{v}_3\rangle &= \int_{-1}^{1} x\cdot x^2\,dx \;=\; \left[\tfrac{1}{4}x^4\right]_{-1}^{1} \;=\; 0,\\
\langle \mathbf{u}_2,\mathbf{u}_2\rangle &= \int_{-1}^{1} x\cdot x\,dx \;=\; \left[\tfrac{1}{3}x^3\right]_{-1}^{1} \;=\; \frac{2}{3}.
\end{aligned}$$

Then

$$\mathbf{u}_3 \;=\; \mathbf{v}_3 - \frac{\langle \mathbf{u}_1,\mathbf{v}_3\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\mathbf{v}_3\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2 \;=\; x^2 - \frac{2/3}{2}\cdot 1 - \frac{0}{2/3}\cdot x \;=\; x^2 - \frac{1}{3}.$$

To calculate $\mathbf{u}_4$, we first evaluate four integrals:

$$\begin{aligned}
\langle \mathbf{u}_1,\mathbf{v}_4\rangle &= \int_{-1}^{1} 1\cdot x^3\,dx \;=\; \left[\tfrac{1}{4}x^4\right]_{-1}^{1} \;=\; 0,\\
\langle \mathbf{u}_2,\mathbf{v}_4\rangle &= \int_{-1}^{1} x\cdot x^3\,dx \;=\; \left[\tfrac{1}{5}x^5\right]_{-1}^{1} \;=\; \frac{2}{5},
\end{aligned}$$

$$\begin{aligned}\langle \mathbf{u}_3, \mathbf{v}_4 \rangle &= \int_{-1}^{1} (x^2 - \tfrac{1}{3}) \cdot x^3 \, dx = \left[\tfrac{1}{6}x^6 - \tfrac{1}{12}x^4\right]_{-1}^{1} = 0, \\ \langle \mathbf{u}_3, \mathbf{u}_3 \rangle &= \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 \, dx = \left[\tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x\right]_{-1}^{1} = \tfrac{8}{45}.\end{aligned}$$

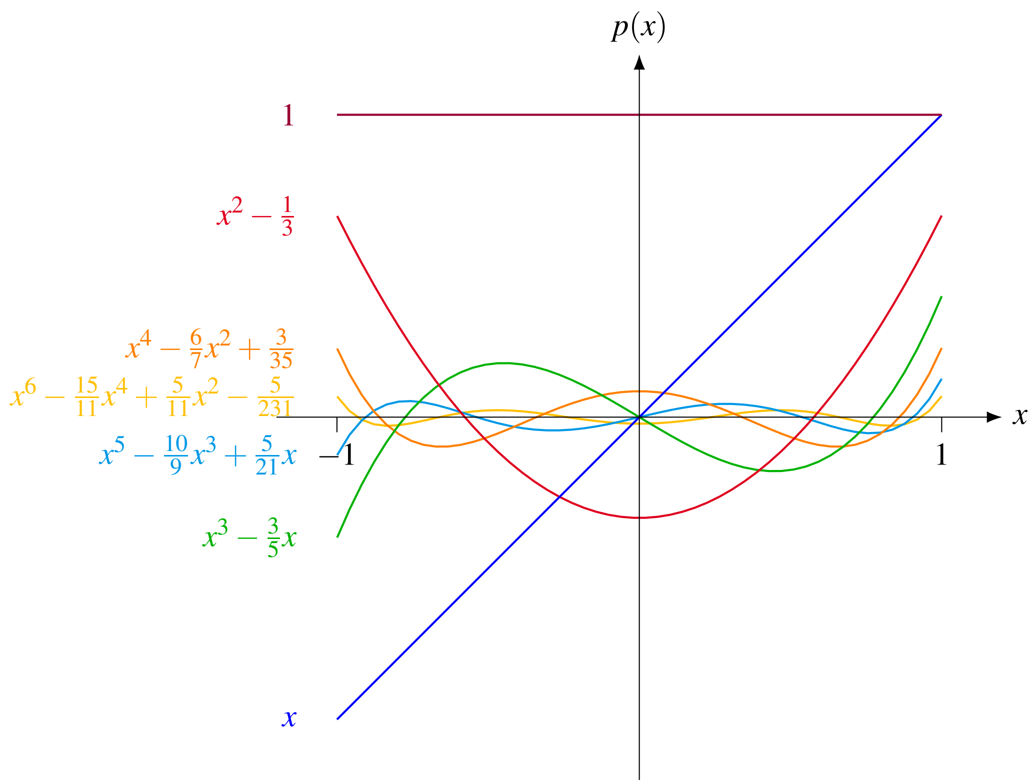
Then

$$\begin{aligned}\mathbf{u}_4 &= \mathbf{v}_4 - \frac{\langle \mathbf{u}_1, \mathbf{v}_4 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_4 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \mathbf{v}_4 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle}\mathbf{u}_3 \\ &= x^3 - \frac{0}{2} \cdot 1 - \frac{2/5}{2/3} \cdot x - \frac{0}{8/45} \cdot (x^2 - \frac{1}{3}) = x^3 - \frac{3}{5}x.\end{aligned}$$

Thus, we obtain the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \left\{1,\ x,\ x^2 - \tfrac{1}{3},\ x^3 - \tfrac{3}{5}x\right\}$. ♠

The orthogonal polynomials from Example 11.35 are known (up to scalar multiples) as **Legendre polynomials**. We can continue in the same fashion applying the Gram-Schmidt procedure to the polynomials $1, x, x^2, x^3, x^4, x^5, x^6, \ldots$ to get an infinite sequence of orthogonal polynomials. The first few elements of this sequence are:

$$\begin{aligned}p_0(x) &= 1, \\ p_1(x) &= x, \\ p_2(x) &= x^2 - \frac{1}{3}, \\ p_3(x) &= x^3 - \frac{3}{5}x, \\ p_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35}, \\ p_5(x) &= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x, \\ p_6(x) &= x^6 - \frac{15}{11}x^4 + \frac{5}{11}x^2 - \frac{5}{231}, \\ p_7(x) &= x^7 - \frac{21}{13}x^5 + \frac{105}{143}x^3 - \frac{35}{429}x, \\ p_8(x) &= x^8 - \frac{28}{15}x^6 + \frac{14}{13}x^4 - \frac{28}{143}x^2 + \frac{7}{1287}.\end{aligned}$$



The Gram-Schmidt procedure yields an *orthogonal* basis. If we want to compute an *orthonormal* basis, we also have to normalize each basis vector. Since normalization usually involves dividing by a square root, it is best to do this at the end, i.e., after the entire Gram-Schmidt procedure is complete, rather than normalizing each $\mathbf{u}_i$ immediately after it is found. Note that the Gram-Schmidt procedure itself does not involve computing any square roots.

Example 11.36: Finding an orthonormal basis

In $\mathbb{R}^4$, find an orthogonal basis for

$$\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}.$$

Solution. In Example 11.32, we already found an orthogonal basis

$$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\} = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1/4\\1/4\\1/4\\-3/4\end{bmatrix},\begin{bmatrix}1/3\\1/3\\-2/3\\0\end{bmatrix}\right\}$$

for this space. So all that is left to do is to normalize each vector. The orthonormal basis is

$$\left\{\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}, \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}\right\} = \left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{2}{\sqrt{3}}\begin{bmatrix}1/4\\1/4\\1/4\\-3/4\end{bmatrix}, \frac{\sqrt{3}}{\sqrt{2}}\begin{bmatrix}1/3\\1/3\\-2/3\\0\end{bmatrix}\right\}.$$

Alternatively, we could have also normalized the orthogonal basis we found in Example 11.33. In that case, we obtain the orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}.$$

♠

Example 11.37: Orthonormal basis of polynomials

Find an orthonormal basis for the space of Example 11.35.

Solution. In Example 11.35, we found the orthogonal basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\} = \left\{1,\ x,\ x^2-\frac{1}{3},\ x^3-\frac{3}{5}x\right\}$. We also computed

$$\langle\mathbf{u}_1,\mathbf{u}_1\rangle = 2, \quad \langle\mathbf{u}_2,\mathbf{u}_2\rangle = \frac{2}{3}, \quad \text{and} \quad \langle\mathbf{u}_3,\mathbf{u}_3\rangle = \frac{8}{45}.$$

We also need to compute $\langle\mathbf{u}_4,\mathbf{u}_4\rangle$:

$$\langle\mathbf{u}_4,\mathbf{u}_4\rangle = \int_{-1}^{1}(x^3-\tfrac{3}{5}x)^2\,dx = \int_{-1}^{1} x^6-\tfrac{6}{5}x^4+\tfrac{9}{25}x^2\,dx = \left[\tfrac{1}{7}x^7-\tfrac{6}{25}x^5+\tfrac{3}{25}x^3\right]_{-1}^{1} = \frac{8}{175}.$$

Therefore, the orthonormal basis is:

$$\left\{\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}, \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}, \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|}\right\} = \left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x,\ \sqrt{\frac{45}{8}}(x^2-\frac{1}{3}),\ \sqrt{\frac{175}{8}}(x^3-\frac{3}{5}x)\right\}.$$

♠

We finish this section by remarking that the formula

$$\frac{\langle\mathbf{u},\mathbf{v}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\mathbf{u}$$

is exactly what we called the **projection of v onto u** in Section 2.6.5, except that we have generalized this concept from $\mathbb{R}^n$ to an arbitrary inner product space. We can define

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle\mathbf{u},\mathbf{v}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\mathbf{u}.$$

With this definition, the Gram-Schmidt procedure can also be expressed more succinctly as follows.

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_2), \\
\mathbf{u}_3 &= \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \operatorname{proj}_{\mathbf{u}_2}(\mathbf{v}_3), \\
&\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_k) - \operatorname{proj}_{\mathbf{u}_2}(\mathbf{v}_k) - \ldots - \operatorname{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k).
\end{aligned}$$

Exercises

Exercise 11.3.1 *In $\mathbb{R}^3$ with the usual dot product, find an orthogonal basis for*

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\6\\0\end{bmatrix}\right\}.$$

Exercise 11.3.2 *In $\mathbb{R}^4$ with the usual dot product, find an orthogonal basis for*

$$\operatorname{span}\left\{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\2\\-1\end{bmatrix}\right\}.$$

Exercise 11.3.3 *In $\mathbb{R}^4$ with the usual dot product, find an orthogonal basis for*

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\2\\2\end{bmatrix}\right\}.$$

Exercise 11.3.4 *Let*

$$A = \begin{bmatrix}3 & -1 & 0\\-1 & 5 & 2\\0 & 2 & 3\end{bmatrix},$$

and consider the vector space $\mathbb{R}^3$ with the inner product given by $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T A\mathbf{w}$. Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}-1\\1\\-5\end{bmatrix}, \quad \textit{and} \quad \mathbf{v}_3 = \begin{bmatrix}2\\2\\3\end{bmatrix}.$$

Apply the Gram-Schmidt procedure to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ with respect to the above inner product.

Exercise 11.3.5 *Let*

$$A = \begin{bmatrix}3 & 2 & 0 & 0\\2 & 5 & 1 & 0\\0 & 1 & 3 & 1\\0 & 0 & 1 & 3\end{bmatrix},$$

and consider the vector space $\mathbb{R}^4$ with the inner product given by $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T A\mathbf{w}$. Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\0\\0\\2\end{bmatrix}, \quad \textit{and} \quad \mathbf{v}_3 = \begin{bmatrix}2\\0\\2\\3\end{bmatrix},$$

and let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*. Apply the Gram-Schmidt procedure to* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *to find an orthogonal basis* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *for* W *with respect to the above inner product.*

Exercise 11.3.6 *Consider the inner product space* $C[0,2]$*, with the inner product given by*

$$\langle p, q \rangle = \int_0^2 f(x)g(x)\,dx.$$

Use the Gram-Schmidt procedure to find an orthogonal basis for $\operatorname{span}\{1, x, x^2\}$.

Exercise 11.3.7 *Find an orthonormal basis for the subspace of* $\mathbb{R}^3$ *from Exercise 11.3.1.*

11.4 Orthogonal projections and Fourier series

Outcomes

A. *Compute the orthogonal projection of a vector onto a subspace.*

B. *Find the least squares approximation of a function by a polynomial of a given degree.*

C. *Calculate the generalized Fourier series of a vector.*

D. *Use Fourier series to approximate a function by polynomials or trigonometric functions.*

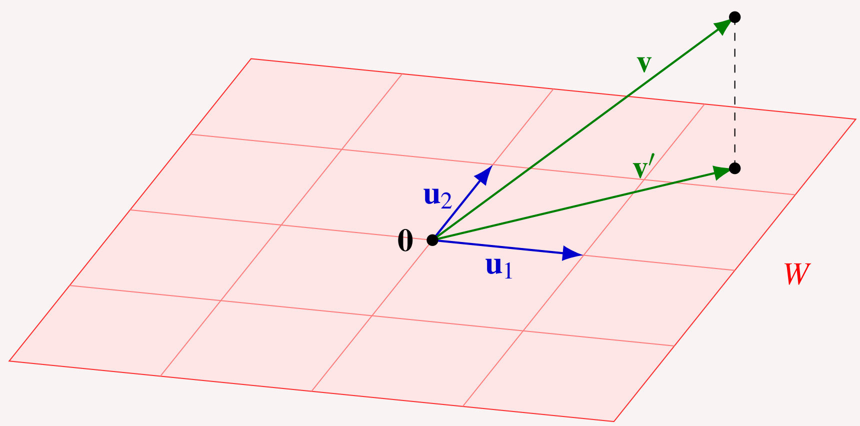
We now reconsider a problem that we briefly encountered, only in the context of $\mathbb{R}^3$, in Section 3.2: how to find the shortest distance between a point and a subspace. The method we used in Section 3.2 (see Example 3.26) relies on the existence of normal vectors and does not generalize beyond $\mathbb{R}^3$. The following proposition gives a much better method for solving this problem, provided that we have an orthogonal basis of the subspace.

Proposition 11.38: Orthogonal projection onto a subspace

Let V be an inner product space, and let W be a subspace of V. Assume $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthogonal basis of W, and $\mathbf{v}\in V$ is any vector. Then the following vector $\mathbf{v}'$ is the element of W that is closest to $\mathbf{v}$, i.e., such that $\|\mathbf{v}-\mathbf{v}'\|$ is as small as possible.

$$\mathbf{v}' = \frac{\langle \mathbf{u}_1,\mathbf{v}\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2,\mathbf{v}\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{u}_k,\mathbf{v}\rangle}{\langle \mathbf{u}_k,\mathbf{u}_k\rangle}\mathbf{u}_k.$$

Moreover, the vector $\mathbf{v}-\mathbf{v}'$ is orthogonal to W.



*The vector $\mathbf{v}'$ is called the **orthogonal projection of v onto** W. We also say that $\mathbf{v}'$ is the **best approximation** of $\mathbf{v}$ in W.*

Proof. In case $\mathbf{v}\in W$, then by Proposition 11.24, we have

$$\mathbf{v} = \frac{\langle \mathbf{u}_1,\mathbf{v}\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2,\mathbf{v}\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{u}_k,\mathbf{v}\rangle}{\langle \mathbf{u}_k,\mathbf{u}_k\rangle}\mathbf{u}_k,$$

and therefore $\mathbf{v}'=\mathbf{v}$. In that case $\|\mathbf{v}-\mathbf{v}'\|=0$, which is clearly as small as possible, and $\mathbf{v}-\mathbf{v}'=\mathbf{0}$ is orthogonal to W, so we are done.

Now assume that $\mathbf{v}\notin W$. Let $W'=\operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k,\mathbf{v}\}$. Apply the Gram-Schmidt procedure to the $k+1$ vectors $\mathbf{u}_1,\ldots,\mathbf{u}_k,\mathbf{v}$ to obtain an orthogonal basis of W'. Since the first k vectors are already orthogonal, the Gram-Schmidt procedure does not change them, and we therefore obtain an orthogonal basis of W' of the form $\mathbf{u}_1,\ldots,\mathbf{u}_k,\mathbf{u}_{k+1}$. Since $\mathbf{v}\in W'$, we can write $\mathbf{v}$ as a linear combination of these basis vectors, i.e., $\mathbf{v}=a_1\mathbf{u}_1+\ldots+a_k\mathbf{u}_k+a_{k+1}\mathbf{u}_{k+1}$. Moreover, by Proposition 11.24, we know that each a_j is the corresponding Fourier coefficient of $\mathbf{v}$, i.e., $a_j=\frac{\langle \mathbf{u}_j,\mathbf{v}\rangle}{\langle \mathbf{u}_j,\mathbf{u}_j\rangle}$.

Now consider any $\mathbf{w}\in W$. Since $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is a basis of W, we can write $\mathbf{w}=x_1\mathbf{u}_1+\ldots+x_k\mathbf{u}_k$. Using Proposition 11.27, we have

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= \|(a_1-x_1)\mathbf{u}_1+\ldots+(a_k-x_k)\mathbf{u}_k+a_{k+1}\mathbf{u}_{k+1}\|^2\\ &= (a_1-x_1)^2\|\mathbf{u}_1\|^2 + \ldots + (a_k-x_k)^2\|\mathbf{u}_k\|^2 + a_{k+1}^2\|\mathbf{u}_{k+1}\|^2.\end{aligned}$$

Here, $a_1,\ldots,a_{k+1}$ are the fixed coordinates of $\mathbf{v}$, and $x_1,\ldots,x_k$ depend on $\mathbf{w}$. Therefore, $\|\mathbf{v}-\mathbf{w}\|$ takes its smallest value when $x_j = a_j$, for $j = 1, \ldots, k$. So the minimum value of $\|\mathbf{v} - \mathbf{w}\|$ occurs when

$$\mathbf{w} = a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k = \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v}\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{u}_k, \mathbf{v}\rangle}{\langle \mathbf{u}_k, \mathbf{u}_k\rangle}\mathbf{u}_k = \mathbf{v}'.$$

This is what had to be shown. Finally, to show that $\mathbf{v} - \mathbf{v}'$ is orthogonal to W, note that $\mathbf{v} - \mathbf{v}' = a_{k+1}\mathbf{u}_{k+1}$, which is orthogonal to each $\mathbf{u}_1, \ldots, \mathbf{u}_k$, and therefore to W. ♠

Example 11.39: Orthogonal projection onto a subspace

Consider $\mathbb{R}^4$ *with the usual dot product. Let* $W = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, *where*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Note that $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are orthogonal. Let*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Find the best approximation of $\mathbf{v}$ *in* W.

Solution. We calculate $\langle \mathbf{u}_1, \mathbf{v}\rangle = 9$, $\langle \mathbf{u}_1, \mathbf{u}_1\rangle = 6$, $\langle \mathbf{u}_2, \mathbf{v}\rangle = 12$, and $\langle \mathbf{u}_2, \mathbf{u}_2\rangle = 12$. Therefore, by Proposition 11.38, the desired vector is

$$\mathbf{v}' = \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v}\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 = \frac{9}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \frac{12}{12}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 4 \\ 3 \end{bmatrix}.$$

♠

Note how straightforward the calculations in this example are. All we had to do is calculate a few inner products. It is not even necessary to solve a system of equations. Such is the power of orthogonal bases.

The next example shows that we can use exactly the same method to find approximations of functions.

Example 11.40: Approximating a function by a polynomial

Let $V = C[-1,1]$ be the vector space of continuous functions, with the inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Consider the function $f \in V$ given by

$$f(x) \;=\; 1-|x| \;=\; \begin{cases} 1+x & \text{if } x<0, \\ 1-x & \text{if } x \geq 0. \end{cases}$$

Find the closest approximation to f by a polynomial of degree at most 2. Graph both f and the approximating polynomial.

Solution. Let $W = \operatorname{span}\{1, x, x^2\}$ be the subspace of V consisting of polynomials of degree at most 2. What we are looking for is an element $g \in W$ such that $\|f-g\|$ is as small as possible. We can solve this problem using Proposition 11.38.

First we need an orthogonal basis for W. We found such an orthogonal basis in Example 11.35, namely the Legendre polynomials $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2 - \frac{1}{3}$. By Proposition 11.38, the desired approximation $g \in W$ is given by:

$$g \;=\; \frac{\langle p_0, f\rangle}{\langle p_0, p_0\rangle}\,p_0 + \frac{\langle p_1, f\rangle}{\langle p_1, p_1\rangle}\,p_1 + \frac{\langle p_2, f\rangle}{\langle p_2, p_2\rangle}\,p_2.$$

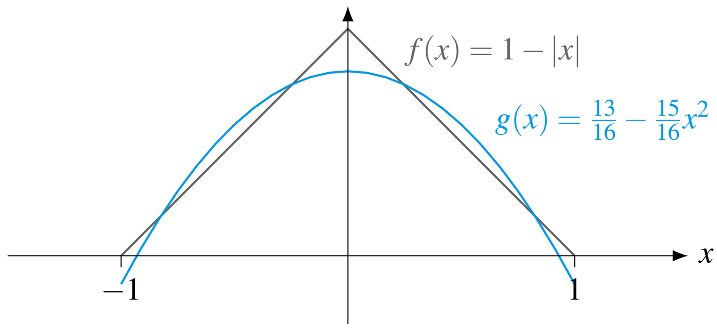
We already computed the inner products $\langle p_0, p_0\rangle = 2$, $\langle p_1, p_1\rangle = \frac{2}{3}$, and $\langle p_2, p_2\rangle = \frac{8}{45}$ in Example 11.35. We calculate the remaining inner products:

$$\begin{aligned}
\langle p_0, f\rangle &= \int_{-1}^{1} 1\cdot f(x)\,dx = \int_{-1}^{0} 1\cdot(1+x)\,dx + \int_{0}^{1} 1\cdot(1-x)\,dx = \frac{1}{2}+\frac{1}{2} = 1, \\
\langle p_1, f\rangle &= \int_{-1}^{1} x\cdot f(x)\,dx = \int_{-1}^{0} x\cdot(1+x)\,dx + \int_{0}^{1} x\cdot(1-x)\,dx = -\frac{1}{6}+\frac{1}{6} = 0, \\
\langle p_2, f\rangle &= \int_{-1}^{1} (x^2-\frac{1}{3})\cdot f(x)\,dx \\
&= \int_{-1}^{0} (x^2-\frac{1}{3})\cdot(1+x)\,dx + \int_{0}^{1} (x^2-\frac{1}{3})\cdot(1-x)\,dx = -\frac{1}{12}-\frac{1}{12} = -\frac{1}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g &= \frac{\langle p_0, f\rangle}{\langle p_0, p_0\rangle}\,p_0 + \frac{\langle p_1, f\rangle}{\langle p_1, p_1\rangle}\,p_1 + \frac{\langle p_2, f\rangle}{\langle p_2, p_2\rangle}\,p_2 \\
&= \frac{1}{2}\,p_0 + \frac{0}{2/3}\,p_1 - \frac{1/6}{8/45}\,p_2 \\
&= \frac{1}{2} - \frac{15}{16}(x^2-\frac{1}{3}) \\
&= \frac{13}{16} - \frac{15}{16}x^2.
\end{aligned}$$

The following graph shows the function $f(x)$ as well as the polynomial $g(x)$:



♠

When approximating a function f by a polynomial, as in the last example, there is no need to stop with polynomials of degree 2. We can also ask what is the best approximation of f by a polynomial of degree 3, of degree 4, of degree 5, and so on. By increasing the degree of the polynomials, we get better and better approximations to f. This leads us to the concept of a generalized Fourier series.

Definition 11.41: Generalized Fourier series

*Let V be an inner product space and let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots\}$ be an infinite orthogonal set of vectors. Let $\mathbf{v} \in V$. The **generalized Fourier series** of $\mathbf{v}$ (with respect to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$) consists of the following sequence of vectors:*

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1, \\
\mathbf{v}_2 &= \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v}\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2, \\
\mathbf{v}_3 &= \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v}\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 + \frac{\langle \mathbf{u}_3, \mathbf{v}\rangle}{\langle \mathbf{u}_3, \mathbf{u}_3\rangle}\mathbf{u}_3, \\
\mathbf{v}_4 &= \frac{\langle \mathbf{u}_1, \mathbf{v}\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v}\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 + \frac{\langle \mathbf{u}_3, \mathbf{v}\rangle}{\langle \mathbf{u}_3, \mathbf{u}_3\rangle}\mathbf{u}_3 + \frac{\langle \mathbf{u}_4, \mathbf{v}\rangle}{\langle \mathbf{u}_4, \mathbf{u}_4\rangle}\mathbf{u}_4, \\
&\vdots
\end{aligned}$$

*The vectors $\mathbf{v}_i$ are also called **generalized Fourier approximations** of $\mathbf{v}$.*

By Proposition 11.38, we know that each $\mathbf{v}_i$ is the best approximation of $\mathbf{v}$ in the subspace $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_i\}$. In particular, $\|\mathbf{v} - \mathbf{v}_{i+1}\| \leq \|\mathbf{v} - \mathbf{v}_i\|$, so each $\mathbf{v}_i$ is potentially a better approximation of $\mathbf{v}$ than the previous one. In a course on analysis[1], you will learn that in many situations, the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ can be shown to converge to $\mathbf{v}$.

Example 11.42: Generalized Fourier series

Proceeding as in Example 11.40, find the generalized Fourier approximations of $f(x) = 1 - |x|$ up to degree 8.

[1]"Algebra" is the name for those subjects where all sums are finite. This includes, for example, linear algebra and abstract algebra. "Analysis" is the name for those subjects where sums are potentially infinite (and may or may not converge). This includes, for example, calculus, complex analysis, and functional analysis.

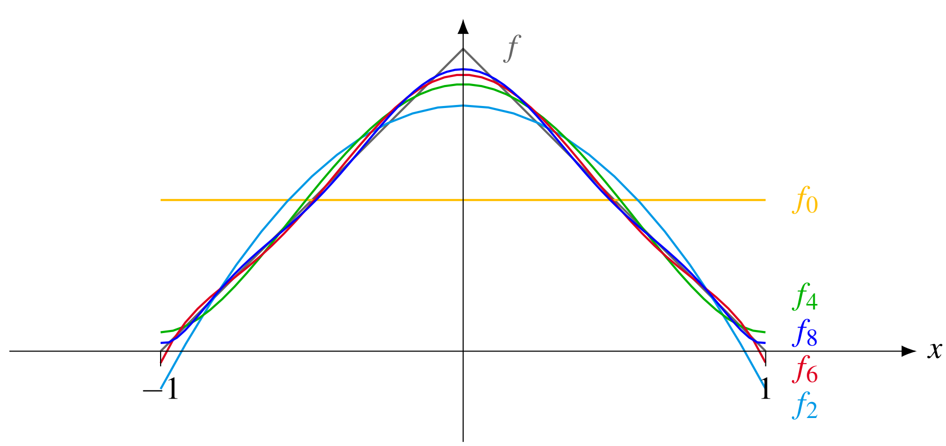
Solution. We use the Legendre polynomials $p_0, \ldots, p_8$ from Section 11.3 and calculate the relevant inner products, each of which requires solving an integral. As these integrals get a bit complicated, it is best to use a computer algebra system to compute them.

$$\begin{array}{rclcrcl}
\langle p_0, f\rangle &=& 1, && \langle p_0, p_0\rangle &=& 2,\\
\langle p_1, f\rangle &=& 0, && \langle p_1, p_1\rangle &=& \frac{2}{3},\\
\langle p_2, f\rangle &=& -\frac{1}{6}, && \langle p_2, p_2\rangle &=& \frac{8}{45},\\
\langle p_3, f\rangle &=& 0, && \langle p_3, p_3\rangle &=& \frac{8}{175},\\
\langle p_4, f\rangle &=& \frac{1}{105}, && \langle p_4, p_4\rangle &=& \frac{128}{11025},\\
\langle p_5, f\rangle &=& 0, && \langle p_5, p_5\rangle &=& \frac{128}{43659},\\
\langle p_6, f\rangle &=& -\frac{1}{924}, && \langle p_6, p_6\rangle &=& \frac{512}{693693},\\
\langle p_7, f\rangle &=& 0, && \langle p_7, p_7\rangle &=& \frac{512}{2760615},\\
\langle p_8, f\rangle &=& \frac{1}{6435}, && \langle p_8, p_8\rangle &=& \frac{32768}{703956825}.
\end{array}$$

We therefore have the following approximations:

$$\begin{array}{rcrcl}
f_0 &=& f_1 &=& \frac{1}{2}p_0,\\
f_2 &=& f_3 &=& \frac{1}{2}p_0 - \frac{15}{16}p_2,\\
f_4 &=& f_5 &=& \frac{1}{2}p_0 - \frac{15}{16}p_2 + \frac{105}{128}p_4,\\
f_6 &=& f_7 &=& \frac{1}{2}p_0 - \frac{15}{16}p_2 + \frac{105}{128}p_4 - \frac{3003}{2048}p_6,\\
f_8 &=& f_9 &=& \frac{1}{2}p_0 - \frac{15}{16}p_2 + \frac{105}{128}p_4 - \frac{3003}{2048}p_6 + \frac{109395}{32768}p_8.
\end{array}$$

The following graph shows the function f as well as its approximations f_0, f_2, f_4, f_6, and f_8. It can be seen that each successive approximation is closer to the function f than the previous one.



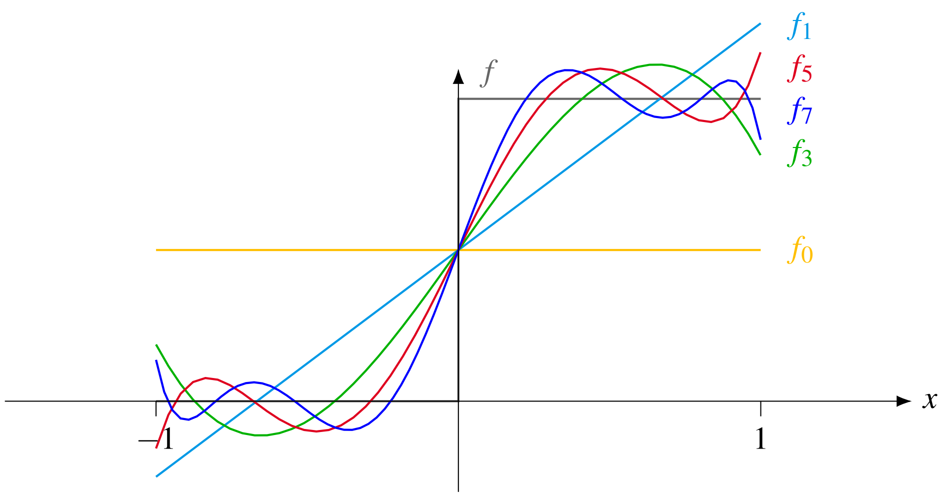
♠

Example 11.43: Generalized Fourier series

The following graph shows the generalized Fourier approximations of the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

by polynomials up to degree 7. (Note: the function f is not continuous, so not technically an element of $C[-1,1]$, but we ignore this here. Instead of the vector space of continuous functions, we can work in the vector space of piecewise continuous functions).



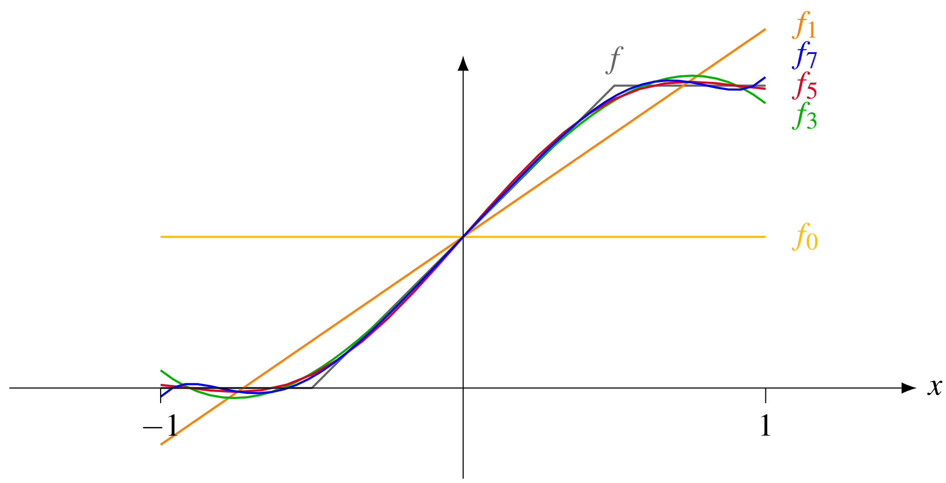
Comparing the last two examples, we see that the function in Example 11.43 is much harder to approximate well by polynomials. This is due to the discontinuity in the latter function. Nevertheless, the sequence of approximations eventually converges to f.

Example 11.44: Generalized Fourier series

The following graph shows the generalized Fourier approximations of the function

$$f(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{2}, \\ x + \frac{1}{2} & \text{if } -\frac{1}{2} \leq x < \frac{1}{2}, \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}$$

by polynomials up to degree 7.



So far, we have worked with Legendre polynomials, but there are of course other examples of orthogonal sets of functions. An important such set is given by sine and cosine waves. We will see that every periodic function can be decomposed into sine and cosine waves of varying frequencies. This was Fourier's original discovery, and the corresponding series are just known as **Fourier series** (i.e., not "generalized").

Example 11.45: Fourier series

Consider the inner product space $C[0,2\pi]$. Recall from Example 11.21 that the following functions form an orthogonal set:

$$\begin{aligned}
\mathbf{u}_0 &= 1 \\
\mathbf{u}_1 &= \sin x \\
\mathbf{u}_2 &= \cos x \\
\mathbf{u}_3 &= \sin 2x \\
\mathbf{u}_4 &= \cos 2x \\
\mathbf{u}_5 &= \sin 3x \\
\mathbf{u}_6 &= \cos 3x \\
&\vdots
\end{aligned}$$

Consider the function $f(x) = x - \pi$, where $x \in [0, 2\pi]$. Find its Fourier series.

Solution. Following Definition 11.41, we must calculate a number of inner products. We have $\langle \mathbf{u}_0, \mathbf{u}_0 \rangle = 2\pi$. Also, for all $i \geq 1$, we have $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = \pi$. We note the following antiderivatives, for $k \geq 1$:

$$\int x \sin kx\, dx \;=\; -\frac{x}{k}\cos kx + \frac{1}{k^2}\sin kx, \qquad \int x \cos kx\, dx \;=\; \frac{x}{k}\sin kx + \frac{1}{k^2}\cos kx.$$

Using these formulas, we can compute the following inner products quite easily:

$$\langle \mathbf{u}_0, f \rangle \;=\; \int_0^{2\pi} (x - \pi) \;=\; 0,$$

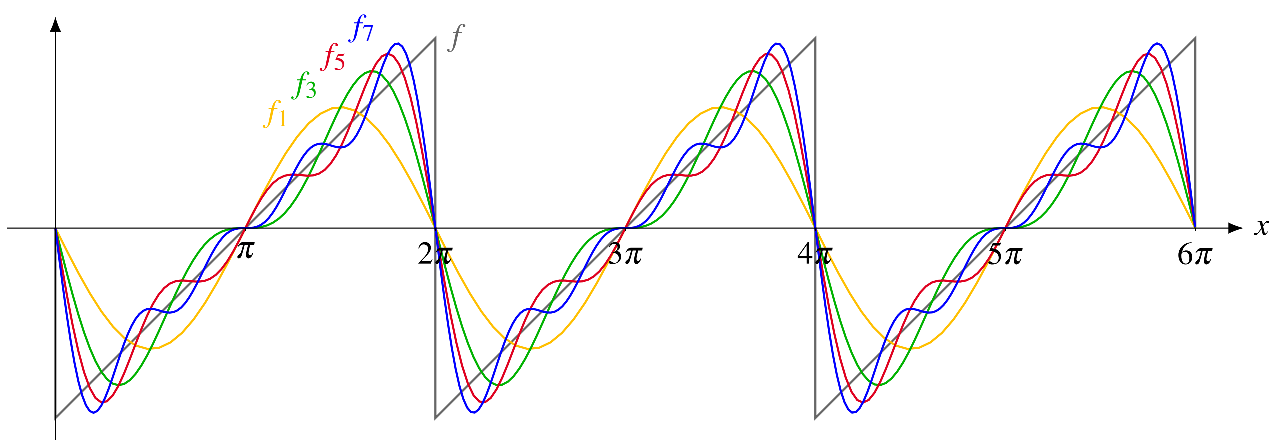
$$
\begin{aligned}
\langle \mathbf{u}_1, f\rangle &= \int_0^{2\pi} (x-\pi)\sin x &= -2\pi, \\
\langle \mathbf{u}_2, f\rangle &= \int_0^{2\pi} (x-\pi)\cos x &= 0, \\
\langle \mathbf{u}_3, f\rangle &= \int_0^{2\pi} (x-\pi)\sin 2x &= -\frac{2\pi}{2}, \\
\langle \mathbf{u}_4, f\rangle &= \int_0^{2\pi} (x-\pi)\cos 2x &= 0, \\
\langle \mathbf{u}_5, f\rangle &= \int_0^{2\pi} (x-\pi)\sin 3x &= -\frac{2\pi}{3}, \\
\langle \mathbf{u}_6, f\rangle &= \int_0^{2\pi} (x-\pi)\cos 3x &= 0,
\end{aligned}
$$

and so on. We therefore find the following Fourier series:

$$
\begin{aligned}
f_1 &= -2\sin x, \\
f_3 &= -2\sin x - \frac{2}{2}\sin 2x, \\
f_5 &= -2\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x, \\
f_7 &= -2\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x, \\
f_9 &= -2\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x - \frac{2}{5}\sin 5x,
\end{aligned}
$$

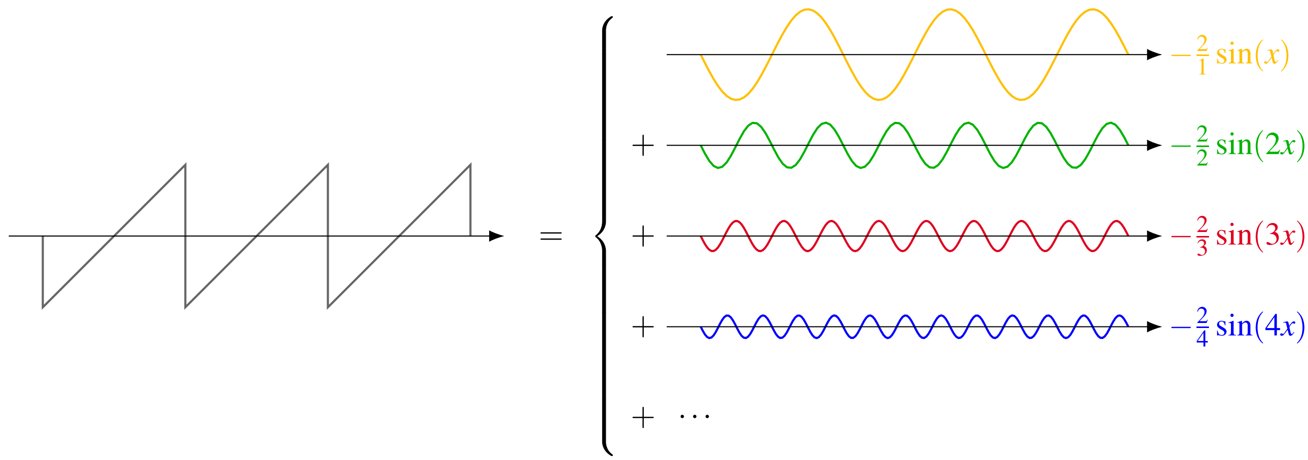
and so on. ♠

The following graph illustrates the successive approximations of this Fourier series.



Note that, although the function f is defined on the interval $[0, 2\pi]$, we have extended it periodically for $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, and so on. This makes sense because all of the orthogonal functions 1, $\sin x$, $\cos x$, $\sin 2x$, and so on, have period 2π.

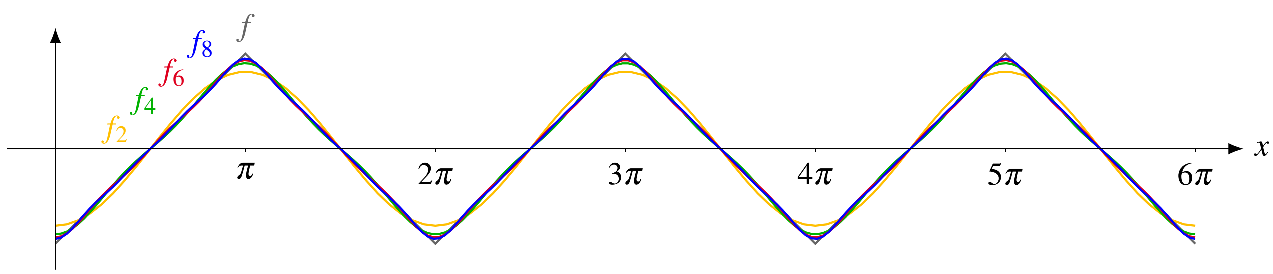
We can also illustrate the same information differently, by showing the individual sine waves making up the wave form of the function $f(x)$. In the context of audio signals, these sine waves are also called the **harmonics** of the signal.



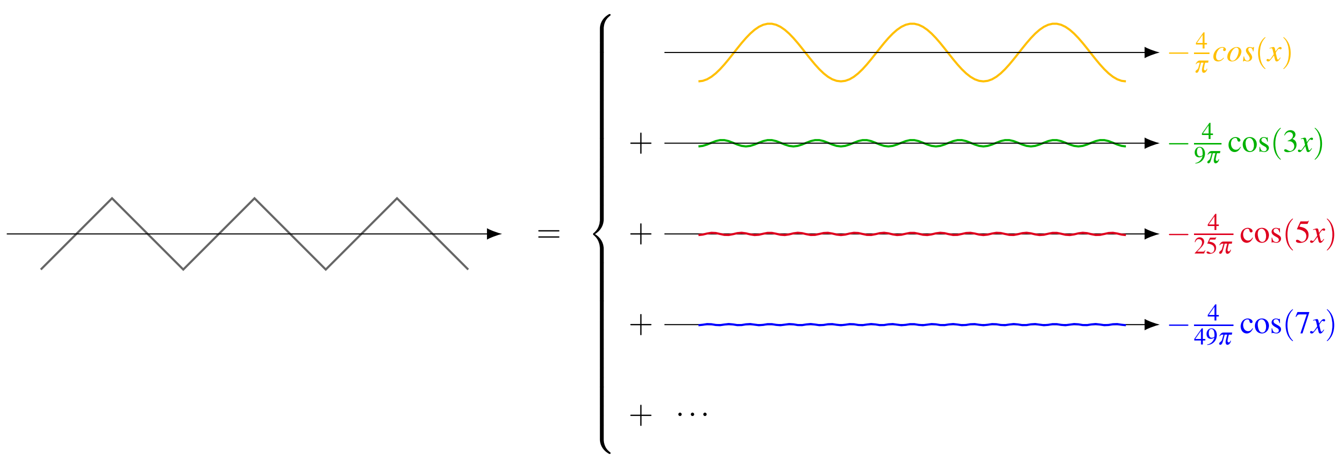
Example 11.46: Fourier series

The following graph shows the first few Fourier approximations of the function

$$f(x) = \begin{cases} x - \frac{\pi}{2} & \text{if } 0 \le x < \pi, \\ \frac{3\pi}{2} - x & \text{if } \pi \le x \le 2\pi. \end{cases}$$



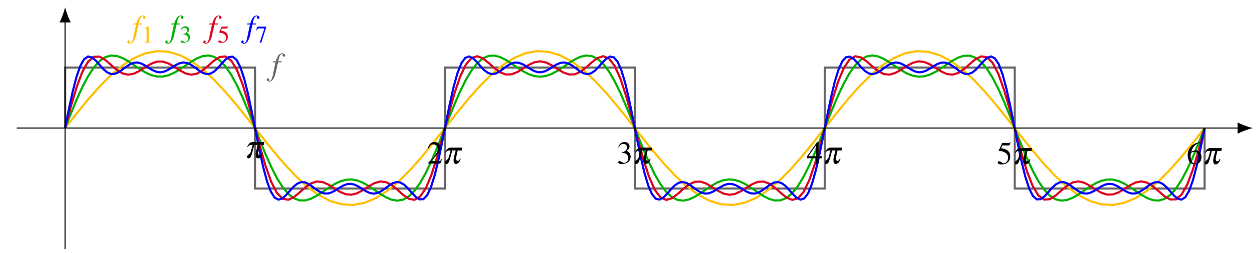
Note how rapidly this Fourier series converges to f. The following image shows the function f (extended periodically outside the interval $[0, 2\pi]$) as a sum of its harmonics. Compared to Example 11.45, we see that the higher harmonics have much smaller amplitudes, which explains the rapid convergence.



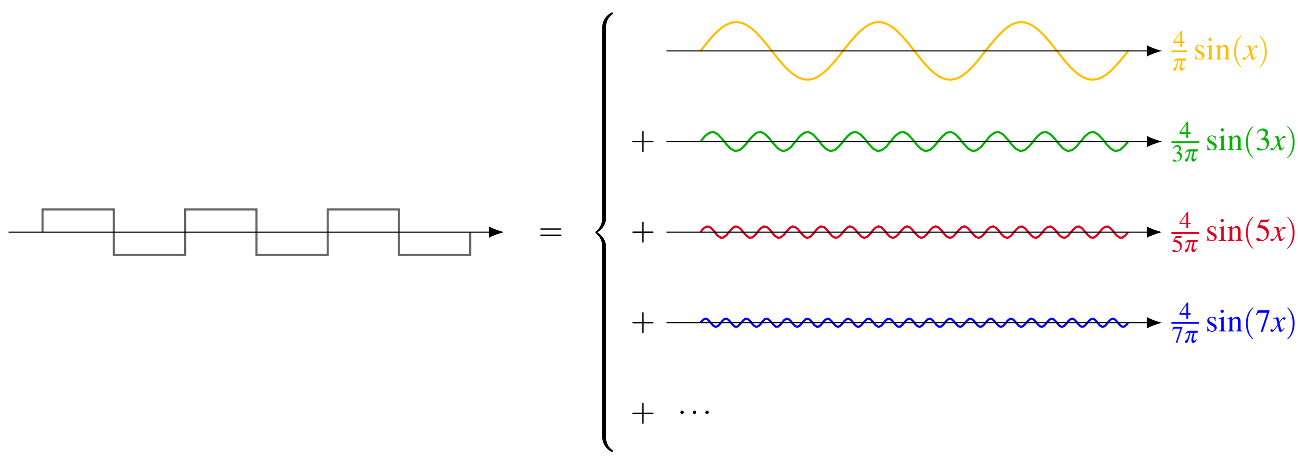
Example 11.47: Fourier series

The following graph shows the first few Fourier approximations of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \pi, \\ -1 & \text{if } \pi \leq x \leq 2\pi. \end{cases}$$



Once again, here is an image showing the function f as a sum of its harmonics:



Watch the video at `https://youtu.be/3IAMpH4xF9Q` for a demonstration of what the functions from Examples 11.45–11.47, and their harmonics, sound like as audio signals.

Exercises

Exercise 11.4.1 *Consider* $\mathbb{R}^3$ *with the usual dot product. Let*

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}.$$

Note that $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are orthogonal. Find the best approximation of* $\mathbf{v}$ *in* $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Exercise 11.4.2 *Consider $\mathbb{R}^4$ with the usual dot product. Let*

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}, \quad \textit{and} \quad \mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ -5 \\ 5 \end{bmatrix}.$$

Note that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are orthogonal. Find the best approximation of $\mathbf{v}$ in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Exercise 11.4.3 *In the inner product space $V = C[-1,1]$, consider the function $f \in V$ given by*

$$f(x) = \begin{cases} 1 & \textit{if } x < 0, \\ 1 - x & \textit{if } x \geq 0. \end{cases}$$

Find the closest approximation to f by a polynomial of degree at most 0, 1, 2, 3, and 4. Graph both f and the approximating polynomials.

Exercise 11.4.4 *In the inner product space $C[-\pi,\pi]$, consider the orthogonal set of functions from Example 11.21: $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots\}$. Let $f(x) = x^2$, where $x \in [-\pi,\pi]$. Find the Fourier series of f.*

11.5 Application: Least squares approximations and curve fitting

Outcomes

A. *Find least squares approximations for a system of equations.*

B. *Find best fit lines and parabolas for a set of data points.*

In this section, we will consider the problem of finding approximate solutions to a system of linear equations $A\mathbf{v} = \mathbf{b}$. This can be useful when the system is inconsistent, but we would still like to find a "best" answer. For example, consider the following system of equations.

$$\begin{array}{rcrcl} 0.1x & + & 0.2y & = & 0.3, \\ 0.2x & + & 0.5y & = & 0.7, \\ 0.7x & + & 0.2y & = & 0.9. \end{array}$$

This system has the solution $(x,y) = (1,1)$, so it is clearly consistent. Now imagine that we introduce some small inaccuracies into the equations. The inaccuracies might perhaps be due to round-off errors, or due to measurement errors if the coefficients are obtained from experimental data. We might end up with the following system of equations:

$$\begin{array}{rcrcl} 0.1000001x & + & 0.2y & = & 0.3, \\ 0.2x & + & 0.5y & = & 0.7, \\ 0.7x & + & 0.2y & = & 0.9. \end{array}$$

Except for a tiny error in one of the coefficients, this is the same system of equations as before. We would expect that such a small error does not affect the result much. However, this last system of equations is inconsistent; it has no solutions at all. You can see this by observing that $(x,y) = (1,1)$ is still the unique solution to the last two equations; substituting this into the first equation, we get $0.3000001 = 0.3$, which almost, but not exactly, true. What we would like to find in a situation like this is an "approximate solution", i.e., numbers x and y such that each of the three equations "almost" holds. We can formulate the problem more precisely as follows:

Problem 11.48: Least squares approximation problem

Given a (possibly inconsistent) system of equations $A\mathbf{v} = \mathbf{b}$, find $\mathbf{v}$ such that

$$\|A\mathbf{v} - \mathbf{b}\|$$

*is as small as possible. We call such a vector $\mathbf{v}$ a **least squares approximation** for the system of equations.*

To see why this is called a "least squares" approximation, consider a system of equations

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2, \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{array}$$

We can write this in matrix form $A\mathbf{v} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then

$$A\mathbf{v} - \mathbf{b} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n - b_1 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n - b_m \end{bmatrix},$$

and therefore

$$\|A\mathbf{v} - \mathbf{b}\|^2 = (a_{11}x_1 + \cdots + a_{1n}x_n - b_1)^2 + \ldots + (a_{m1}x_1 + \cdots + a_{mn}x_n - b_m)^2.$$

Therefore, minimizing $\|A\mathbf{v} - \mathbf{b}\|$ is the same as minimizing the sum of the squares of the errors of all the equations, where the error of each equation is defined to be the difference between its left-hand side and right-hand side.

We note that $\|A\mathbf{v}-\mathbf{b}\| = 0$ if and only if $A\mathbf{v} = \mathbf{b}$. Therefore, if the system of equations $A\mathbf{v} = \mathbf{b}$ is consistent, then its least squares approximations are exactly the solutions of the system of equations in the usual sense.

The least squares approximation problem has a very elegant solution, provided by the following proposition.

Proposition 11.49: Solution of the least squares approximation problem

A vector $\mathbf{v}$ *is a least squares approximation of the system of equations* $A\mathbf{v} = \mathbf{b}$ *if and only if*

$$A^T A\mathbf{v} = A^T\mathbf{b}.$$

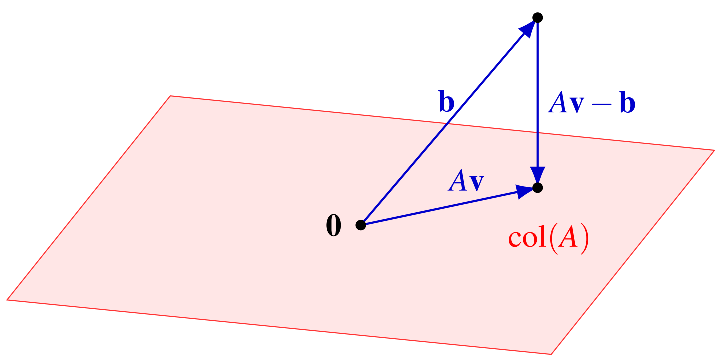
Proof. Let $\mathbf{a}_1, \ldots \mathbf{a}_n$ be the columns of the matrix A. Recall that span$\{\mathbf{a}_1, \ldots \mathbf{a}_n\}$ is called the **column space** of A, which we write as $\text{col}(A)$. If

$$\mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

is any vector, then by the definition of matrix multiplication, we have

$$A\mathbf{v} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n.$$

Therefore, a vector is of the form $A\mathbf{v}$ if and only if it is an element of $\text{col}(A)$. In particular, the equation $A\mathbf{v} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is an element of the column space of A. For $\mathbf{v}$ to be a least squares approximation, we want $\|A\mathbf{v}-\mathbf{b}\|$ to be as small as possible. This means that we are looking for the element of $\text{col}(A)$ that is closest to $\mathbf{b}$.



From Proposition 11.38, we know that this happens when $A\mathbf{v}-\mathbf{b}$ is orthogonal to $\text{col}(A)$. Since $\text{col}(A) = \text{span}\{\mathbf{a}_1, \ldots \mathbf{a}_n\}$, this is equivalent to saying that $A\mathbf{v}-\mathbf{b}$ is orthogonal to each of the vectors $\mathbf{a}_1, \ldots \mathbf{a}_n$. Therefore,

$$\mathbf{a}_i^T(A\mathbf{v}-\mathbf{b}) = 0$$

for $i = 1, \ldots, n$. Since $\mathbf{a}_1^T, \ldots \mathbf{a}_n^T$ are the rows of the matrix A^T, this system of n equations is equivalent to the single equation

$$A^T(A\mathbf{v}-\mathbf{b}) = \mathbf{0},$$

or equivalently,

$$A^T A\mathbf{v} = A^T\mathbf{b}.$$

♠

Example 11.50: Least squares approximation

Find the least squares approximation for the system of equations

$$\begin{aligned} 2x + 2y + 2z &= 1, \\ x - y - z &= -2, \\ -x - y + 2z &= 4, \\ 2x + 2y - z &= -8. \end{aligned}$$

Solution. We first write the system in matrix form $A\mathbf{v} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

By Proposition 11.49, the least squares approximation is given by the solution of the system of equations $A^T A\mathbf{v} = A^T\mathbf{b}$. We calculate:

$$A^T A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 2 & -1 & -1 & 2 \\ 2 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 8 & -1 \\ 8 & 10 & 1 \\ -1 & 1 & 10 \end{bmatrix},$$

$$A^T\mathbf{b} = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 2 & -1 & -1 & 2 \\ 2 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} -20 \\ -16 \\ 20 \end{bmatrix}.$$

Therefore, we must solve the system of equations

$$\begin{bmatrix} 10 & 8 & -1 \\ 8 & 10 & 1 \\ -1 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ -16 \\ 20 \end{bmatrix}.$$

After some row operations, we find that the unique solution is $(x,y,z) = (-1,-1,2)$. We can double-check this answer as follows. We calculate

$$A\mathbf{v} - \mathbf{b} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 6 \\ -6 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

and check that this vector is orthogonal to every column of A. Since this is the case, our answer is correct.

♠

An important application of least squares approximations is **curve fitting**: finding the "best" function of a given type (for example, linear, quadratic) to fit a given series of data points. The next two examples show how to use least square approximations to solve curve fitting problems.

Example 11.51: Least squares line

A company has collected daily data on temperature and ice cream sales over a period of one week. The data is as follows:

Date	*Peak temperature*	*Sales*
July 1	17 °C	\$320
July 2	22 °C	\$570
July 3	26 °C	\$850
July 4	20 °C	\$470
July 5	24 °C	\$750
July 6	23 °C	\$620
July 7	23 °C	\$680

The company is interested in predicting how temperature will affect future sales. Find a function of the form $y = a + bx$ that best fits the data, where x is temperature in degrees Celsius and y is sales in dollars. By "best fit", we mean that the sum of the square of the errors should be as small as possible (where each error is the difference between the dollar amount predicted by the formula $y = a + bx$ and the actual dollar amount). Such a function is called a ***least squares line*** *or a* ***linear regression*** *for the data.*

Solution. We first write down a system of equations that expresses the relationship $y = a + bx$ for each of the seven data points (x, y):

$$\begin{aligned} a + 17b &= 320, \\ a + 22b &= 570, \\ a + 26b &= 850, \\ a + 20b &= 470, \\ a + 24b &= 750, \\ a + 23b &= 620, \\ a + 23b &= 680. \end{aligned}$$

This is a system of seven equations in two variables (the variables are a and b). This system is likely inconsistent, because there are more equations than variables, and also because it is unlikely that the relationship between temperature and sales is exactly (as opposed to approximately) linear. Instead, we find the least squares approximation for the system of equations. We let

$$A = \begin{bmatrix} 1 & 17 \\ 1 & 22 \\ 1 & 26 \\ 1 & 20 \\ 1 & 24 \\ 1 & 23 \\ 1 & 23 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 320 \\ 570 \\ 850 \\ 470 \\ 750 \\ 620 \\ 680 \end{bmatrix}.$$

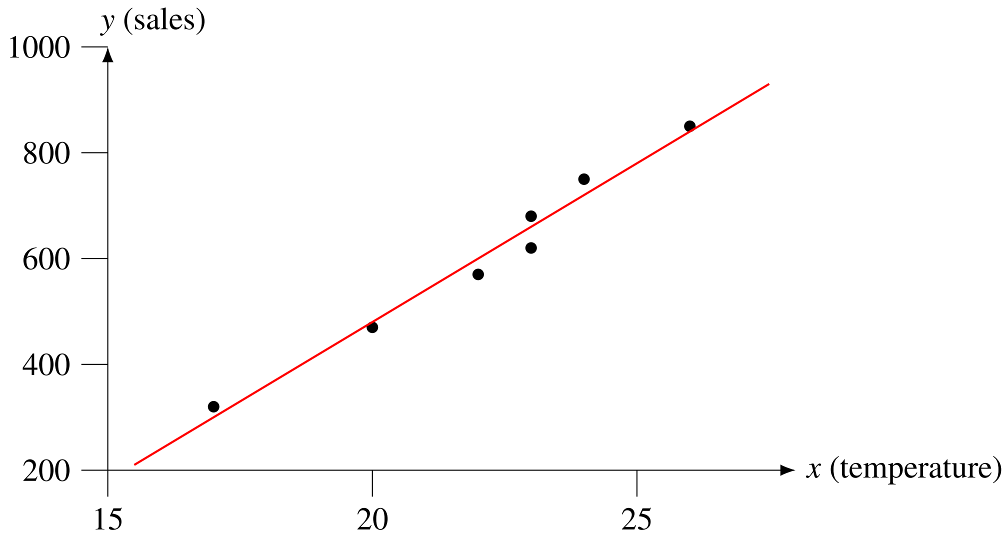
By Proposition 11.49, we must solve the equation $A^T A\mathbf{v} = A^T\mathbf{b}$. We calculate

$$A^T A = \begin{bmatrix} 7 & 155 \\ 155 & 3483 \end{bmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{bmatrix} 4260 \\ 97380 \end{bmatrix}.$$

So the system of equations we must solve is

$$\begin{bmatrix} 7 & 155 \\ 155 & 3483 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4260 \\ 97380 \end{bmatrix}.$$

After some row operations, we find the unique solutions $(a, b) = (-720, 60)$. This means that the desired linear approximation is $y = -720 + 60x$. The following plot shows this function along with the original data points.



The following table compares the observed data to the computed linear regression (sorted by increasing temperature). It also shows the error for each data point.

Temperature	Actual sales	Best linear fit	Error
17	320	300	-20
20	470	480	$+10$
22	570	600	$+30$
23	620	660	$+40$
23	680	660	-20
24	750	720	-30
26	850	840	-10

The sum of the squares of the errors is $20^2 + 10^2 + 30^2 + 40^2 + 20^2 + 30^2 + 10^2 = 4400$. ♠

Example 11.52: Least squares parabola

Find a quadratic polynomial that is the best fit for the following data points:

$$\begin{aligned}
(x_1, y_1) &= (3, 23.5), \\
(x_2, y_2) &= (4, 13.5), \\
(x_3, y_3) &= (5, 12.5), \\
(x_4, y_4) &= (6, 5.5), \\
(x_5, y_5) &= (7, 9.0), \\
(x_6, y_6) &= (8, 8.0), \\
(x_7, y_7) &= (9, 19.0).
\end{aligned}$$

Solution. We are looking for an equation of the form $y = a + bx + cx^2$. Substituting each of the seven data points into the equation, we obtain 7 equations in the unknowns a, b, and c:

$$\begin{aligned} a + 3b + 9c &= 23.5, \\ a + 4b + 16c &= 13.5, \\ a + 5b + 25c &= 12.5, \\ a + 6b + 36c &= 5.5, \\ a + 7b + 49c &= 9.0, \\ a + 8b + 64c &= 8.0, \\ a + 9b + 81c &= 19.0. \end{aligned}$$

We write this in matrix form at $A\mathbf{v} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \\ 1 & 9 & 81 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 23.5 \\ 13.5 \\ 12.5 \\ 5.5 \\ 9.0 \\ 8.0 \\ 19.0 \end{bmatrix}.$$

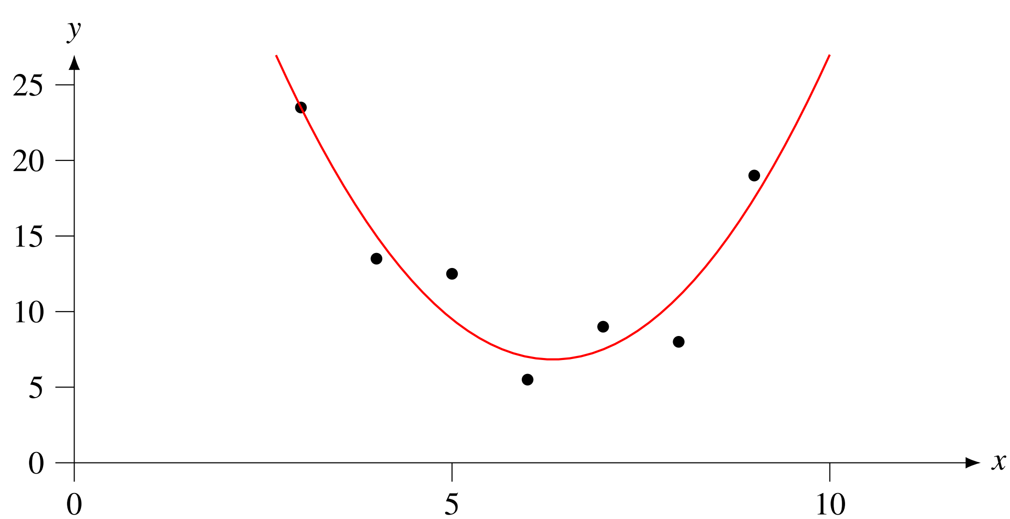
To find the least squares approximation, we calculate

$$A^T A = \begin{bmatrix} 7 & 42 & 280 \\ 42 & 280 & 2016 \\ 280 & 2016 & 15316 \end{bmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{bmatrix} 91 \\ 518 \\ 3430 \end{bmatrix}$$

and solve the system of equations $A^T A\mathbf{v} = A^T\mathbf{b}$, i.e.,

$$\begin{bmatrix} 7 & 42 & 280 \\ 42 & 280 & 2016 \\ 280 & 2016 & 15316 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 91 \\ 518 \\ 3430 \end{bmatrix}.$$

After doing some row operations, we find that the unique solution is $(a,b,c) = (67,-19,1.5)$. Therefore, the desired quadratic approximation is $y = 67 - 19x + 1.5x^2$. The following plot shows this function along with the original data points.



The following table shows the original data, the values of the best fit parabola, and the error for each data point.

x	Actual y	Best fit parabola	Error
3	23.5	23.5	0.0
4	13.5	15.0	$+1.5$
5	12.5	9.5	-3.0
6	5.5	7.0	$+1.5$
7	9.0	7.5	-1.5
8	8.0	11.0	$+3.0$
9	19.0	17.5	-1.5

The sum of the squares of the errors is $0^2 + 1.5^2 + 3^2 + 1.5^2 + 1.5^2 + 3^2 + 1.5^2 = 27$. ♠

Exercises

Exercise 11.5.1 *Find the least squares approximation for the system of equations*

$$\begin{aligned} x + 2y + 2z &= 5, \\ x + y - z &= 11, \\ x + 2y - z &= -18, \\ 2x - y + 2z &= 0. \end{aligned}$$

Exercise 11.5.2 *Find the least squares approximation for the system of equations*

$$\begin{bmatrix} -1 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}.$$

Exercise 11.5.3 *Consider the points* $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (0, 3)$, $(x_3, y_3) = (1, 3)$, $(x_4, y_4) = (2, 5)$, $(x_5, y_5) = (3, 9)$. *Find the least squares line for these points.*

Exercise 11.5.4 *Consider the points* $(x_1, y_1) = (-1, 4)$, $(x_2, y_2) = (0, -2)$, $(x_3, y_3) = (1, 4)$, $(x_4, y_4) = (2, 2)$. *Find the least squares parabola for these points.*

11.6 Orthogonal functions and orthogonal matrices

Outcomes

A. *Determine whether a linear transformation is an isometry and/or orthogonal.*

B. *Determine whether a matrix is orthogonal.*

Definition 11.53: Isometries and orthogonal maps

Let V,W be real inner product spaces, and let $T:V\to W$ be a linear transformation. We say that T is an ***isometry*** *if for all $\mathbf{u},\mathbf{v}\in V$,*

$$\langle T(\mathbf{u}),T(\mathbf{v})\rangle = \langle \mathbf{u},\mathbf{v}\rangle.$$

Moreover, T is called an ***orthogonal transformation****, or simply* ***orthogonal****, if it is an isometry and invertible.*

In particular, if T is an isometry, we have $\|T\mathbf{v}\| = \sqrt{\langle T\mathbf{v},T\mathbf{v}\rangle} = \sqrt{\langle \mathbf{v},\mathbf{v}\rangle} = \|\mathbf{v}\|$, so isometries preserve the norm. (This is fact where the name "isometry" comes from: from Greek "isos", meaning "equal", and "metron", meaning "measure"). Conversely, any norm-preserving linear function is an isometry, as the following proposition shows.

Proposition 11.54: Norm-preserving linear maps are isometries

Let $T:V\to W$ be a linear function such that for all $\mathbf{v}\in V$, $\|T\mathbf{v}\| = \|\mathbf{v}\|$. Then T is an isometry.

Proof. We first note that for all $\mathbf{u},\mathbf{v}\in V$,

$$\langle \mathbf{u},\mathbf{v}\rangle = \frac{1}{2}(\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle) = \frac{1}{2}(\langle \mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}\rangle - \langle \mathbf{u},\mathbf{u}\rangle - \langle \mathbf{v},\mathbf{v}\rangle) = \frac{1}{2}(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2).$$

Now suppose T is norm-preserving and linear. Then for all $\mathbf{u},\mathbf{v}\in V$,

$$\begin{aligned}
\langle T(\mathbf{u}),T(\mathbf{v})\rangle &= \frac{1}{2}(\|T(\mathbf{u})+T(\mathbf{v})\|^2 - \|T(\mathbf{u})\|^2 - \|T(\mathbf{v})\|^2) \\
&= \frac{1}{2}(\|T(\mathbf{u}+\mathbf{v})\|^2 - \|T(\mathbf{u})\|^2 - \|T(\mathbf{v})\|^2) \\
&= \frac{1}{2}(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}^2\|) \\
&= \langle \mathbf{u},\mathbf{v}\rangle.
\end{aligned}$$

Therefore, T is an isometry. ♠

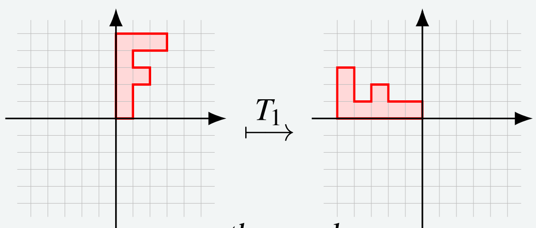
We also note that if T is an isometry and $\mathbf{u}\perp\mathbf{v}$, then $T(\mathbf{u})\perp T(\mathbf{v})$, because $\langle T(\mathbf{u}),T(\mathbf{v})\rangle = \langle \mathbf{u},\mathbf{v}\rangle = 0$. So isometries preserve right angles. In fact, isometries also preserve arbitrary angles, due to the formula $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\langle T(\mathbf{u}),T(\mathbf{v})\rangle}{\|T(\mathbf{u})\|\,\|T(\mathbf{v})\|}$.

Example 11.55: Orthogonal transformations on R^n

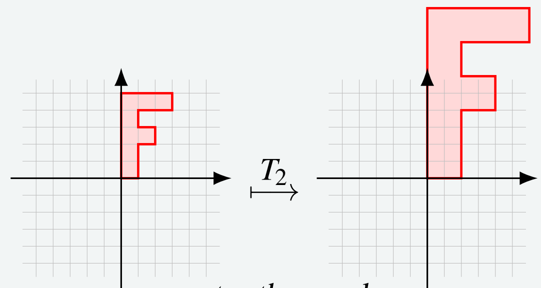
Let $T_1, T_2, T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations with matrices

$$[T_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad [T_2] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad [T_3] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

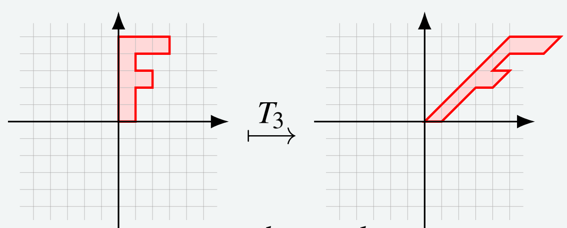
respectively. T_1 is a counterclockwise rotation by 90°, T_2 is a scaling by a factor of 2, and T_3 is a shearing. Then T_1 is orthogonal, but T_2 and T_3 are not. Note that T_1, being a rotation, preserves lengths and angles. T_2 preserves angles but not lengths. T_3 preserves neither lengths nor angles.



orthogonal



not orthogonal



not orthogonal

Proposition 11.56: The matrix of an orthogonal transformation

Let V, W be finite-dimensional real inner product spaces and let $T : V \to W$ be a linear transformation. Let B and C be orthonormal bases of V and W, respectively, and let $P = [T]_{C,B}$ be the matrix of T with respect to the bases B and C. Then

(a) T is an isometry if and only if $P^T P = I$.

(b) T is orthogonal if and only if $P^T P = I$ and $\dim V = \dim W$.

Proof. (a) Recall from Proposition 11.28 that for all $\mathbf{u}, \mathbf{v} \in V$, we have $\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_B \cdot [\mathbf{v}]_B$. Consider two vectors $\mathbf{u}, \mathbf{v} \in V$ and let $\mathbf{t} = [u]_B$ and $\mathbf{s} = [v]_B$. We have:

$$\begin{aligned} \langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \quad &\iff \quad [T(\mathbf{u})]_C \cdot [T(\mathbf{v})]_C = [\mathbf{u}]_B \cdot [\mathbf{v}]_B \\ &\iff \quad ([T]_{C,B}[\mathbf{u}]_B) \cdot ([T]_{C,B}[\mathbf{v}]_B) = [\mathbf{u}]_B \cdot [\mathbf{v}]_B \\ &\iff \quad (P\mathbf{t}) \cdot (P\mathbf{s}) = \mathbf{t} \cdot \mathbf{s} \\ &\iff \quad \mathbf{t}^T P^T P \mathbf{s} = \mathbf{t}^T \mathbf{s} \\ &\iff \quad \mathbf{t}^T P^T P \mathbf{s} = \mathbf{t}^T I \mathbf{s}. \end{aligned}$$

Therefore, T is an isometry if and only if $\mathbf{t}^T P^T P\mathbf{s} = \mathbf{t}^T I\mathbf{s}$ holds for all $\mathbf{t},\mathbf{s} \in \mathbb{R}^m$. If $P^T P = I$, then this is clearly true. Conversely, assume $\mathbf{t}^T P^T P\mathbf{s} = \mathbf{t}^T I\mathbf{s}$ holds for all $\mathbf{t},\mathbf{s} \in \mathbb{R}^m$. Then by taking $\mathbf{t}$ and $\mathbf{s}$ to be the i^{th} and j^{th} standard basis vector, it follows that the (i,j)-entry of $P^T P$ is equal to the (i,j)-entry of I, for all i and j. Therefore, $P^T P = I$.

(b) By definition, T is orthogonal if and only if it is an isometry and invertible. Therefore, if T is orthogonal, then $P^T P = I$ by part (a) and $\dim V = \dim W$ because every invertible matrix is square (see Theorem 4.48). Conversely, assume that $P^T P = I$ and $\dim V = \dim W$. Then T is an isometry by part (a). Also, by Theorem 4.64, since P is square and P^T is a left inverse of P, the matrix P, and therefore the linear transformation T, is invertible. Therefore, T is orthogonal. ♠

Example 11.57: Orthogonal transformations

Which of the following matrices define orthogonal transformations? Which ones define isometries?

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}.$$

Solution. We have

$$A^T A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

and therefore A defines an isometry and (since it is also a square matrix) an orthogonal transformation. We have

$$B^T B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq I,$$

so the linear transformation defined by B is neither an isometry nor orthogonal. A similar pair of calculations shows that $C^T C = I$ and $D^T D = I$, so both C and D are isometries. Since C is a square matrix, it is also orthogonal. D is not a square matrix, and therefore not orthogonal. ♠

In light of Proposition 11.56, we say that an $n \times n$-matrix P is orthogonal if $P^T P = I$.

Definition 11.58: Orthogonal matrix

An $n \times n$-matrix P is called ***orthogonal*** *if $P^T P = I$.*

The following proposition gives several equivalent ways of checking whether an $n \times n$-matrix P is orthogonal.

Proposition 11.59: Conditions for orthogonal matrices

The following are equivalent for an $n \times n$-matrix P:

(a) P *is orthogonal.*

(b) $P^T P = I$.

(c) P *is invertible and* $P^{-1} = P^T$.

(d) $PP^T = I$.

(e) P^T *is orthogonal.*

(f) *The columns of* P *form an orthonormal set of vectors.*

(g) *The rows of* P *form an orthonormal set of vectors.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is just the definition of orthogonality, bearing in mind that P is assumed to be a square matrix. The equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follow from Theorem 4.64, since a square matrix is invertible if and only if it is left invertible if and only if it is right invertible. The equivalence (d) $\Leftrightarrow$ (e) is again just the definition of orthogonality, this time applied to P^T. To show (b) $\Leftrightarrow$ (f), let $\mathbf{a}_1,\ldots,\mathbf{a}_n$ be the columns of P, and let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be the standard basis vectors of $\mathbb{R}^n$. Then we have $P^T P = I$ if and only if for all i, j, the (i, j)-entry of $P^T P$ is equal to the (i, j)-entry of I. But the (i, j)-entry of $P^T P$ is $\mathbf{e}_i^T P^T P\mathbf{e}_j = \mathbf{a}_i^T \mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j$. On the other hand, the (i, j)-entry of I is 1 if $i = j$ and 0 if $i \neq j$. It follows that $P^T P = I$ if and only if for all i, j,

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 1 & \text{when } i = j \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

But this is exactly what it means for $\mathbf{a}_1,\ldots,\mathbf{a}_n$ to form an orthonormal set. The proof of (d) $\Leftrightarrow$ (g) is completely analogous, using rows instead of columns. ♠

Example 11.60: Orthogonal matrices

Determine which of the following matrices are orthogonal by checking whether the columns form an orthonormal set.

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad D = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Solution. By Proposition 11.59, it suffices to check whether the columns of each matrix are orthonormal. This is the case for A and B. For example, the columns of B are

$$\mathbf{b}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix},$$

and we have

$$\mathbf{b}_1 \cdot \mathbf{b}_1 = \frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 1, \qquad \mathbf{b}_1 \cdot \mathbf{b}_2 = \frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 0,$$

$$\mathbf{b}_2 \cdot \mathbf{b}_2 = \frac{1}{9}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 1, \qquad \mathbf{b}_1 \cdot \mathbf{b}_3 = \frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 0,$$

$$\mathbf{b}_3 \cdot \mathbf{b}_3 = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 1, \qquad \mathbf{b}_2 \cdot \mathbf{b}_3 = \frac{1}{9}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 0.$$

So the columns of B are orthonormal. The columns of C are orthogonal, but not normalized, so C is not an orthogonal matrix. The columns of D are normalized, but not orthogonal, so D is not an orthogonal matrix either. ♠

Proposition 11.61: Properties of orthogonal matrices

(a) If P, Q are orthogonal $n \times n$-matrices, then PQ is orthogonal.

(b) The identity matrix is orthogonal.

(c) If P is orthogonal, then so is P^{-1}.

(d) If P is an orthogonal $n \times n$-matrix and Q is an orthogonal $m \times m$-matrix, then the $(n+m) \times (n+m)$-matrix

$$\left[\begin{array}{c|c} P & 0 \\ \hline 0 & Q \end{array}\right]$$

is orthogonal.

Exercises

Exercise 11.6.1 *Which of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are orthogonal?*

(a) A rotation by 30 degrees about the z-axis.

(b) A reflection about the plane $x = y$.

(c) A scaling by a factor of 2.

(d) A scaling by a factor of -1.

Exercise 11.6.2 *Which of the following matrices define orthogonal transformations? Which ones define isometries?*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad D = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

Exercise 11.6.3 *Determine which of the following matrices are orthogonal by checking whether the columns form an orthonormal set.*

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad C = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix}, \quad D = \frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

11.7 Diagonalization of real symmetric matrices

Outcomes

A. *Compute the eigenvalues of a real symmetric matrix.*

B. *Compute an orthogonal basis of eigenvectors for a real symmetric matrix.*

C. *Orthogonally diagonalize a real symmetric matrix.*

In Chapter 8, we saw that some matrices are diagonalizable, and others are not. In this section, we will see that the theory of diagonalization is much nicer when the matrix to be diagonalized is symmetric. Recall that a matrix A is **symmetric** if $A = A^T$. We begin with some observations about the eigenvalues and eigenvectors of a real symmetric matrix.

Proposition 11.62: Eigenvalues and eigenvectors of real symmetric matrices

Let A be a symmetric matrix with real entries. Then

(a) *All eigenvalues of A are real (i.e., not complex).*

(b) *Eigenvectors for distinct eigenvalues of A are orthogonal.*

Proof. (a) Suppose λ is a (possibly complex) eigenvalue of A, with (possibly complex) eigenvector $\mathbf{v}$. We will show that λ is in fact real. Let $\overline{\lambda}$ be the complex conjugate of λ, and let $\overline{\mathbf{v}}$ be the complex conjugate of $\mathbf{v}$, i.e., the vector obtained from $\mathbf{v}$ by taking the complex conjugate of each of its entries. Since $\mathbf{v}$ is an eigenvector for eigenvalue λ of A, we have

$$\overline{\mathbf{v}}^T A\mathbf{v} = \lambda\overline{\mathbf{v}}^T\mathbf{v}. \tag{11.2}$$

Taking the complex conjugate of both sides of the equation, and using the fact that $\overline{A} = A$ (since A is real), we get

$$\mathbf{v}^T A\overline{\mathbf{v}} = \overline{\lambda}\mathbf{v}^T\overline{\mathbf{v}}. \tag{11.3}$$

Then, taking the transpose of both sides of the equation, and using the fact that $A^T = A$ (since A is symmetric), we have

$$\overline{\mathbf{v}}^T A\mathbf{v} = \overline{\lambda}\overline{\mathbf{v}}^T\mathbf{v}. \tag{11.4}$$

Comparing equations (11.2) and (11.4), we find that $\lambda\overline{\mathbf{v}}^T\mathbf{v} = \overline{\lambda}\overline{\mathbf{v}}^T\mathbf{v}$. Since $\overline{\mathbf{v}}^T\mathbf{v}$ is a non-zero scalar, it follows that $\lambda = \overline{\lambda}$, i.e., λ is real.

(b) Suppose $\mathbf{v}$ is an eigenvector for eigenvalue λ, $\mathbf{w}$ is an eigenvector for eigenvalue μ, and $\lambda \neq \mu$. We evaluate $\mathbf{v}^T A\mathbf{w}$ in two different ways. On the on hand, we have

$$\mathbf{v}^T A\mathbf{w} = \mathbf{v}^T(A\mathbf{w}) = \mathbf{v}^T(\mu\mathbf{w}) = \mu\mathbf{v}^T\mathbf{w}.$$

On the other hand, we have

$$\mathbf{v}^T A\mathbf{w} = (\mathbf{v}^T A)\mathbf{w} = (A^T\mathbf{v})^T\mathbf{w} = (A\mathbf{v})^T\mathbf{w} = (\lambda\mathbf{v})^T\mathbf{w} = \lambda\mathbf{v}^T\mathbf{w}.$$

Therefore, $\mu\mathbf{v}^T\mathbf{w} = \lambda\mathbf{v}^T\mathbf{w}$, or equivalently $(\lambda-\mu)\mathbf{v}^T\mathbf{w} = 0$. Since by assumption, $\lambda - \mu \neq 0$, we must have $\mathbf{v}^T\mathbf{w} = 0$, i.e., $\mathbf{v} \perp \mathbf{w}$. ♠

Recall that an $n\times n$-matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$. We say that A is **orthogonally diagonalizable** if P can, moreover, be chosen to be orthogonal. Orthogonal diagonalizability is a convenient property, because when P is orthogonal, then $P^{-1} = P^T$, and we can interchangeably write $D = P^{-1}AP$ or $D = P^TAP$. The following is the main theorem about the diagonalization of real symmetric matrices.

Theorem 11.63: Diagonalization of real symmetric matrices

Every real symmetric matrix A is orthogonally diagonalizable.

Proof. By induction on the size of the matrix. For a 1×1-matrix, there is nothing to show, as it is already diagonal. Now consider a real symmetric $n\times n$-matrix A with $n \geq 2$. By the fundamental theorem of algebra, the characteristic polynomial has at least one root, so that A has at least one (possibly complex) eigenvalue λ. By Proposition 11.62(a), λ is real. Since we can solve the system of equations $(A-\lambda I)\mathbf{v} = \mathbf{0}$ over the real numbers, there exists a real eigenvector $\mathbf{v}$ for the eigenvalue λ. We can assume without loss of generality that $\mathbf{v}$ is normalized, because else we could replace $\mathbf{v}$ by $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$. By the Gram-Schmidt method, we can find an orthonormal basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ of $\mathbb{R}^n$ such that $\mathbf{u}_1 = \mathbf{v}$. Let Q be the orthogonal matrix that has $\mathbf{u}_1,\ldots,\mathbf{u}_n$ as its columns, and consider $B = Q^{-1}AQ$. Since $B\mathbf{e}_1 = Q^{-1}AQ\mathbf{e}_1 = Q^{-1}A\mathbf{u}_1 = \lambda Q^{-1}\mathbf{u}_1 = \lambda\mathbf{e}_1$, the matrix B is of the form

$$B = \begin{bmatrix} \lambda & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Moreover, since $B = Q^{-1}AQ = Q^TAQ$, the matrix B is symmetric. Therefore $b_{12}, \ldots, b_{1n} = 0$, and B is of the form

$$B = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right],$$

where C is a symmetric matrix of dimension $n-1$. By induction hypothesis, C is orthogonally diagonalizable, i.e., there exists an orthogonal matrix R such that $R^{-1}CR = D$ is diagonal. Let

$$S = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & R & \\ 0 & & & \end{array}\right].$$

Then S is orthogonal by Proposition 11.61(4). Let $E = S^{-1}BS$. Then

$$E = S^{-1}BS = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & R^{-1}CR & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D & \\ 0 & & & \end{array}\right].$$

Therefore, E is diagonal. Let $P = QS$. Then P is orthogonal by Proposition 11.61(1). Moreover, we have

$$P^{-1}AP = S^{-1}Q^{-1}AQS = S^{-1}BS = E.$$

Therefore, A is orthogonally diagonalizable. ♠

Example 11.64: Diagonalization of real symmetric matrices

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix},$$

i.e., find an orthogonal matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution. We proceed in much the same way as in Chapter 8, except that at a crucial moment, we ensure that P is orthogonal. We start by calculating the characteristic polynomial of A:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) - 4 = \lambda^2 - 9\lambda + 14.$$

We find the roots using the quadratic formula. The roots of the characteristic polynomial, and therefore the eigenvalues of A, are $\lambda = 7$ and $\lambda = 2$. For the eigenvalue $\lambda = 7$, we find the eigenvector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and for the eigenvalue $\lambda = 2$, we find the eigenvector

$$\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

We note that these two eigenvectors are orthogonal to each other, exactly as predicted by Proposition 11.62. So the vectors $\{\mathbf{v}, \mathbf{w}\}$ form an orthogonal set. We turn this into an orthonormal set by normalizing each eigenvector, i.e., the normalized eigenvectors are

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

We let P be the matrix that has columns $\mathbf{u}_1$ and $\mathbf{u}_2$, i.e.,

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Note that since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal, P is automatically orthogonal by Proposition 11.59. Moreover, we have

$$P^{-1} = P^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Finally,

$$P^{-1}AP = P^TAP = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}.$$

♠

Example 11.65: Diagonalization of real symmetric matrices

Find an orthogonal matrix P and a diagonal matrix D such $D = P^{-1}AP$, where

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{bmatrix}.$$

Solution. Again, we start by calculating the characteristic polynomial and its roots:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ 1 & 3-\lambda & 2 \\ -2 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 32.$$

The roots are $\lambda = 4$ and $\lambda = -2$. To find the eigenvectors for $\lambda = 4$, we solve the system of equations

$$(A - 4I)\mathbf{v} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}\mathbf{v} = \mathbf{0}.$$

The solution space is 2-dimensional, with basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

To find the eigenvectors for $\lambda = -2$, we solve the system of equations

$$(A+2I)\mathbf{v} = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

The solution space is 1-dimensional, with basis

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

As predicted by Proposition 11.62, $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. However, there is a slight complication: $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal to each other. This is because they are eigenvectors for the *same* eigenvalue ($\lambda = 4$), not for *distinct* eigenvalues. We found basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the eigenspace for $\lambda = 4$, but it doesn't happen to be an orthogonal basis. However, we can fix this by applying the Gram-Schmidt method to $\{\mathbf{v}_1, \mathbf{v}_2\}$. This yields the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for the eigenspace for $\lambda = 4$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

We now have an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3\}$ of eigenvectors of the matrix A. We normalize the three vectors and use them as the columns of P:

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

It seems that inverting P will not be all that easy, but in fact, since P is orthogonal, the inverse is just the transpose:

$$P^{-1} = P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

We have

$$P^{-1}AP = D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

♠

Exercises

Exercise 11.7.1 *For each of the following symmetric matrices, find the eigenvalues, an orthonormal basis for each eigenspace, and then orthogonally diagonalize the matrix.*

$$(a) \quad A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}, \quad (b) \quad A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad (c) \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad (d) \quad A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{bmatrix}.$$

Exercise 11.7.2 *Prove the converse of Theorem 11.63: if a matrix A is orthogonally diagonalizable, then A is symmetric.*

11.8 Positive semidefinite and positive definite matrices

Outcomes

A. *Determine whether a matrix is positive semidefinite and/or positive definite, either directly or by looking at the eigenvalues.*

B. *Determine whether a matrix is positive semidefinite and/or positive definite using Descartes' rule of signs.*

C. *Determine whether a matrix defines an inner product.*

In Example 11.3, we saw that it is sometimes possible to define an inner product on $\mathbb{R}^n$ from an $n \times n$-matrix A by the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}.$$

We will now explore in more detail under what conditions this formula defines an inner product.

Definition 11.66: Positive semidefinite and positive definite matrices

Let A be an $n \times n$-matrix over the real numbers.

- *A is called* ***symmetric*** *if $A = A^T$.*
- *A is called* ***positive semidefinite*** *if it is symmetric and for all $\mathbf{v} \in \mathbb{R}^n$, we have $\mathbf{v}^T A \mathbf{v} \geq 0$.*
- *A is called* ***positive definite*** *if it is positive semidefinite and $\mathbf{v}^T A \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.*

Equivalently, the positive definite property can also be stated as follows: A is positive definite if it is symmetric and for all $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$, we have $\mathbf{v}^T A \mathbf{v} > 0$.

Example 11.67: Positive semidefinite and positive definite matrices

Which of the following matrices are positive semidefinite? Which ones are positive definite?

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}.$$

Solution. The matrix A is positive definite. To see why, consider any vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$\mathbf{v}^T A\mathbf{v} \;=\; \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; 2x^2 - 2xy + y^2 \;=\; x^2 + (x-y)^2 \;\geq\; 0.$$

This inequality implies that A is positive semidefinite. Moreover, $\mathbf{v}^T A\mathbf{v} = 0$ if and only if $x = 0$ and $x - y = 0$, which implies $x = y = 0$. So $\mathbf{v} = \mathbf{0}$ is the only solution of $\mathbf{v}^T A\mathbf{v} = 0$, and A is positive definite.

The matrix B is not positive semidefinite (and therefore not positive definite either). For example, consider $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then

$$\mathbf{v}^T B\mathbf{v} \;=\; \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;=\; 1 - 2 - 2 + 1 \;=\; -2 \;<\; 0,$$

showing that B is not positive semidefinite.

The matrix C is positive semidefinite, but not positive definite. To see why, consider $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$\mathbf{v}^T C\mathbf{v} \;=\; \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; x^2 - 2xy + y^2 \;=\; (x-y)^2 \;\geq\; 0.$$

Therefore, C is positive semidefinite. However, it is not positive definite because, for example, for $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $\mathbf{v} \neq \mathbf{0}$ but $\mathbf{v}^T C\mathbf{v} = 0$.

The matrix D is not symmetric, and therefore neither positive semidefinite nor positive definite. ♠

The interest of positive definite matrices lies in the following proposition.

Proposition 11.68: Positive definite matrices and inner products on $\mathbb{R}^n$

Let A be an $n \times n$-matrix over the real numbers. Then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$$

defines an inner product on $\mathbb{R}^n$ if and only if A is positive definite. Moreover, every inner product on $\mathbb{R}^n$ arises from some positive definite matrix A in this way.

Proof. Define $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$. We will check each of the three properties of an inner product from Definition 11.1. Linearity holds for all matrices A, because

$$\langle \mathbf{u}, k\mathbf{v} \rangle \;=\; \mathbf{u}^T A(k\mathbf{v}) \;=\; k(\mathbf{u}^T A\mathbf{v}) \;=\; k\langle \mathbf{u}, \mathbf{v} \rangle$$

and

$$\langle \mathbf{u}, \mathbf{v}+\mathbf{v}' \rangle \;=\; \mathbf{u}^T A(\mathbf{v}+\mathbf{v}') \;=\; \mathbf{u}^T A\mathbf{v}+\mathbf{u}^T A\mathbf{v}' \;=\; \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{v}'\rangle.$$

For symmetry, first observe that for all $\mathbf{u},\mathbf{v}$,

$$\langle \mathbf{v},\mathbf{u}\rangle \;=\; \mathbf{v}^T A\mathbf{u} \;=\; (\mathbf{v}^T A\mathbf{u})^T \;=\; \mathbf{u}^T A^T \mathbf{v}.$$

Therefore $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle$ if and only if $\mathbf{u}^T A\mathbf{v} = \mathbf{u}^T A^T \mathbf{v}$. This holds for all $\mathbf{u},\mathbf{v}$ if and only if $A = A^T$. Therefore, symmetry holds if and only if A is symmetric. Finally, the positive definite property of the inner product holds, by definition, if and only if A is positive definite.

To prove the second part, consider any inner product $\langle \mathbf{u},\mathbf{v}\rangle$ on $\mathbb{R}^n$. Let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be the standard basis vectors, and define $a_{ij} = \langle \mathbf{e}_i,\mathbf{e}_j\rangle$. Then $A = \left[a_{ij}\right]$ is a matrix. We claim that $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$ for all $\mathbf{u},\mathbf{v}$. Indeed, let $\mathbf{u} = u_1\mathbf{e}_1+\ldots+u_n\mathbf{e}_n$ and $\mathbf{v} = v_1\mathbf{e}_1+\ldots+v_n\mathbf{e}_n$. Then

$$\begin{aligned}
\langle \mathbf{u},\mathbf{v}\rangle &= \langle u_1\mathbf{e}_1+\ldots+u_n\mathbf{e}_n,\; v_1\mathbf{e}_1+\ldots+v_n\mathbf{e}_n\rangle \\
&= u_1v_1\langle \mathbf{e}_1,\mathbf{e}_1\rangle + u_1v_2\langle \mathbf{e}_1,\mathbf{e}_2\rangle+\ldots+u_nv_n\langle \mathbf{e}_n,\mathbf{e}_n\rangle \\
&= u_1v_1a_{11}+u_1v_2a_{12}+\ldots+u_nv_na_{nn} \\
&= \mathbf{u}^T A\mathbf{v}.
\end{aligned}$$

By the first part, A is positive definite. ♠

Given a symmetric matrix A, it is not always an easy task to determine whether A is positive definite (or semidefinite) by using the definition directly. This would require checking the condition $\mathbf{v}^T A\mathbf{v} \geq 0$ for *all* vectors $\mathbf{v}$, of which there are infinitely many. The following proposition gives us a more practical method for determining whether a matrix is positive (semi)definite.

Proposition 11.69: Characterization of positive (semi)definite matrices using eigenvalues

Let A be a symmetric $n\times n$-matrix, and let $\lambda_1,\ldots,\lambda_n$ be its eigenvalues. Then A is positive semidefinite if and only if $\lambda_1,\ldots,\lambda_n \geq 0$. Moreover, A is positive definite if and only if $\lambda_1,\ldots,\lambda_n > 0$.

Proof. By Theorem 11.63, we know that A is orthogonally diagonalizable, i.e., $D = P^T AP$, where P is an orthogonal matrix and

$$D = \left[\begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{array}\right].$$

Let $\mathbf{v} = [x_1,\ldots,x_n]^T$ be any vector, and let $\mathbf{w} = P\mathbf{v}$. Then $\mathbf{w}^T A\mathbf{w} = \mathbf{v}^T P^T AP\mathbf{v} = \mathbf{v}^T D\mathbf{v}$, so that A is positive (semi)definite if and only if D is positive (semi)definite. Also, we have $\mathbf{v}^T D\mathbf{v} = \lambda_1x_1^2+\ldots+\lambda_nx_n^2$. Therefore $\mathbf{v}^T D\mathbf{v} \geq 0$ for all $\mathbf{v}$ if and only if $\lambda_1,\ldots,\lambda_n \geq 0$. Moreover, $\mathbf{v}^T D\mathbf{v} > 0$ for all $\mathbf{v} \neq \mathbf{0}$ if and only if $\lambda_1,\ldots,\lambda_n > 0$, as claimed. ♠

Example 11.70: Using eigenvalues to check whether a matrix is positive definite

Use eigenvalues to determine whether the matrix

$$A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

is positive definite, positive semidefinite, or neither.

Solution. The characteristic polynomial is

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 5-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3+11\lambda^2-30\lambda = -\lambda(\lambda-5)(\lambda-6).$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 5$, and $\lambda_3 = 6$. Therefore, the matrix A is positive semidefinite, but not positive definite. ♠

While Proposition 11.69 gives us a method to determine whether a matrix A is positive definite (or semidefinite), it still requires finding all of the eigenvalues of A, i.e., to factor the characteristic polynomial. This can be a difficult calculation, especially if the degree of the polynomial is large or if the roots are not integers. Fortunately, there is a better way to determine whether a matrix is positive definite (or semidefinite) by directly looking at the characteristic polynomial, without having to calculate its roots. The method was found by René Descartes in 1637 and is called *Descartes' rule of signs*. The general form of Descartes' rule of signs is actually more complicated than what we consider here. We state a version that has been specialized to characteristic polynomials of symmetric matrices.

Let $a_0,\ldots,a_n$ be a sequence of real numbers. We say that $a_0,\ldots,a_n$ have **strongly alternating signs** if $a_0 > 0$, $a_1 < 0$, $a_2 > 0$, and so on (i.e., $a_i > 0$ when i is even, and $a_i < 0$ when i is odd). We say that $a_0,\ldots,a_n$ have **weakly alternating signs** if $a_0 \geq 0$, $a_1 \leq 0$, $a_2 \geq 0$, and so on (i.e., $a_i \geq 0$ when i is even, and $a_i \leq 0$ when i is odd).

Proposition 11.71: Descartes' rule of signs for positive (semi)definite matrices

Let A be a symmetric $n\times n$-matrix, and let

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

be its characteristic polynomial. Then:

- *A is positive definite if and only if $a_0,\ldots,a_n$ have strongly alternating signs.*
- *A is positive semidefinite if and only if $a_0,\ldots,a_n$ have weakly alternating signs.*

Proof. We first prove a general fact about polynomials. Suppose $d_1,\ldots,d_n$ are real numbers and

$$(x+d_1)(x+d_2)\cdots(x+d_n) = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0.$$

Then $d_1,\ldots,d_n > 0$ if and only if $b_0,\ldots,b_n > 0$. Moreover, $d_1,\ldots,d_n \geq 0$ if and only if $b_0,\ldots,b_n \geq 0$.

Proof: To prove the first claim, assume $d_1,\ldots,d_n > 0$. It is easy to see that by multiplying out $(x+d_1)(x+d_2)\cdots(x+d_n)$, we can only obtain positive coefficients, so $b_0,\ldots,b_n > 0$. Conversely, assume $b_0,\ldots,b_n > 0$. Then for every $x \geq 0$, we clearly have $b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0 > 0$, and therefore no such $x \geq 0$ can be a root of this polynomial. In other words, all of the roots must be negative. Since the roots are $-d_1,\ldots,-d_n$, it follows that $d_1,\ldots,d_n > 0$. The proof of the second claim (using "$\geq$" instead of "$>$") is similar.

We are now ready to prove Proposition 11.71. By Theorem 11.63, we know that A is diagonalizable, i.e., $A = PDP^{-1}$ for some real diagonal matrix D. Note that A and D have the same characteristic polynomial. If $d_1,\ldots,d_n$ are the diagonal entries of D, the characteristic polynomial can therefore be written in two different ways:

$$p(\lambda) = (d_1 - \lambda)(d_2 - \lambda)\cdots(d_n - \lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots + a_n\lambda^n.$$

Now let $\lambda = -x$ and consider

$$p(-x) = (d_1 + x)(d_2 + x)\cdots(d_n + x) = a_0 - a_1x + a_2x^2 - a_3x^3 + \ldots \pm a_nx^n.$$

We have: A is positive definite if and only if $d_1,\ldots,d_n > 0$, if and only if $a_0,-a_1,a_2,-a_3,\ldots > 0$, if and only if $a_0,\ldots,a_n$ are strongly alternating. Moreover, A is positive semidefinite if and only if $d_1,\ldots,d_n \geq 0$, if and only if $a_0,-a_1,a_2,-a_3,\ldots \geq 0$, if and only if $a_0,\ldots,a_n$ are weakly alternating. ♠

Example 11.72: Using Descartes' rule of signs to check if a matrix is positive definite

Use Descartes' rule of signs to check whether the matrix

$$A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

is positive definite, positive semidefinite, or neither.

Solution. We already found the characteristic polynomial in Example 11.70: it is $-\lambda^3 + 11\lambda^2 - 30\lambda$. The coefficients are $a_0 = 0$, $a_1 = -30$, $a_2 = 11$, and $a_3 = -1$. Note that we have included all of the coefficients, even ones that are zero. Since the signs are weakly alternating, but not strongly alternating, the matrix is positive semidefinite, but not positive definite. ♠

Example 11.73: Using Descartes' rule of signs to check if a matrix is positive definite

Use Descartes' rule of signs to determine which of the following matrices are positive definite and/or positive semidefinite.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}.$$

Solution. The characteristic polynomials are:

$$\begin{aligned}\det(A-\lambda I) &= -\lambda^3+5\lambda^2-3\lambda-2,\\ \det(B-\lambda I) &= -\lambda^3+5\lambda^2-2\lambda+0,\\ \det(C-\lambda I) &= -\lambda^3+7\lambda^2-11\lambda+1.\end{aligned}$$

For A, the coefficients are not weakly alternating, so A is not positive semidefinite. For B, the coefficients are weakly, but not strongly alternating (note that $a_0=0$), and therefore B is positive semidefinite, but not positive definite. For C, the coefficients are strongly alternating, so C is positive definite. ♠

Exercises

Exercise 11.8.1 *Determine by direct calculation (i.e., without calculating the characteristic polynomial or the eigenvalues) which of the following matrices are positive definite, positive semidefinite, or neither.*

$$A=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad B=\begin{bmatrix}1&-2\\-2&5\end{bmatrix},\quad C=\begin{bmatrix}2&3\\1&3\end{bmatrix},\quad D=\begin{bmatrix}2&1&1\\1&1&0\\1&0&1\end{bmatrix},\quad E=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}.$$

Exercise 11.8.2 *Calculate the eigenvalues of each symmetric matrix, then determine for each matrix whether it is positive definite, positive semidefinite, or neither.*

$$A=\begin{bmatrix}2&2\\2&5\end{bmatrix},\quad B=\begin{bmatrix}4&-6\\-6&9\end{bmatrix},\quad C=\begin{bmatrix}2&1&0\\1&1&-1\\0&-1&2\end{bmatrix},\quad D=\begin{bmatrix}3&-4&2\\-4&4&0\\2&0&4\end{bmatrix}.$$

is positive definite, positive semidefinite, or neither.

Exercise 11.8.3 *Which of the following formulas define an inner product on $\mathbb{R}^3$? Here, $\mathbf{u}=[u_1,u_2,u_3]^T$ and $\mathbf{v}=[v_1,v_2,v_3]^T$.*

(a) $\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1+2u_1v_2+2u_2v_1+3u_2v_2+3u_3v_3$.

(b) $\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1-u_1v_2-u_2v_1+2u_2v_2+u_3v_3$.

(c) $\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1-u_1v_2-u_2v_1+u_2v_2+4u_3v_3$.

(d) $\langle\mathbf{u},\mathbf{v}\rangle=u_1v_1-u_1v_2-u_2v_1+3u_2v_2-u_2v_3-u_3v_2+u_3v_3$.

Exercise 11.8.4 *Use Descartes' rule of signs to determine which of the following matrices are positive definite and/or positive semidefinite.*

$$A=\begin{bmatrix}2&2\\2&1\end{bmatrix},\quad B=\begin{bmatrix}2&-1&0\\-1&1&0\\0&0&1\end{bmatrix},\quad C=\begin{bmatrix}2&1&-1&1\\1&1&0&0\\-1&0&2&-1\\1&0&-1&1\end{bmatrix}.$$

11.9 Application: Simplification of quadratic forms

Outcomes

A. Determine whether or not a function of several variables is a quadratic form.

B. Convert a quadratic form to and from matrix form.

C. Apply a change of variables to a quadratic form.

D. Diagonalize a quadratic form.

E. Find the principal axes of a quadratic form.

F. Sketch quadratic curves, such as circles and ellipses.

In this section, we will explore an application of the diagonalization of symmetric matrices, namely, the simplification of quadratic forms. Quadratic forms are special kinds of functions that arise, for example, in calculus when we approximate some quantity up to terms of second order.

Definition 11.74: Quadratic form

*A **quadratic form** is a polynomial in n variables in which each term is of degree 2. For example, the following is a quadratic form in 3 variables:*

$$f(x,y,z) = 3x^2+3y^2+2xy-4xz+4yz.$$

More generally, a quadratic form is a function of the form

$$f(x_1,\ldots,x_n) = q_1x_1^2+\ldots+q_nx_n^2+q_{12}x_1x_2+\ldots+q_{ij}x_ix_j+\ldots+q_{n-1,n}x_{n-1}x_n.$$

*The numbers $q_1,\ldots,q_n,q_{12},\ldots,q_{n-1,n}$, which may be positive, negative, or zero, are called the **coefficients** of the quadratic form.*

Example 11.75: Quadratic forms

Which of the following are quadratic forms?

(a) $f(x,y,z) = 2xy+3xz-5yz$.

(b) $g(x,y,z) = x^2+2xy+y^2+3$.

(c) $h(x,y,z) = (x+y)^2-z^2$.

Solution. The function f is a quadratic form. The function g is not a quadratic form, because the constant term, $+3$, is not of degree 2. It should be either a coefficient times the square of a variable, or a coefficient times the product of two variables. The function h is a quadratic form. We can simplify it to $h(x,y,z) = x^2+2xy+y^2-z^2$. ♠

Definition 11.76: Matrix form of a quadratic form

Let A be a symmetric $n\times n$-matrix, and let $\mathbf{v} = [x_1,\ldots,x_n]^T$. Then

$$f(x_1,\ldots,x_n) = \mathbf{v}^T A\mathbf{v}$$

*is a quadratic form in n variables. Conversely, every quadratic form in n variables can be uniquely written in this way. We call this the **matrix form** of the quadratic form.*

Example 11.77: Matrix form of a quadratic form

Find the coefficients of the quadratic form $f(x,y,z) = \mathbf{v}^T A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -1 \\ 4 & -1 & -2 \end{bmatrix}.$$

Solution. We have

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -1 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2-2z^2+4xy+8xz-2yz.$$

Note that there is a term $2xy$ and a term $2yx$, which together yield $4xy$. Similarly the term $4xz$ and $4zx$ are combined into $8xz$, and the terms $-yz$ and $-zy$ are combined into $-2yz$. ♠

Example 11.78: Matrix form of a quadratic form

Write the quadratic form

$$f(x,y,z) = 5x^2-y^2+z^2+2xy-4xz+3yz$$

in matrix form.

Solution. We can write this as

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 5 & 1 & -2 \\ 1 & -1 & 1.5 \\ -2 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Note: since the matrix A must be symmetric, we have no choice but to split the term $2xy$ evenly into $1xy$ and $1yx$. This explains why the $(1,2)$- and $(2,1)$-entries of the matrix are 1. Also, $-4xz$ has been split into $-2xz$ and $-2zx$, and $3yz$ has been split into $1.5yz$ and $1.5zy$. In general, the matrix of the quadratic form

$$f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz$$

is

$$A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix}.$$

♠

We will now turn to the question of how to simplify quadratic forms. The primary tool we have for doing so is a **change of variables**. This means replacing the variables $x_1, \ldots, x_n$ by new variables $y_1, \ldots, y_n$ that are linear combinations of $x_1, \ldots, x_n$.

Example 11.79: Change of variables

Apply the change of variables

$$\begin{aligned} x &= u \\ y &= v - w \\ z &= w \end{aligned}$$

to the quadratic form

$$3x^2 + y^2 + 2xy + 2xz + 2yz$$

Solution. We have

$$\begin{aligned} & 3x^2 + y^2 + 2xy + 2xz + 2yz \\ = \;& 3u^2 + (v-w)^2 + 2u(v-w) + 2uw + 2(v-w)w \\ = \;& 3u^2 + v^2 - w^2 + 2uv. \end{aligned}$$

♠

The simplest kind of quadratic form is one that involves only squared variables, and no products of two different variables. We call such quadratic forms **diagonal**.

Definition 11.80: Diagonal quadratic form

A quadratic form is ***diagonal*** *if it is of the form*

$$f(x_1, \ldots, x_n) = q_1 x_1^2 + \ldots + q_n x_n^2.$$

Proposition 11.81: Diagonalization of quadratic forms

Every quadratic form can be made diagonal by a change of variables.

Proof. We first write the quadratic form in matrix form, i.e.,

$$f(x_1, \ldots, x_n) = \mathbf{v}^T A \mathbf{v},$$

where A is a symmetric matrix and $\mathbf{v} = [x_1, \ldots, x_n]^T$. By Theorem 11.63, A is orthogonally diagonalizable. So let P be an orthogonal matrix such that

$$P^{-1}AP = P^TAP = D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{nn} \end{bmatrix}$$

is diagonal. Let $\mathbf{w} = [y_1, \ldots, y_n]^T$ be a new set of variables such that $\mathbf{v} = P\mathbf{w}$. Then

$$\mathbf{v}^TA\mathbf{v} = \mathbf{w}^TP^TAP\mathbf{w} = \mathbf{w}^TD\mathbf{w} = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2.$$

So the quadratic form is diagonal in the variables $y_1, \ldots, y_n$. ♠

Example 11.82: Diagonalization of quadratic forms

Consider the quadratic form

$$f(x,y) = 3x^2 + 4xy + 6y^2.$$

Perform a change of variables so that the quadratic form becomes diagonal.

Solution. We first write $f(x,y)$ in matrix form:

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Next, we orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$. In Example 11.64, we found that $P^{-1}AP = D$, where

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since P is orthogonal, we also have $P^TAP = D$. Next, let u, v be new variables such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (u-2v)/\sqrt{5} \\ (2u+v)/\sqrt{5} \end{bmatrix}$$

Then the change of variables gives

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} P^TAP \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} D \begin{bmatrix} u \\ v \end{bmatrix} = 7u^2 + 2v^2.$$

♠

Example 11.83: Diagonalization of quadratic forms

Diagonalize the quadratic form

$$f(x,y,z) = 3x^2 + 3y^2 + 2xy - 4xz + 4yz.$$

Solution. The matrix form of $f(x,y,z)$ is

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{v}^T A \mathbf{v}.$$

We orthogonally diagonalize the matrix A. In Example 11.64, we found that $P^{-1}AP = D$, where

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Let u, v, w be new variables such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Then the change of variables gives

$$\begin{aligned} f(x,y,z) &= \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} u & v & w \end{bmatrix} P^T A P \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= \begin{bmatrix} u & v & w \end{bmatrix} D \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= 4u^2 + 4v^2 - 2w^2. \end{aligned}$$

♠

Next, we turn our attention to the task of sketching the solutions of quadratic equations in 2 or more variables.

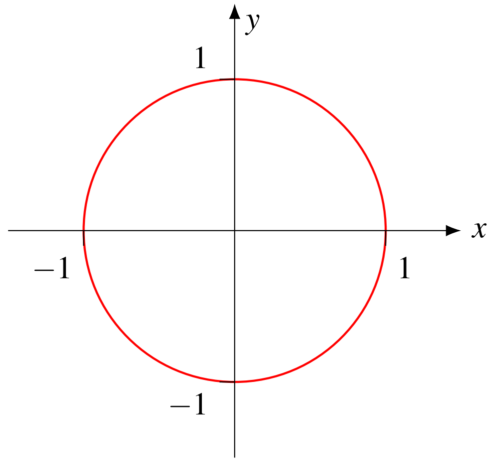
Example 11.84: Sketching circles and ellipses

Sketch the following curves:

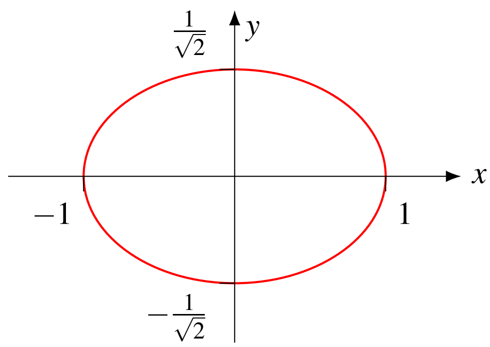
(a) $x^2 + y^2 = 1$,

(b) $x^2 + 2y^2 = 1$.

Solution. (a) The curve $x^2+y^2=1$ is the familiar unit circle.

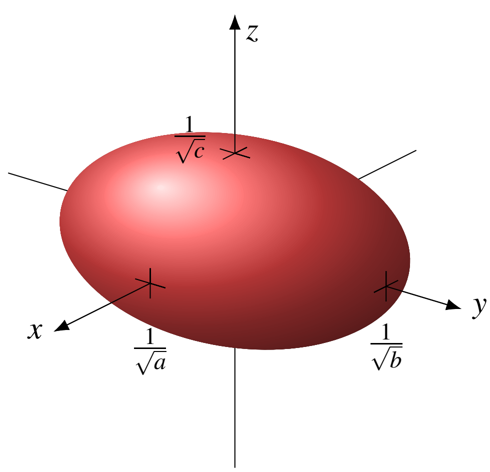


(b) The curve $x^2+2y^2=1$ is the same, except that it has been stretched by a factor of $1/\sqrt{2}$ in the y-direction. In other words, it is an ellipse with x-intercepts ± 1 and y-intercepts $\pm 1/\sqrt{2}$.



♠

In general, for $a,b>0$, the curve $ax^2+by^2=1$ is an ellipse with x-intercepts $\pm 1/\sqrt{a}$ and y-intercepts $\pm 1/\sqrt{b}$. Similarly, for $a,b,c>0$, the equation $ax^2+by^2+cz^2=1$ describes a 3-dimensional ellipsoid with x-intercepts $\pm 1/\sqrt{a}$, y-intercepts $\pm 1/\sqrt{b}$, and z-intercepts $\pm 1/\sqrt{c}$:



Each ellipse or ellipsoid has a set of **principal axes**, which are its axes of symmetry. When the quadratic forms are diagonal, as in the above examples, the principal axes are the standard coordinate axes. When the quadratic forms are not diagonal, the principal axes are the eigenvectors of the matrix A. This is the content of the following proposition.

Proposition 11.85: Principal axes of a quadratic form

Consider a quadratic form in matrix form, $f(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$, where A is a positive definite $n\times n$-matrix. Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ be an orthonormal set of eigenvectors of A, and let $\lambda_1,\ldots,\lambda_n$ be the corresponding eigenvalues. Then the solutions of the equation

$$\mathbf{v}^T A\mathbf{v} = 1$$

form an n-dimensional ellipsoid whose principal axes are parallel to $\mathbf{u}_1,\ldots,\mathbf{u}_n$ and whose $\mathbf{u}_i$-intercepts are $\pm\frac{1}{\sqrt{\lambda_i}}$.

Proof. Let P be the orthogonal matrix whose columns are $\mathbf{u}_1,\ldots,\mathbf{u}_n$. From the proof of Proposition 11.81, we know that the equation $\mathbf{v}^T A\mathbf{v} = 1$ is equivalent to $\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2 = 1$, where $y_1,\ldots,y_n$ are variables such that

$$\mathbf{v} = P\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}. \tag{11.5}$$

Since A is positive definite, we have $\lambda_1,\ldots,\lambda_n > 0$ by Proposition 11.69. We therefore know that in the $(y_1,\ldots,y_n)$-coordinate system, the equation $\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2 = 1$ describes an ellipsoid whose y_1-intercepts are $\pm 1/\sqrt{\lambda_1}$, whose y_2-intercepts are $\pm 1/\sqrt{\lambda_2}$, and so on. The only thing that remains to do is to figure out the direction of the coordinate axes. The y_1-axis points in the direction of the point with coordinates $(y_1,\ldots,y_n) = (1,0,\ldots,0)$. Using the change of variables formula (11.5), we find that this corresponds to the first column of P, i.e., $\mathbf{v} = \mathbf{u}_1$. Similarly, the y_2-axis points in the direction of $\mathbf{u}_2$, and so on. ♠

Example 11.86: Sketching a quadratic curve

Sketch the curve $3x^2 + 4xy + 6y^2 = 1$.

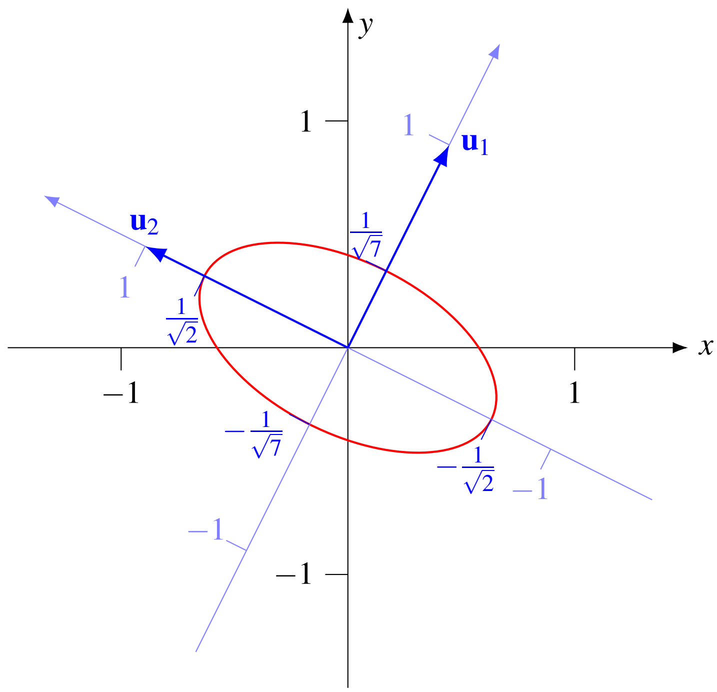
Solution. The matrix for the quadratic form $3x^2 + 4xy + 6y^2$ is

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

From Example 11.82, we know that the normalized eigenvectors of A are

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

with respective eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 2$. Therefore, by Proposition 11.85, the curve $3x^2 + 4xy + 6y^2 = 1$ is an ellipse with principal axes $\mathbf{u}_1$ and $\mathbf{u}_2$, and with respective intercepts $\pm 1/\sqrt{7}$ and $\pm 1/\sqrt{2}$.



♠

Exercises

Exercise 11.9.1 *Which of the following are quadratic forms?*

(a) $f_1(x,y,z) = (x+y+z)^2$.

(b) $f_2(x,y,z) = x^2 + y^2 + z^2$.

(c) $f_3(x,y,z) = (x+y)^2 + (x+z)^2 + (y+z)^2$.

(d) $f_4(x,y,z) = x^2 - y^2$.

(e) $f_5(x,y,z) = x^2 + 2xyz + (y+z)^2$.

(f) $f_6(x,y,z) = x(y+z)$.

(g) $f_7(x,y,z) = x^2y^2z^2$.

Exercise 11.9.2 *Find the coefficients of the quadratic form* $f(x,y,z) = \mathbf{v}^T A\mathbf{v}$, *where*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -3 & -4 \\ 2 & -4 & 5 \end{bmatrix}.$$

Exercise 11.9.3 *Write the quadratic form*

$$f(x,y,z) = (x+y)^2 + 2(x-z)^2 + 3yz$$

in matrix form.

Exercise 11.9.4 *Apply the change of variables* $x = u + 2w$, $y = v$, $z = w - v$ *to the quadratic form*

$$x^2 + 7y^2 + 5z^2 - 4xy - 4xz + 11yz.$$

Exercise 11.9.5 *Perform a change of variables so that the quadratic form* $f(x,y) = 3x^2 - 2xy + 3y^2$ *becomes diagonal.*

Exercise 11.9.6 *Diagonalize the quadratic form*

$$f(x,y,z) = 3x^2 + 3y^2 + 4z^2 + 2xz - 2yz.$$

Exercise 11.9.7 *Find the principal axes of the following curves, and sketch them:*

(a) $x^2 + \frac{1}{2}y^2 = 1$,

(b) $2x^2 + 4xy + 5y^2 = 1$.

(c) $3x^2 + 2xy + 3y^2 = 1$.

Exercise 11.9.8 *Find the principal axes of the ellipsoid* $2x^2 + 2y^2 + 3z^2 + 2xz - 2yz$.

11.10 Complex inner product spaces

Outcomes

A. *Compute dot products in* $\mathbb{C}^n$.

B. *Use properties of the complex dot product to prove equalities and inequalities.*

C. *Compute the adjoint of a matrix.*

D. *Check whether an operation is a complex inner product.*

E. *Determine whether vectors in a complex inner product space are orthogonal.*

F. *Calculate the complex Fourier coefficients of a vector.*

G. *Use the Gram-Schmidt procedure to find an orthogonal basis of a subspace of a complex inner product space.*

H. *Compute the orthogonal projection of a complex vector onto a subspace.*

So far, in this chapter, the field K was always $\mathbb{R}$, the set of real numbers. The reason we have not considered inner products over other fields K is that the positive definite property requires $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and the requirement that a scalar is "greater or equal to 0" does not make sense if K is, say, the field of integers modulo p.

In this section, we will consider inner product spaces over the complex numbers. It turns out that the theory of complex inner product spaces is similar, but not completely identical, to that of real inner product spaces. To explain the difference, consider the definition of the dot product. In $\mathbb{R}^n$, the dot product of two vectors

$$\mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

is defined to be

$$\mathbf{v} \bullet \mathbf{w} = x_1 y_1 + \ldots + x_n y_n.$$

One of the most important properties of the dot product is positivity: for all $\mathbf{v}$, we have

$$\mathbf{v} \bullet \mathbf{v} = x_1^2 + \ldots + x_n^2 \geq 0.$$

The reason positivity holds is that the square of a real number is always greater or equal to 0. If we blindly replaced $x_1, \ldots, x_n$ by complex numbers and kept the same definition of dot product, positivity would no longer hold. This is because for a complex number z, it is not in general true that $z^2 \geq 0$. In fact, z^2 may not be a real number, and even in cases where z^2 is real, it may not be positive. For example, if $z = i$, then $z^2 = -1$.

Fortunately, all is not lost: the complex numbers actually do have a useful positivity property. Namely, if $z = a + bi$ is a complex number and $\bar{z} = a - bi$ is its complex conjugate, then

$$\bar{z}z = (a - bi)(a + bi) = a^2 + b^2 \geq 0.$$

So instead of squaring a complex number, we should multiply it by its conjugate. With this in mind, we arrive at the following definition of dot product on $\mathbb{C}^n$:

Definition 11.87: The dot product on $\mathbb{C}^n$

Let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

be vectors in $\mathbb{C}^n$. Their ***(complex) dot product*** *is defined to be*

$$\mathbf{v} \bullet \mathbf{w} = \overline{v_1} w_1 + \ldots + \overline{v_n} w_n.$$

The complex dot product satisfies properties that are similar to, but not exactly the same as, the properties satisfied by the real dot product.

Proposition 11.88: Properties of the complex dot product

The dot product satisfies the following properties, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ *and* $k, \ell \in \mathbb{C}$.

- *Conjugate symmetry:* $\mathbf{u} \bullet \mathbf{v} = \overline{\mathbf{v} \bullet \mathbf{u}}$.
- *Linearity on the right:* $\mathbf{u} \bullet (k\mathbf{v} + \ell\mathbf{w}) = k(\mathbf{u} \bullet \mathbf{v}) + \ell(\mathbf{u} \bullet \mathbf{w})$.
- *Antilinearity on the left:* $(k\mathbf{u} + \ell\mathbf{v}) \bullet \mathbf{w} = \overline{k}(\mathbf{u} \bullet \mathbf{w}) + \overline{\ell}(\mathbf{v} \bullet \mathbf{w})$.
- *The positive definite property:* $\mathbf{u} \bullet \mathbf{u} \geq 0$, *and* $\mathbf{u} \bullet \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

We note that the complex dot product can be equivalently expressed as a matrix product, namely

$$\mathbf{v} \bullet \mathbf{w} = \begin{bmatrix} \overline{v_1} & \cdots & \overline{v_n} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \overline{\mathbf{v}}^T \mathbf{w}.$$

Here, $\overline{\mathbf{v}}$ denotes the complex conjugate of a vector (i.e., taking the complex conjugate of each component of a vector), and $(-)^T$ denotes the transpose as usual. As a matter of fact, when working with complex vectors and matrices, it turns out that we should almost *always* take the complex conjugate at the same time as taking the transpose. For this reason, we introduce a special name and notation for the conjugate transpose of a vector or matrix.

Definition 11.89: Adjoint of a matrix

*Let A be a complex $m \times n$-matrix. The **adjoint** of A, denoted A^*, is the transpose of the complex conjugate of A. In symbols:*

$$A^* = \overline{A}^T.$$

With this definition, we can also write the dot product as

$$\mathbf{v} \bullet \mathbf{w} = \mathbf{v}^* \mathbf{w}.$$

We are now ready to state the definition of a complex inner product, which is a generalization of the complex dot product.

Definition 11.90: Complex inner product space

*A **complex inner product space** is a complex vector space V equipped with an operation that assigns to any pair of vectors $\mathbf{u}, \mathbf{v} \in V$ a complex number $\langle \mathbf{u}, \mathbf{v} \rangle$, called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$, satisfying the following properties:*

1. *Conjugate symmetry:* $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.
2. *Linearity on the right:* $\langle \mathbf{u}, k\mathbf{v} + \ell\mathbf{w} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle + \ell\langle \mathbf{u}, \mathbf{w} \rangle$.
3. *Antilinearity on the left:* $\langle k\mathbf{u} + \ell\mathbf{v}, \mathbf{w} \rangle = \overline{k}\langle \mathbf{u}, \mathbf{w} \rangle + \overline{\ell}\langle \mathbf{v}, \mathbf{w} \rangle$.
4. *The positive definite property:* $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, *and* $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Note that conjugate symmetry implies that $\langle \mathbf{u}, \mathbf{u} \rangle$ is a real number for every vector $\mathbf{u}$. Namely, let $z = \langle \mathbf{u}, \mathbf{u} \rangle$. Then by conjugate symmetry, we have

$$z = \langle \mathbf{u}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{u} \rangle} = \bar{z}.$$

Since z is equal to its own conjugate, it must be a real number. Therefore, the positive definite property makes sense: when we require that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, we are talking about a real number that is greater than or equal to 0. (It would not in general make sense to ask whether a complex number is greater than or equal to 0).

The space $\mathbb{C}^n$ with the complex dot product is evidently an example of a complex inner product space. Here is another example:

Example 11.91: Complex-valued continuous functions

Let $a < b$ be real numbers, and let $C[a,b]$ be the space of continuous, complex-valued functions $f : [a,b] \to \mathbb{C}$. Given two such functions $f, g \in C[a,b]$, we define their inner product as

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) \, dx.$$

With this operation, $C[a,b]$ is a complex inner product space.

Armed with this definition of complex inner products, we can now pretty much redo everything we did for real inner products in the complex case. The only thing we have to be careful about is to put the complex conjugate operation in the correct places.

- The **norm** of a vector in a complex inner product space is defined to be $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. This definition makes sense because $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.
- The **Cauchy-Schwarz inequality** $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$ and the **triangle inequality** $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ hold in complex inner product spaces.
- Two vectors $\mathbf{u}, \mathbf{v}$ in a complex inner product space are called **orthogonal**, in symbols $\mathbf{u} \perp \mathbf{v}$, if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
- A vector $\mathbf{u}$ in a complex inner product space is called **normalized** if $\|\mathbf{u}\| = 1$.
- A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is called an **orthogonal set** if the vectors are non-zero and pairwise orthogonal, and an **orthonormal set** if the vectors are moreover normalized.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are orthogonal, then $\|a_1\mathbf{u}_1 + \ldots + a_k\mathbf{u}_k\|^2 = |a_1|^2 \|\mathbf{u}_1\|^2 + \ldots + |a_k|^2 \|\mathbf{u}_k\|^2$. The absolute value signs are necessary because $\overline{a_i} a_i = |a_i|^2$.

Example 11.92: Orthogonal vectors

Consider $\mathbb{C}^2$ *with the complex dot product, and the following vectors:*

$$\mathbf{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Are $\mathbf{u}$ *and* $\mathbf{v}$ *orthogonal? Are* $\mathbf{u}$ *and* $\mathbf{w}$ *orthogonal?*

Solution. We have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* \mathbf{v} = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i - i = -2i \neq 0,$$

so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. We have

$$\langle \mathbf{u}, \mathbf{w} \rangle = \mathbf{u}^* \mathbf{w} = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i - i = 0,$$

so $\mathbf{u}$ and $\mathbf{w}$ are orthogonal. Note that it is crucial here that we did not forget to take the complex conjugate of $\mathbf{u}$, or else we would have gotten a different answer. ♠

In some formulas, we must be careful about whether we use $\langle \mathbf{v}, \mathbf{w} \rangle$ or $\langle \mathbf{w}, \mathbf{v} \rangle$. Although this did not make any difference in the case of real inner products, it does make a difference for complex inner products, because in general, $\langle \mathbf{v}, \mathbf{w} \rangle \neq \langle \mathbf{w}, \mathbf{v} \rangle$. In particular, we have to be careful about this in the formulas for Fourier coefficients, projections, and the Gram-Schmidt procedure.

Example 11.93: Complex Fourier coefficients

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *be an orthogonal set of vectors in a complex inner product space, and let*

$$\mathbf{v} = a_1 \mathbf{u}_1 + \ldots + a_n \mathbf{u}_n.$$

Which of the following formulas is correct?

$$a_i = \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \quad \text{or} \quad a_i = \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \quad ?$$

Solution. To check whether the first formula is correct, we calculate

$$\begin{aligned} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} &= \frac{\langle a_1 \mathbf{u}_1 + \ldots + a_i \mathbf{u}_i + \ldots + a_n \mathbf{u}_n, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \\ &= \overline{a_1} \frac{\langle \mathbf{u}_1, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} + \ldots + \overline{a_i} \frac{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} + \ldots + \overline{a_n} \frac{\langle \mathbf{u}_n, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \\ &= \overline{a_1} \cdot 0 + \ldots + \overline{a_i} \cdot 1 + \ldots + \overline{a_n} \cdot 0 \\ &= \overline{a_i}. \end{aligned}$$

Note that because of antilinearity, this formula came out to be $\overline{a_i}$, and not a_i. Therefore, the first formula is not correct. The check the second formula, we calculate

$$\frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} = \frac{\langle \mathbf{u}_i, a_1 \mathbf{u}_1 + \ldots + a_i \mathbf{u}_i + \ldots + a_n \mathbf{u}_n \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}$$

$$\begin{aligned} &= a_1 \frac{\langle \mathbf{u}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_i \rangle} + \ldots + a_i \frac{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} + \ldots + a_n \frac{\langle \mathbf{u}_i, \mathbf{u}_n \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \\ &= a_1 \cdot 0 + \ldots + a_i 1 + \ldots + a_n \cdot 0 \\ &= a_i. \end{aligned}$$

Therefore, the second formula is correct. ♠

Since this is an important result, we state it as a proposition.

Proposition 11.94: Complex Fourier coefficients

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors in a complex inner product space and

$$\mathbf{v} = a_1 \mathbf{u}_1 + \ldots + a_n \mathbf{u}_n,$$

then

$$a_i = \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \quad \text{and} \quad \overline{a_i} = \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}.$$

Example 11.95: Calculating complex Fourier coefficients

Suppose that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for a complex inner product space V, such that $\|\mathbf{u}_1\| = 1$, $\|\mathbf{u}_2\| = \sqrt{5}$, and $\|\mathbf{u}_3\| = 2$. Moreover, suppose that $\mathbf{v} \in V$ is a vector such that $\langle \mathbf{u}_1, \mathbf{v} \rangle = i$, $\langle \mathbf{u}_2, \mathbf{v} \rangle = -2$, and $\langle \mathbf{u}_3, \mathbf{v} \rangle = 1 - 2i$. Find the coordinates of $\mathbf{v}$ with respect to B.

Solution. We need to find a_1, a_2, a_3 such that $\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3$. By Proposition 11.94, we have

$$\begin{aligned} a_1 &= \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\|\mathbf{u}_1\|^2} = \frac{i}{1} = i, \\ a_2 &= \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} = \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\|\mathbf{u}_2\|^2} = \frac{-2}{5} = -\frac{2}{5}, \\ a_3 &= \frac{\langle \mathbf{u}_3, \mathbf{v} \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} = \frac{\langle \mathbf{u}_3, \mathbf{v} \rangle}{\|\mathbf{u}_3\|^2} = \frac{1-2i}{4} = \frac{1}{4} - \frac{1}{2} i. \end{aligned}$$

♠

The Gram-Schmidt orthogonalization procedure works without changes in complex inner product spaces, as long as we are careful not to confuse $\langle \mathbf{u}_i, \mathbf{v}_j \rangle$ with $\langle \mathbf{v}_j, \mathbf{u}_i \rangle$.

Proposition 11.96: Complex Gram-Schmidt orthogonalization procedure

Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ *be a basis for some subspace* W *of a complex inner product space* V*. Define vectors* $\mathbf{u}_1,\ldots,\mathbf{u}_k$ *as follows:*

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1, \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{u}_1,\mathbf{v}_3\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\mathbf{v}_3\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2, \\
&\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \frac{\langle \mathbf{u}_1,\mathbf{v}_k\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\mathbf{v}_k\rangle}{\langle \mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2 - \ldots - \frac{\langle \mathbf{u}_{k-1},\mathbf{v}_k\rangle}{\langle \mathbf{u}_{k-1},\mathbf{u}_{k-1}\rangle}\mathbf{u}_{k-1}.
\end{aligned}$$

Then $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ *is an orthogonal basis of* W*.*

Example 11.97: Complex Gram-Schmidt orthogonalization procedure

Consider $\mathbb{C}^3$ *with the complex dot product. Let*

$$\mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1 \\ i \end{bmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{bmatrix} i \\ -1 \\ -3 \end{bmatrix}$$

Use the Gram-Schmidt procedure to find an orthogonal basis for $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$.

Solution. We start with

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1 \\ i \end{bmatrix}.$$

Next, we calculate

$$\langle \mathbf{u}_1,\mathbf{v}_2\rangle = \mathbf{u}_1^*\mathbf{v}_2 = \begin{bmatrix} 1-i & 1 & -i \end{bmatrix}\begin{bmatrix} i \\ -1 \\ -3 \end{bmatrix} = (1-i)i + 1(-1) + (-i)(-3) = 4i$$

and

$$\langle \mathbf{u}_1,\mathbf{u}_1\rangle = \mathbf{u}_1^*\mathbf{u}_1 = \begin{bmatrix} 1-i & 1 & -i \end{bmatrix}\begin{bmatrix} 1+i \\ 1 \\ i \end{bmatrix} = (1-i)(1+i) + 1\cdot 1 + (-i)(i) = 4.$$

Therefore

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\mathbf{v}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 = \mathbf{v}_2 - \frac{4i}{4}\mathbf{u}_1 = \mathbf{v}_2 - i\mathbf{u}_1 = \begin{bmatrix} i \\ -1 \\ -3 \end{bmatrix} - i\begin{bmatrix} 1+i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ -1-i \\ -2 \end{bmatrix}.$$

The desired orthogonal basis is $\{\mathbf{u}_1, \mathbf{u}_2\}$. We double-check that $\mathbf{u}_1$ and $\mathbf{u}_2$ are indeed orthogonal:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \mathbf{u}_1^* \mathbf{u}_2 = \begin{bmatrix} 1-i & 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -1-i \\ -2 \end{bmatrix} = (1-i)1 + 1(-1-i) + (-i)(-2) = 0.$$

♠

Orthogonal projections also work in complex inner product spaces.

Proposition 11.98: Orthogonal projection onto a subspace

Let V be a complex inner product space, and let W be a subspace of V. Assume $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of W, and $\mathbf{v} \in V$ is any vector. Let

$$\mathbf{v}' = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 + \ldots + \frac{\langle \mathbf{u}_k, \mathbf{v} \rangle}{\langle \mathbf{u}_k, \mathbf{u}_k \rangle} \mathbf{u}_k.$$

*Then $\mathbf{v}'$ is the best approximation of $\mathbf{v}$ in W, i.e., it is the element of W such that $\|\mathbf{v} - \mathbf{v}'\|$ is as small as possible. Moreover, the vector $\mathbf{v} - \mathbf{v}'$ is orthogonal to W. We say that $\mathbf{v}'$ is the **orthogonal projection of $\mathbf{v}$ onto W**.*

Example 11.99: Orthogonal projection onto a subspace

Consider the subspace of $\mathbb{C}^3$ spanned by

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}.$$

With respect to the complex dot product, find the best approximation of $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ in this subspace.

Solution. First notice that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$, so that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. Therefore, the best approximation is given by

$$\mathbf{v}' = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2.$$

We calculate the relevant inner products:

$$\langle \mathbf{u}_1, \mathbf{v} \rangle = \begin{bmatrix} 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = -2i,$$

$$\langle \mathbf{u}_2, \mathbf{v} \rangle = \begin{bmatrix} -i & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = 6,$$

$$\begin{aligned}\langle \mathbf{u}_1, \mathbf{u}_1 \rangle &= \begin{bmatrix} 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = 2, \\ \langle \mathbf{u}_2, \mathbf{u}_2 \rangle &= \begin{bmatrix} -i & 1 & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} = 3.\end{aligned}$$

Therefore,

$$\mathbf{v}' = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{v} \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2. = \frac{-2i}{2}\mathbf{u}_1 + \frac{6}{3}\mathbf{u}_2 = -i \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} + 2 \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 3 \\ 2 \end{bmatrix}.$$

To double-check the answer, we can check that $\mathbf{v} - \mathbf{v}'$ is indeed orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. We have

$$\mathbf{v} - \mathbf{v}' = \begin{bmatrix} -i \\ -1 \\ 2 \end{bmatrix}$$

and

$$\begin{aligned}\langle \mathbf{u}_1, \mathbf{v} - \mathbf{v}' \rangle &= \begin{bmatrix} 1 & -i & 0 \end{bmatrix} \begin{bmatrix} -i \\ -1 \\ 2 \end{bmatrix} = 0, \\ \langle \mathbf{u}_2, \mathbf{v} - \mathbf{v}' \rangle &= \begin{bmatrix} -i & 1 & 1 \end{bmatrix} \begin{bmatrix} -i \\ -1 \\ 2 \end{bmatrix} = 0.\end{aligned}$$

♠

We finish this section with some remarks on the differences between the notations used in mathematics and in physics. Complex inner product spaces are very important in physics because they are the foundation of quantum mechanics. The adjoint of a matrix A is usually denoted A^* in mathematics and $A^\dagger$ in physics. In quantum mechanics, a column vector $\mathbf{v}$ is often written $|v\rangle$, and the corresponding row vector $\mathbf{v}^*$ is then written as $\langle v|$. This is the so-called **Dirac notation**. With this convention, an inner product $\mathbf{v}^*\mathbf{w}$ is $\langle v||w\rangle$, which is usually written as $\langle v|w\rangle$. Also, the matrix $\mathbf{v}\mathbf{w}^*$ is written $|v\rangle\langle w|$ and is called an **outer product**. In mathematics, it is customary for inner products to be linear in the left component and antilinear in the right component. In physics, it is customary to use the opposite convention, i.e., inner products are antilinear in the left component and linear in the right component. In this book, we have used the physics convention of antilinearity in the left component, because it is the better convention.

Exercises

Exercise 11.10.1 *Compute the following complex dot products in* $\mathbb{C}^n$*:*

$$(a) \begin{bmatrix} 1 \\ i \end{bmatrix} \bullet \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \quad (b) \begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix} \bullet \begin{bmatrix} i \\ 1 \\ -3 \end{bmatrix}, \quad (c) \begin{bmatrix} i \\ 1 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix}, \quad (d) \begin{bmatrix} i \\ i+1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1+2i \\ -2 \\ -i \end{bmatrix}.$$

Exercise 11.10.2 *Compute the norm of each of the following vectors in $\mathbb{C}^n$:*

$$(a)\begin{bmatrix} 1 \\ i \end{bmatrix}, \quad (b)\begin{bmatrix} 1 \\ 0 \\ 1+2i \end{bmatrix}, \quad (c)\begin{bmatrix} 1+i \\ 1-i \\ 3i \end{bmatrix}.$$

Exercise 11.10.3 *Suppose* $\mathbf{u},\mathbf{v} \in \mathbb{C}^n$ *and* $\mathbf{u} \bullet \mathbf{u} = 1$, $\mathbf{v} \bullet \mathbf{v} = 2$, *and* $\mathbf{u} \bullet \mathbf{v} = 2i$. *Then compute:*

(a) $(2\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}-\mathbf{v})$.

(b) $(i\mathbf{u}-\mathbf{v}) \bullet (\mathbf{u}-2i\mathbf{v})$.

Exercise 11.10.4 *Compute the adjoint of the following matrices.*

$$(a)\begin{bmatrix} i \\ i-1 \\ 2 \end{bmatrix}, \quad (b)\begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}, \quad (c)\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 3 & 2+2i \end{bmatrix}.$$

Exercise 11.10.5 *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a complex inner product space such that* $\langle \mathbf{u},\mathbf{u} \rangle = 3$, $\langle \mathbf{v},\mathbf{v} \rangle = 2$, *and* $\langle \mathbf{u},\mathbf{v} \rangle = i+1$. *Then compute:*

(a) $\langle \mathbf{u}+2\mathbf{v}, \mathbf{u}-i\mathbf{v} \rangle$.

(b) $\langle \mathbf{u}-i\mathbf{v}, 2\mathbf{u}+\mathbf{v} \rangle$.

Exercise 11.10.6 *In* $C[0,2\pi]$, *consider the functions* $f(x) = \sin x + i\cos x$, $g(x) = 1$, *and* $h(x) = x$. *Compute the following:*

$$(a)\ \langle f,g \rangle, \quad (b)\ \langle f,h \rangle, \quad (c)\ \langle f,f \rangle, \quad (d)\ \|f\|.$$

Exercise 11.10.7 *Which of the following vectors, if any, are orthogonal?*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Exercise 11.10.8 *Suppose that* $B = \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ *is an orthogonal basis for a complex inner product space* V, *such that* $\|\mathbf{u}_1\| = \sqrt{2}$, $\|\mathbf{u}_2\| = 2$, *and* $\|\mathbf{u}_3\| = \sqrt{3}$. *Moreover, suppose that* $\mathbf{v} \in V$ *is a vector such that* $\langle \mathbf{u}_1,\mathbf{v} \rangle = 1$, $\langle \mathbf{u}_2,\mathbf{v} \rangle = 3i$, *and* $\langle \mathbf{u}_3,\mathbf{v} \rangle = 1-i$. *Find the coordinates of* $\mathbf{v}$ *with respect to* B.

Exercise 11.10.9 *Consider* $\mathbb{C}^3$ *with the complex dot product. Let*

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ i \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1+i \\ 3i+2 \end{bmatrix}$$

Use the Gram-Schmidt procedure to find an orthogonal basis for span$\{\mathbf{v}_1, \mathbf{v}_2\}$. *Then find an orthonormal basis.*

Exercise 11.10.10 *Consider the subspace of* $\mathbb{C}^3$ *spanned by the orthogonal vectors*

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ i \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{u}_2 = \begin{bmatrix} i \\ 3 \\ i \end{bmatrix}.$$

Find the best approximation of $\mathbf{v} = \begin{bmatrix} -2i \\ 3 \\ 4i \end{bmatrix}$ *in this subspace.*

11.11 Unitary and hermitian matrices

Outcomes

A. *Determine whether a complex linear transformation is an isometry and/or unitary.*

B. *Determine whether a complex matrix is unitary.*

C. *Determine whether a complex matrix is hermitian.*

D. *Calculate the eigenvalues and eigenvectors of a hermitian matrix.*

E. *Compute an orthogonal basis of eigenvectors for a hermitian matrix.*

F. *Unitarily diagonalize a hermitian matrix.*

In the context of real inner product spaces, we studied orthogonal functions, orthogonal matrices, and symmetric matrices. The corresponding concepts in the context of complex inner product spaces are unitary functions, unitary matrices, and hermitian matrices. We will introduce these concepts in this section. Since the proofs are similar to those in Section 11.6, we omit most of them.

Definition 11.100: Isometries and unitary maps of complex inner product spaces

Let V, W *be complex inner product spaces. A linear transformation* $T : V \to W$ *is called an* ***isometry*** *if for all* $\mathbf{u}, \mathbf{v} \in V$,

$$\langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

An isometry that is also invertible is called a ***unitary transformation***, *or simply* ***unitary***.

In the case of real inner product spaces, we found in Proposition 11.56 that a square matrix P is the matrix of an orthogonal transformation (with respect to orthonormal bases) if and only if $P^T P = I$. In the complex case, we have an analogous property, except that we must use the adjoint instead of the transpose.

Proposition 11.101: The matrix of a unitary transformation

Let $T : V \to W$ be a linear transformation between finite-dimensional complex inner product spaces V and W. Let B and C be orthonormal bases of V and W, respectively, and let $P = [T]_{C,B}$ be the matrix of T with respect to the bases B and C. Then

(a) T is an isometry if and only if $P^*P = I$.

(b) T is unitary if and only if $P^*P = I$ and $\dim V = \dim W$.

This motivates the following definition. We therefore define a **unitary** matrix to be an $n \times n$-matrix satisfying $P^*P = I$.

Definition 11.102: Unitary matrix

An $n \times n$-matrix P is called **unitary** *if $P^*P = I$.*

We have the following analogue of Proposition 11.59:

Proposition 11.103: Conditions for unitary matrices

The following are equivalent for a complex $n \times n$-matrix P:

(a) P is unitary.

(b) $P^*P = I$.

(c) P is invertible and $P^{-1} = P^*$.

(d) $PP^* = I$.

(e) P^* is unitary.

(f) The columns of P form an orthonormal set of vectors.

(g) The rows of P form an orthonormal set of vectors.

Example 11.104: Unitary matrices

Determine which of the following matrices are unitary.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

Solution. We have

$$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

so A is unitary. Similarly, we have

$$B^*B = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I,$$

so B is unitary as well. On the other hand,

$$C^*C = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix} \neq I,$$

so C is not unitary. Equivalently, we could have checked whether the columns of A, B, and C form an orthonormal set of vectors (they do, in the case of A and B, but don't, in the case of C. See also Example 11.92). ♠

Of course, if P happens to be a matrix with real entries, then P is unitary if and only if it is orthogonal, because in that case $P^* = \overline{P}^T = P^T$.

Recall that a matrix A is called **symmetric** if $A = A^T$. In the complex world, we are more often interested in the property $A = A^*$. A matrix with this property is called **hermitian** (after the French mathematician Charles Hermite, 1822–1901).

Definition 11.105: Hermitian matrix

A complex $n \times n$-matrix A is called ***hermitian*** *if $A = A^*$.*

Example 11.106: Hermitian vs. symmetric matrices

Which of the following matrices are hermitian? Which ones are symmetric?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2i \\ 2i & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}, \quad D = \begin{bmatrix} i & 2 \\ 2 & i \end{bmatrix}.$$

Solution. The matrix A is symmetric and also hermitian. The matrix B is symmetric but not hermitian. In fact, we have

$$B^* = \begin{bmatrix} 1 & -2i \\ -2i & 1 \end{bmatrix} \neq B.$$

The matrix C is hermitian but not symmetric. The matrix D is symmetric but not hermitian. ♠

A matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is hermitian if and only if $a_{ij} = \overline{a_{ji}}$, for all i, j. In particular, the diagonal entries of a hermitian matrix are always real, and the off-diagonal entries come in complex conjugate pairs. If all of the entries in a matrix A are real, then A is hermitian if and only if it is symmetric.

Hermitian matrices are of interest, among other things, because their eigenvalues are always real. Moreover, eigenvectors for distinct eigenvalues are orthogonal. The following proposition is analogous to Proposition 11.62.

Proposition 11.107: Eigenvalues and eigenvectors of hermitian matrices

Let A be a hermitian matrix. Then

(a) All eigenvalues of A are real.

(b) Eigenvectors for distinct eigenvalues of A are orthogonal.

Proof. (a) Suppose λ is an eigenvalue of A, with eigenvector $\mathbf{v}$. We will evaluate $\mathbf{v}^*A\mathbf{v}$ in two different ways:

$$\begin{aligned}\mathbf{v}^*A\mathbf{v} &= \mathbf{v}^*(A\mathbf{v}) = \mathbf{v}^*(\lambda\mathbf{v}) = \lambda\mathbf{v}^*\mathbf{v},\\ \mathbf{v}^*A\mathbf{v} &= (\mathbf{v}^*A)\mathbf{v} = (\mathbf{v}^*A^*)\mathbf{v} = (A\mathbf{v})^*\mathbf{v} = (\lambda\mathbf{v})^*\mathbf{v} = \overline{\lambda}\mathbf{v}^*\mathbf{v}.\end{aligned}$$

Therefore, $\lambda\mathbf{v}^*\mathbf{v} = \overline{\lambda}\mathbf{v}^*\mathbf{v}$. Since $\mathbf{v}^*\mathbf{v}$ is non-zero, this implies $\lambda = \overline{\lambda}$, i.e., λ is real.

(b) Suppose $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for eigenvalues λ and μ, respectively, and $\lambda \neq \mu$. From part (a), we know that both λ and μ are real. By evaluating $\mathbf{v}^*A\mathbf{w}$ in two different ways, we find that

$$\begin{aligned}\mathbf{v}^*A\mathbf{w} &= \mathbf{v}^*(A\mathbf{w}) = \mathbf{v}^*(\mu\mathbf{w}) = \mu\mathbf{v}^*\mathbf{w},\\ \mathbf{v}^*A\mathbf{w} &= (\mathbf{v}^*A)\mathbf{w} = (A^*\mathbf{v})^*\mathbf{w} = (A\mathbf{v})^*\mathbf{w} = (\lambda\mathbf{v})^*\mathbf{w} = \lambda\mathbf{v}^*\mathbf{w}.\end{aligned}$$

Therefore, $\mu\mathbf{v}^*\mathbf{w} = \lambda\mathbf{v}^*\mathbf{w}$, or equivalently $(\lambda-\mu)\mathbf{v}^*\mathbf{w} = 0$. Since by assumption, $\lambda - \mu \neq 0$, we must have $\mathbf{v}^*\mathbf{w} = 0$, i.e., $\mathbf{v} \perp \mathbf{w}$. ♠

We say that a matrix A is **unitarily diagonalizable** if there exists a unitary matrix P and a diagonal matrix D such that $D = P^{-1}AP$. The following is the main theorem about the diagonalization of hermitian matrices. It is analogous to Theorem 11.63 in the real case.

Theorem 11.108: Diagonalization of hermitian matrices

Every hermitian matrix A is unitarily diagonalizable as $D = P^{-1}AP$. Moreover, the entries of D are real.

Example 11.109: Diagonalization of hermitian matrices

The matrix

$$A = \begin{bmatrix} 3 & -2i \\ 2i & 6 \end{bmatrix}$$

is hermitian. Unitarily diagonalize A, i.e., find a unitary matrix P and a real diagonal matrix D such that $D = P^{-1}AP$.

Solution. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2i \\ 2i & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) + 4i^2 = \lambda^2 - 9\lambda + 14.$$

Its roots are $\lambda_1 = 2$ and $\lambda_2 = 7$. For the eigenvalue $\lambda_1 = 2$, we find the normalized eigenvector

$$\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2i \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 7$, we find the normalized eigenvector

$$\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2i \end{bmatrix}.$$

Note that these eigenvectors are orthogonal to each other, confirming Proposition 11.107. We therefore have $D = P^{-1}AP$, where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \quad \text{and} \quad P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2i & 1 \\ 1 & 2i \end{bmatrix}.$$

Note that P is unitary and D is real diagonal. ♠

Exercises

Exercise 11.11.1 *Which of the following linear transformations $T : \mathbb{C}^2 \to \mathbb{C}^2$ is unitary?*

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ iy \end{bmatrix}$.

(b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} iy \\ -ix \end{bmatrix}$.

(c) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+iy \\ ix+y \end{bmatrix}$.

(d) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}}x + \frac{i}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x - \frac{i}{\sqrt{2}}y \end{bmatrix}$.

Exercise 11.11.2 *Determine which of the following matrices are unitary.*

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad B = \frac{1}{5}\begin{bmatrix} 4 & 3i \\ 3 & 4i \end{bmatrix}, \quad C = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2i \\ 2i & 1 \end{bmatrix}.$$

Exercise 11.11.3 *Determine which of the following matrices are unitary.*

$$A = \frac{1}{2}\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & i \\ i & 0 & 1 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2i \\ -2i & 2i & 1 \\ 2i & i & 2 \end{bmatrix}, \quad C = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Exercise 11.11.4 *Assume A and B are unitary* 3×3*-matrices. Then which of the following matrices are unitary?*

$$(a)\ AB \quad (b)\ A+B \quad (c)\ iA \quad (d)\ -A \quad (e)\ 2A \quad (f)\ B^{-1} \quad (g)\ A^T \quad (h)\ A^* \quad (i)\ A^2 \quad (j)\ ABA^{-1}$$

Exercise 11.11.5 *Assume A and B are* 3×3*-matrices, and assume A is unitary and B is invertible. Then which of the following matrices are unitary?*

$$(a)\ AB \quad (b)\ A+B \quad (f)\ BAB^{-1} \quad (g)\ ABA^{-1} \quad (h)\ BB^* \quad (i)\ BB^{-1}$$

Exercise 11.11.6 *Which of the following matrices are hermitian? Which ones are symmetric?*

$$A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad B=\begin{bmatrix} i & 2 \\ 2 & -i \end{bmatrix},\quad C=\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\quad D=\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}.$$

Exercise 11.11.7 *Which of the following matrices are hermitian? Which ones are symmetric?*

$$A=\begin{bmatrix} 0 & -2 & 1+i \\ -2 & 3 & i \\ 1-i & i & 2 \end{bmatrix},\quad B=\begin{bmatrix} 3 & i & -1+2i \\ -i & 0 & -1+i \\ -1-2i & -1-i & 2 \end{bmatrix},\quad C=\begin{bmatrix} i & i & i \\ i & i & i \\ i & i & i \end{bmatrix}.$$

Exercise 11.11.8 *Assume A and B are hermitian* 3×3*-matrices, and A is invertible. Then which of the following matrices are hermitian?*

$$(a)\ AB \quad (b)\ A+B \quad (c)\ iA \quad (d)\ -A \quad (e)\ 2A \quad (f)\ A^*BA \quad (g)\ A^{-1}BA \quad (h)\ BB^* \quad (i)\ A^2$$

Exercise 11.11.9 *Unitarily diagonalize the hermitian matrix* $A=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$.

Exercise 11.11.10 *Unitarily diagonalize the hermitian matrix* $A=\begin{bmatrix} 0 & -1+2i \\ -1-2i & 4 \end{bmatrix}$.

Exercise 11.11.11 *Unitarily diagonalize the hermitian matrix* $A=\begin{bmatrix} 2 & 1+i & 1-i \\ 1-i & 3 & -2i \\ 1+i & 2i & 3 \end{bmatrix}$.

11.12 Application: Principal component analysis

Outcomes

A. *Compute the principal components of a matrix A.*

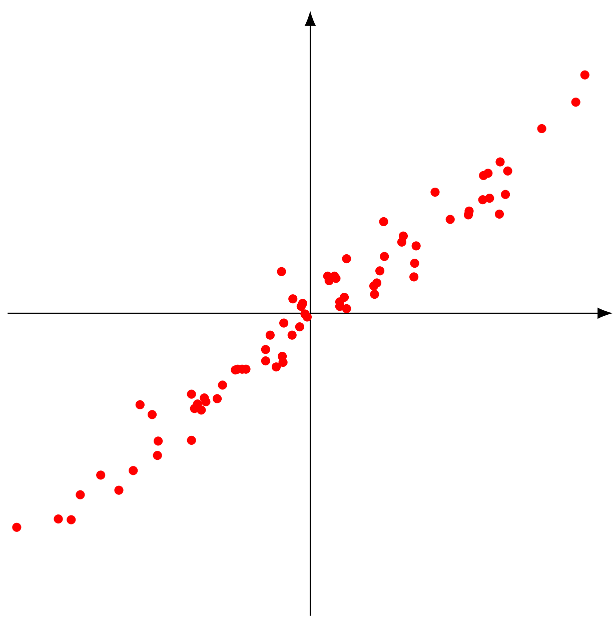
B. *Compute the centroid of a collection of data points.*

C. *Find the k-dimensional subspace that best approximates a given collection of data points.*

D. *Find the k-dimensional affine subspace that best approximates a given collection of data points.*

E. *Compute the total squared distance of the data points to the best fit subspace (or best fit affine subspace).*

In this section, we will explore an application of the diagonalization of symmetric matrices called **principal component analysis**. Imagine we are given a collection of data points such as the following:



(11.6)

Although these points are spread out in two dimensions, they seem to be located pretty close to a 1-dimensional subspace. Probably the best way to interpret this particular data set is to think of the points as being "essentially" on a line, up to some small random errors.

More generally, suppose we are given a collection of data points in n-dimensional space, and we are looking for a k-dimensional subspace that all data points are close to. This is an important way to make sense of high-dimensional data. For example, it would be very difficult to visualize data in a 100-dimensional space. However, it we knew that the data points lie very close to a 2-dimensional subspace, then we could project all of the points to the subspace to obtain a 2-dimensional image of the data.

To state the problem more precisely, let us introduce the following notation. If W is a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ is a vector, let us write $d(\mathbf{v}, W)$ for the shortest distance from $\mathbf{v}$ to W (i.e., the distance from $\mathbf{v}$

to W along a line that is perpendicular to W). Moreover, if W is a subspace of $\mathbb{R}^n$ and $\mathbf{v}_1,\ldots,\mathbf{v}_m \in \mathbb{R}^n$ are the position vectors of m points, we define the **total squared distance** of the points to the subspace to be the quantity

$$D = d(\mathbf{v}_1, W)^2 + \ldots + d(\mathbf{v}_m, W)^2.$$

Then the problem we would like to solve can be stated as follows:

Problem 11.110: Subspace fitting problem

Given vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m \in \mathbb{R}^n$ and given an integer $k \leq n$, find the k-dimensional subspace $W \subseteq \mathbb{R}^n$ that minimizes the total squared distance, i.e., such that D is as small as possible.

The following proposition gives us a method for solving the subspace fitting problem. It turns out that the key ingredient in solving this problem is the diagonalization of symmetric matrices. The method was discovered by Gale Young in 1937.

Proposition 11.111: Solution of the subspace fitting problem

Given vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m \in \mathbb{R}^n$ and $k \leq n$, the optimal solution to the subspace fitting problem can be computed as follows:

1. *Let A be the $m \times n$-matrix whose rows are $\mathbf{v}_1^T,\ldots,\mathbf{v}_m^T$. (Or equivalently, A^T is the $n \times m$-matrix whose columns are $\mathbf{v}_1,\ldots,\mathbf{v}_m$.)*
2. *Let $B = A^T A$. Then B is a positive semidefinite $n \times n$-matrix.*
3. *By Proposition 11.69, all eigenvalues of B are real and non-negative. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of B, listed according to their multiplicity and in decreasing order, i.e., so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Let $\mathbf{u}_1,\ldots,\mathbf{u}_n$ be the corresponding eigenvectors.*
4. *Then $W = \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is the solution to the subspace fitting problem. Moreover, the total squared distance of the points to this subspace is*

$$D = \lambda_{k+1} + \ldots + \lambda_n.$$

Example 11.112: Subspace fitting in $\mathbb{R}^2$

Consider the following collection of points in $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \end{bmatrix} \right\}.$$

Find the 1-dimensional subspace that best approximates this collection of points. What is the total squared distance of the points to the subspace?

Solution. We follow the steps outlined in Proposition 11.111.

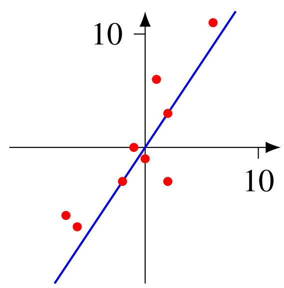
1. We have
$$A^T = \begin{bmatrix} 2 & -1 & 2 & -6 & 6 & 0 & 1 & -2 & -7 \\ -3 & 0 & 3 & -7 & 11 & -1 & 6 & -3 & -6 \end{bmatrix}.$$

2. We calculate
$$B = A^TA = \begin{bmatrix} 135 & 162 \\ 162 & 270 \end{bmatrix}.$$

3. The eigenvalues of B are $\lambda_1 = 378$ and $\lambda_2 = 27$, with corresponding eigenvectors
$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

4. The desired subspace W is spanned by the eigenvector corresponding to the largest eigenvalue, i.e., $W = \operatorname{span}\{\mathbf{u}_1\}$. The total squared distance is $\lambda_2 = 27$.

The space W is shown in the following illustration, along with the original points:



Of course, the example was rigged to ensure that the eigenvalues are integers. In real life, the entries of A and B, as well as the eigenvalues and components of the eigenvectors are usually arbitrary real numbers. ♠

Example 11.113: Subspace fitting in $\mathbb{R}^3$

Consider the following collection of points in $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} -7 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 10 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ -8 \end{bmatrix} \right\}.$$

(a) *Find the 1-dimensional subspace that best approximates this collection of points.*

(b) *Find the 2-dimensional subspace that best approximates this collection of points.*

(c) *What is the 3-dimensional subspace that best approximates this collection of points?*

In each case, what is the total squared distance of the points to the subspace?

Solution. Again, we follow the steps from Proposition 11.111. We can do the calculations for parts (a), (b), and (c) at the same time.

1. We have

$$A^T = \begin{bmatrix} -7 & 0 & 2 & 10 & -2 & -8 & 5 & -6 & 9 & -2 \\ 4 & 3 & -5 & -4 & 5 & -1 & 4 & 9 & -6 & -7 \\ 5 & 3 & -4 & 1 & 4 & -5 & 2 & 6 & 3 & -8 \end{bmatrix}.$$

2. We calculate

$$B = A^T A = \begin{bmatrix} 367 & -154 & 16 \\ -154 & 274 & 170 \\ 16 & 170 & 205 \end{bmatrix}.$$

3. The eigenvalues of B are $\lambda_1 = 513$, $\lambda_2 = 306$, and $\lambda_3 = 27$, with corresponding eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}.$$

For part (a), the desired 1-dimensional subspace is spanned by the eigenvector corresponding to the largest eigenvalue, i.e., it is span$\{\mathbf{u}_1\}$. The total squared distance is $\lambda_2 + \lambda_3 = 306 + 27 = 333$.

For part (b), the desired 2-dimensional subspace is spanned by the eigenvectors corresponding to the two largest eigenvalues, i.e., it is span$\{\mathbf{u}_1, \mathbf{u}_2\}$. The total squared distance is $\lambda_3 = 27$.

Finally, in part (c), the desired 3-dimensional subspace is spanned by all three eigenvectors; it is of course $\mathbb{R}^3$ itself, since it is the only 3-dimensional subspace. The total squared distance is 0, since all points lie in the subspace. ♠

The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ that appear in the solution of the subspace fitting problem are called the **principal components** of the matrix A.

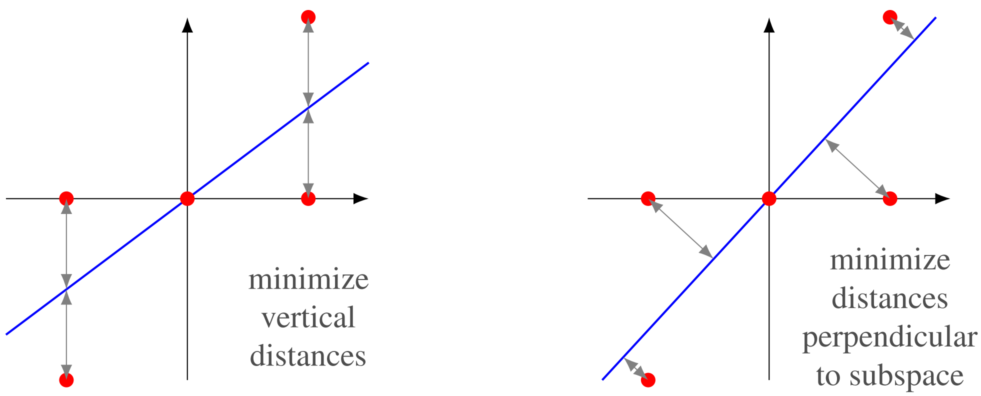
Definition 11.114: Principal components

*Let A be an $m \times n$-matrix. The **principal components** of A are the (normalized, orthogonal) eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the positive semidefinite $n \times n$-matrix $A^T A$. They are usually listed in order of decreasing eigenvalues.*

The first principal component $\mathbf{u}_1$ gives the direction in which the rows of A show the most variability. The second principal component $\mathbf{u}_2$ gives the direction in which the rows of A show the most remaining variability that is orthogonal to $\mathbf{u}_1$. The third principal component $\mathbf{u}_3$ gives the direction of most variability that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, and so on.

Subspace fitting vs. curve fitting

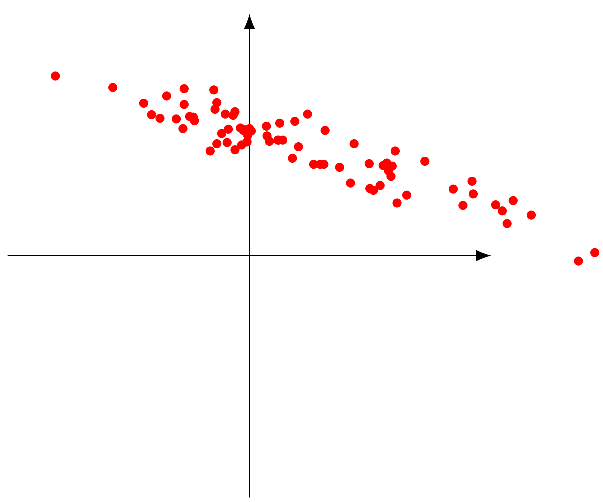
In the particular case where $n = 2$ and $k = 1$, we are looking for a 1-dimensional subspace, i.e., a line through the origin, which best fits the given 2-dimensional data, as in the illustration (11.6) above or as in Example 11.112. On its face, the subspace fitting problem in this case seems similar to the linear curve fitting problem we solved in Section 11.5. However, there is a subtle but important difference: in linear curve fitting, we were seeking to minimize the distances of the points from the line in the y-direction, whereas in subspace fitting, we are seeking to minimize the distances of the points from the subspace in the direction perpendicular to the subspace. The following pair of pictures illustrates the difference:



Linear curve fitting Subspace fitting

Affine fitting

So far, we have been looking to approximate a given collection of points by a *subspace*, which necessarily passes through the origin. But sometimes the points may not be near the origin, as in this example:



In this case, approximating the points by a subspace passing through the origin does not make much sense. Instead, we should be looking for an **affine subspace**. An affine subspace is similar to a subspace, except it does not necessarily contain the origin.

Definition 11.115: Affine subspace

Let V be a vector space. A subset $A \subseteq V$ is called an ***affine subspace*** *of V if A is either empty, or else of the form*

$$A = \mathbf{v} + W = \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\},$$

where $\mathbf{v} \in V$ and W is a subspace of V.

For example, in Chapter 3, we considered lines and planes in $\mathbb{R}^n$ that pass through a given point (not necessarily the origin). These are examples of affine subspaces of $\mathbb{R}^n$. The affine subspace fitting problem is analogous to the subspace fitting problem:

Problem 11.116: Affine subspace fitting problem

Given the position vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of m points in $\mathbb{R}^n$, and given an integer $k \leq n$, find the k-dimensional affine subspace $A \subseteq \mathbb{R}^n$ that minimizes the total squared distance from the points to A.

It turns out that the optimal solution to the affine subspace fitting problem can be computed by first computing the **centroid** of the points, shifting the whole problem so that the centroid is at the origin, and then solving an ordinary subspace fitting problem.

Definition 11.117: Centroid

Given m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, their ***centroid*** *is the vector*

$$\overline{\mathbf{v}} = \frac{1}{m}(\mathbf{v}_1 + \ldots + \mathbf{v}_m).$$

It is also sometimes called the ***average*** *or the* ***center of mass*** *of the vectors.*

Proposition 11.118: Solution of the affine subspace fitting problem

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ and $k \leq n$, the optimal solution to the affine subspace fitting problem can be computed as follows:

1. *Compute the centroid $\overline{\mathbf{v}} = \frac{1}{m}(\mathbf{v}_1 + \ldots + \mathbf{v}_m)$ of the vectors.*
2. *Let $\mathbf{w}_i = \mathbf{v}_i - \overline{\mathbf{v}}$, for all $i = 1, \ldots, n$.*
3. *Compute the solution W to the (ordinary) subspace fitting problem for $\mathbf{w}_1, \ldots, \mathbf{w}_m$, as in Proposition 11.111.*

Then the best solution to the affine subspace problem is $\overline{\mathbf{v}} + W$.

Example 11.119: Affine subspace fitting problem

Consider the following collection of points in $\mathbb{R}^2$*:*

$$\left\{ \begin{bmatrix} 10 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 12 \end{bmatrix} \right\}.$$

Find the 1-dimensional affine subspace that best approximates this collection of points. What is the total squared distance of the points to the subspace?

Solution. We start by computing the centroid:

$$\bar{\mathbf{v}} = \frac{1}{9}(\mathbf{v}_1 + \ldots + \mathbf{v}_9) = \frac{1}{9}\begin{bmatrix} 45 \\ 36 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Next, we shift all vectors by $-\bar{\mathbf{v}}$ to get a new collection of vectors $\mathbf{w}_1, \ldots, \mathbf{w}_9$ centered at the origin. For example,

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 - \bar{\mathbf{v}} &= \begin{bmatrix} 10 \\ -6 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} &= \begin{bmatrix} 5 \\ -10 \end{bmatrix}, \\ \mathbf{w}_2 &= \mathbf{v}_2 - \bar{\mathbf{v}} &= \begin{bmatrix} 2 \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} &= \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \end{aligned}$$

and so on. We get

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_9\} = \left\{ \begin{bmatrix} 5 \\ -10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 8 \end{bmatrix} \right\}.$$

Next we, we proceed as in Proposition 11.111 to find the best subspace fitting $\mathbf{w}_1, \ldots, \mathbf{w}_9$. We have

$$A^T = \begin{bmatrix} 5 & -3 & 0 & 3 & -3 & -2 & -1 & 5 & -4 \\ -10 & 6 & -5 & -1 & 1 & -1 & 7 & -5 & 8 \end{bmatrix}$$

and

$$B = A^T A = \begin{bmatrix} 98 & -136 \\ -136 & 302 \end{bmatrix}.$$

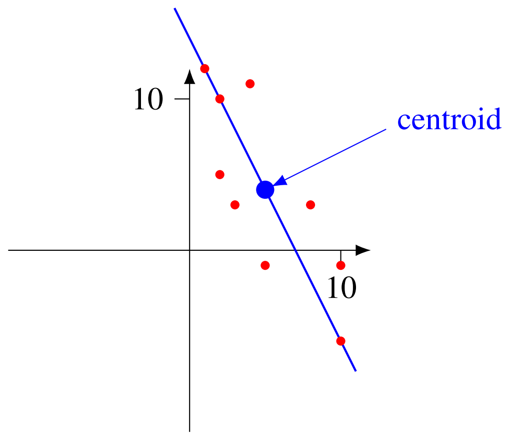
The eigenvalues of B are $\lambda_1 = 370$ and $\lambda_2 = 30$, with corresponding eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, the best-fitting 1-dimensional subspace for $\mathbf{w}_1, \ldots, \mathbf{w}_9$ is $W = \operatorname{span}\{\mathbf{u}_1\}$, and the best-fitting 1-dimensional affine subspace for $\mathbf{v}_1, \ldots, \mathbf{v}_9$ is

$$\bar{\mathbf{v}} + W = \{\bar{\mathbf{v}} + \mathbf{w} \mid \mathbf{w} \in W\} = \left\{ \begin{bmatrix} 5 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Note that this is the equation of a line passing through the centroid $\bar{\mathbf{v}}$, and with direction vector $\mathbf{u}_1$. The points $\mathbf{v}_1, \ldots, \mathbf{v}_9$, their centroid, and the affine subspace $\bar{\mathbf{v}} + W$ are shown in the following illustration:



♠

Application to U.S. Senate voting data

The United States Senate votes on a lot of things: motions, resolutions, amendments, and bills, among other things. Many of these votes are roll call votes, which means that the vote of every individual senator is recorded (as opposed to a voice vote, where only the outcome is recorded). Roll call data for the last 3 decades is publicly available and can be downloaded from the U.S. Senate website at `https://www.senate.gov/legislative/votes.htm`.

We will now explore how to use linear algebra, and in particular principal component analysis, to gain some useful information from the voting records.[2] I have made a spreadsheet containing the votes of 99 senators for the first 200 roll call votes of 2007. Each row in the spreadsheet corresponds to a senator, listed in alphabetical order from Daniel Akaka of Hawaii to Ron Wyden of Oregon. I omitted one senator who died during 2007. Each column of the spreadsheet corresponds to a vote. For example, the first roll call vote of 2007 was on a resolution to honour President Gerald Ford (it passed 88 to 0). Each cell of the spreadsheet contains the number 1 if the senator voted "yes", the number -1 if the senator voted "no", and the number 0 if the senator did not vote. The spreadsheet is available from `https://www.mathstat.dal.ca/~selinger/linear-algebra/` under "Supplementary materials". Here are the first few rows and columns of the spreadsheet.

Akaka, Daniel (D) HI	1	1	1	1	1	-1	1	…
Alexander, Lamar (R) TN	0	1	-1	1	-1	-1	1	…
Allard, A. (R) CO	1	1	-1	-1	-1	1	1	…
Baucus, Max (D) MT	1	1	1	1	1	-1	1	…
Bayh, Evan (D) IN	1	1	1	-1	1	-1	1	…
Bennett, Robert (R) UT	1	1	-1	1	1	-1	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

The human mind is not very well equipped to deal with such massive amounts of data. Rather than listing 122 motions that Senator X supported and 78 motions that she opposed, we like to come up with

[2]This example was inspired by Examples 11.2.13 and 11.3.15 of *"Coding the Matrix: Linear Algebra through Computer Science Applications"* by Philip N. Klein.

abstractions, such as Senator X is “conservative”, “pro choice”, “pro business”, “hawkish”, etc. However, the problem with abstractions is that they do not necessarily mean anything in the real world. In the real world, a senator’s record is just a sequence of votes.

We will represent each senator by a vector in $\mathbb{R}^{200}$, which corresponds to a row of the above table. For example, to Senator Akaka, we associate the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} \in \mathbb{R}^{200}.$$

Thus, we can represent each senator (or more precisely, each senator’s voting record) as a point in 200-dimensional space. In this way, the voting data can be interpreted as 99 points in $\mathbb{R}^{200}$.

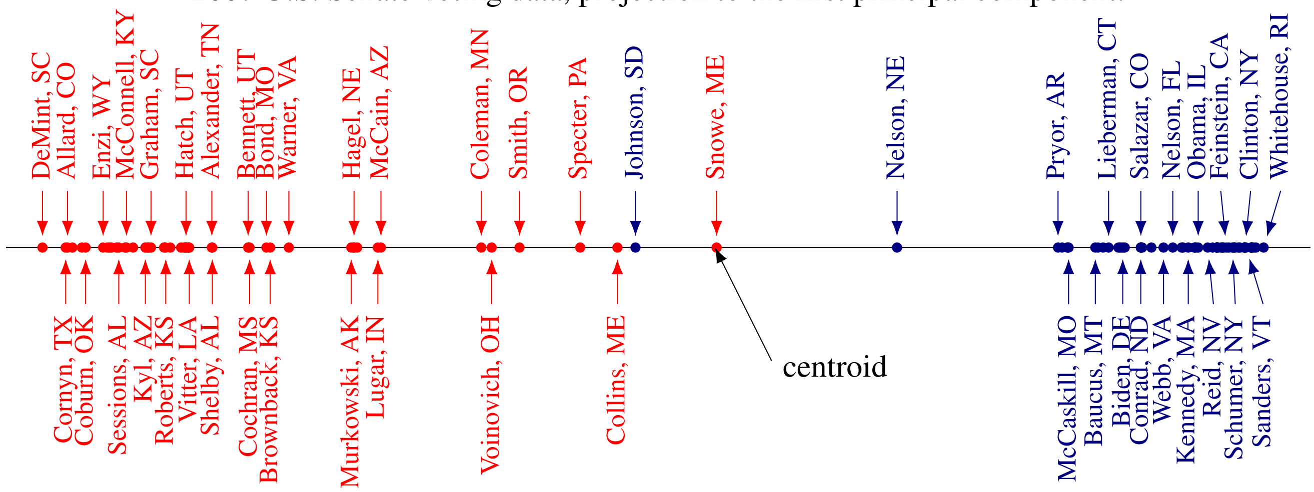
Unfortunately, 200 dimensions are impossible to visualize. But what if the voting records of all the senators lie on (or at least close to) a much-smaller-dimensional affine subspace? This is actually not an unreasonable expectation; after all, there are probably only a handful of issues most senators care about. For example, if a certain senator supports gun control, he will be likely to vote a certain way on measures that affect gun control. If another senator supports the gun lobby, she is likely to vote the opposite way.

We can thus consider this as an instance of the affine subspace problem: we are looking for a low-dimensional affine subspace that is close to all 99 points. Following the method of Proposition 11.118, we first find the centroid of the points, and then we compute a certain 99×200-matrix A and a positive semidefinite 200×200-matrix $B = A^T A$. Using software, we can find the eigenvalues and -vectors of B. The first few eigenvalues (in decreasing order) are:

$$\lambda_1 = 7255.65, \quad \lambda_2 = 519.16, \quad \lambda_3 = 430.60, \quad \lambda_4 = 278.05, \quad \text{and} \quad \lambda_5 = 230.56.$$

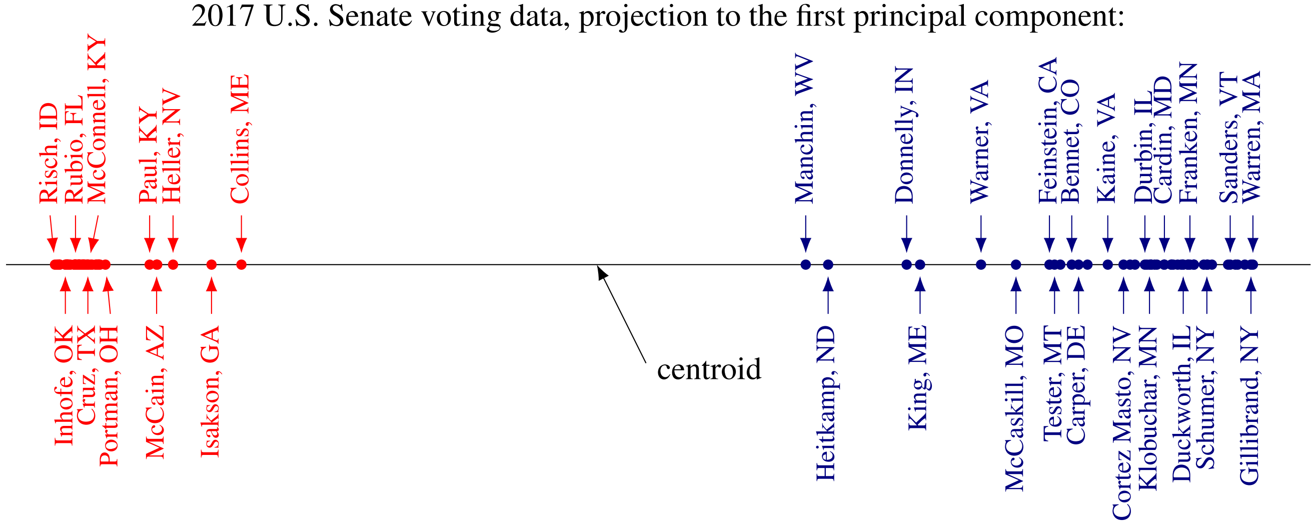
All of the remaining eigenvalues are less than 200, and the sum of the remaining eigenvalues is $\lambda_6 + \ldots + \lambda_{200} = 3913.46$. This means that the vast majority of the voting behavior of each senator is determined by a single dimension, given by the eigenvector corresponding to the eigenvalue λ_1. In other words, there is a 1-dimensional affine subspace that all 99 points are pretty close to. If we project each senator to this affine subspace, we get the following picture:

2007 U.S. Senate voting data, projection to the first principal component:



For convenience, Republican senators have been shown in red and Democratic and independent senators in blue. Not all senators have been named, because in some areas they are clustered very densely. An interpretation of the principal component then immediately suggests itself: it appears to be the "conservative" vs. "liberal" axis. We can use this picture to assist in answering questions such as: *"Which party votes more uniformly?"*, *"Which state are the most liberal Republicans from?"*, *"Which states are the most conservative Democrats from?"*, *"Was Obama really a radical?"*, and *"Was McCain really a maverick?"*. If we repeat the same calculation for the 2017 senate, we get the following picture:

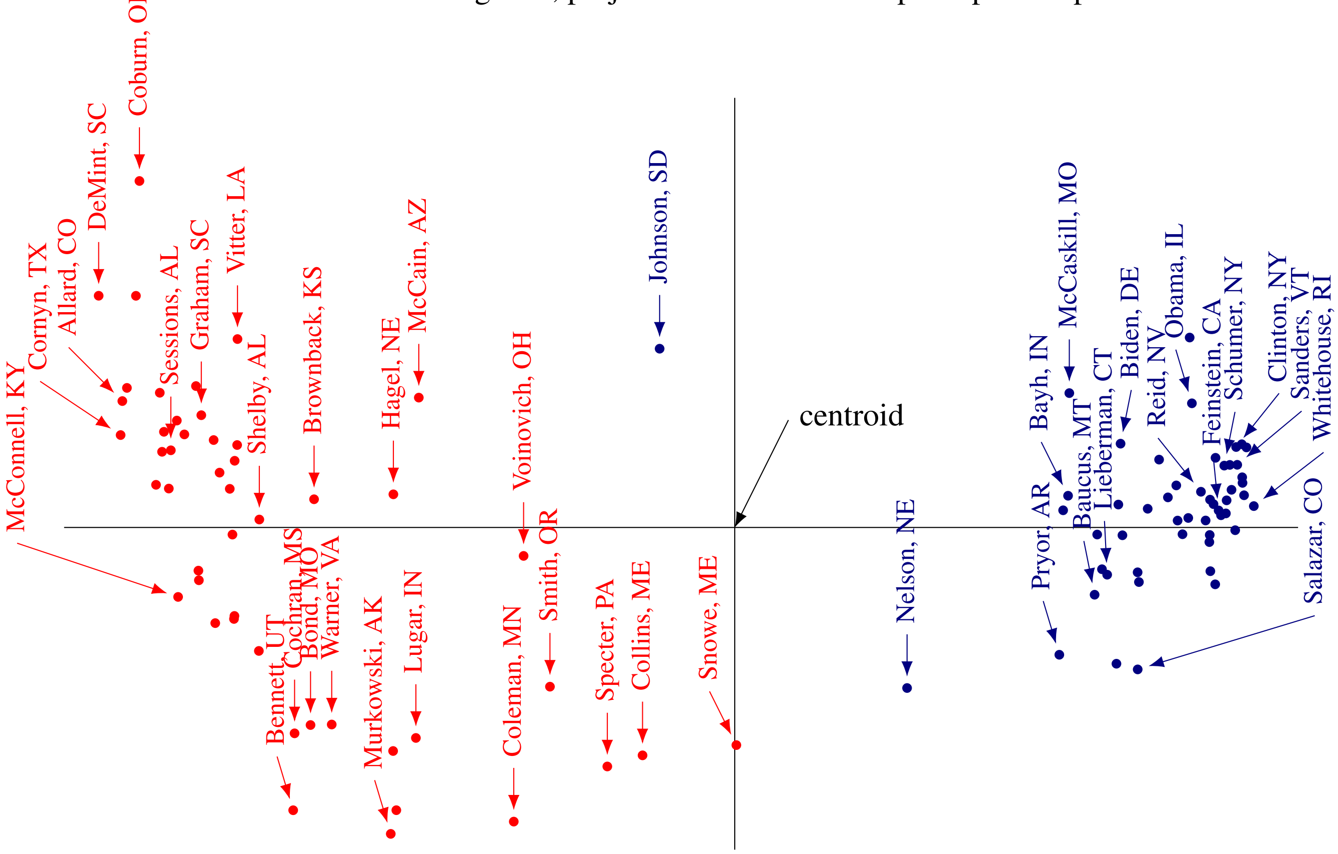
2017 U.S. Senate voting data, projection to the first principal component:



We can use this to help answer questions such as *"Has the senate become more partisan between 2007 and 2017?"*.

If we instead project the data onto the first two principal components, we get the following picture for the 2007 data:

2007 U.S. Senate voting data, projection to the first two principal components:



The picture clearly shows senators clustering in certain areas. We can use this to help answer certain questions, for example, *"How different was Sanders's voting record from Clinton's?"*. However, although the 2-dimensional picture seems to reveal more detail, its interpretation is less clear. While it seems obvious that the horizontal axis corresponds to a conservative vs. liberal world view, it is much less obvious what the political meaning of the vertical axis is. Maybe it is related to some issue that does not typically follow party lines, such as North vs. South, rich states vs. poor states, pro-immigration vs. anti-immigration, and so on. To find a convincing interpretation of the vertical axis, further investigation of the data would be required (such as, looking at the actual content of the votes in question).

Finally, a word of caution. Whenever we use mathematics to try to draw real-world conclusions from data, these conclusions should be taken with an extra-large grain of salt. People have an outsized tendency to trust mathematics and to take its results as infallible. We therefore have a special responsibility not to overstate any conclusions, and to point out potential pitfalls with the analysis. No matter how wonderful principal complement analysis is, we must keep in mind that what we are still only looking at a 2-dimensional projection of a 200-dimensional space. Therefore it is inevitable that lots of details and nuances are lost. We could get a completely different picture by looking at a different 2-dimensional projection.

To see how the data can sometimes be misleading, consider the question *"How similar is Senator Tim Johnson, Democrat of South Dakota, to Senators Olympia Snowe and Susan Collins of Maine?"*. In the 1-dimensional picture, it looked as if they were very similar. We could easily rationalize this by pointing out that Johnson is the most conservative Democrat, and Snowe and Collins are the most liberal

Republicans. However, the 2-dimensional picture reveals an interesting nuance, which is that the voting records of Johnson is not all that similar to that of Snowe and Collins. It is entirely possible that if we add a third or fourth dimension to the picture, many more additional such details will emerge. In summary, while principal component analysis is a useful tool, it is just one tool among many, and we always need to exercise our best judgement in drawing conclusions from data.

Exercises

Exercise 11.12.1 *Consider the following collection of points in* $\mathbb{R}^2$*:*

$$\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-2\\1\end{bmatrix},\begin{bmatrix}3\\-2\end{bmatrix},\begin{bmatrix}2\\-2\end{bmatrix},\begin{bmatrix}-2\\1\end{bmatrix},\begin{bmatrix}3\\-1\end{bmatrix},\begin{bmatrix}-3\\0\end{bmatrix},\begin{bmatrix}0\\3\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\-2\end{bmatrix}\right\}.$$

Find the 1-dimensional subspace that best approximates this collection of points. What is the total squared distance of the points to the subspace? Sketch the subspace and the points.

Exercise 11.12.2 *Consider the following collection of points in* $\mathbb{R}^3$*:*

$$\left\{\begin{bmatrix}0\\0\\9\end{bmatrix},\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0\\5\\-4\end{bmatrix},\begin{bmatrix}6\\-2\\13\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}4\\0\\0\end{bmatrix},\begin{bmatrix}1\\-3\\0\end{bmatrix},\begin{bmatrix}0\\0\\4\end{bmatrix}\right\}.$$

(a) Find the 1-dimensional subspace that best approximates this collection of points.

(b) Find the 2-dimensional subspace that best approximates this collection of points.

(c) What is the 3-dimensional subspace that best approximates this collection of points?

In each case, what is the total squared distance of the points to the subspace?

Exercise 11.12.3 *Find the principal components of the matrix*

$$A=\begin{bmatrix}0&0&0&0&-1&0&3&0&1\\0&1&0&0&2&3&0&2&-1\\1&-1&-3&3&-1&0&0&-2&2\end{bmatrix}.$$

Exercise 11.12.4 *Compute the centroid of the following collection of points:*

$$\left\{\begin{bmatrix}2\\-1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}4\\-3\end{bmatrix},\begin{bmatrix}-4\\-4\end{bmatrix},\begin{bmatrix}0\\-2\end{bmatrix},\begin{bmatrix}-2\\-1\end{bmatrix},\begin{bmatrix}0\\-2\end{bmatrix},\begin{bmatrix}-2\\-4\end{bmatrix},\begin{bmatrix}3\\-1\end{bmatrix},\begin{bmatrix}-5\\-5\end{bmatrix}\right\}.$$

Exercise 11.12.5 *Consider the following collection of points in* $\mathbb{R}^2$*:*

$$\left\{\begin{bmatrix}4\\-3\end{bmatrix},\begin{bmatrix}10\\-9\end{bmatrix},\begin{bmatrix}4\\-7\end{bmatrix},\begin{bmatrix}-2\\3\end{bmatrix},\begin{bmatrix}10\\-3\end{bmatrix},\begin{bmatrix}4\\-3\end{bmatrix},\begin{bmatrix}5\\-3\end{bmatrix},\begin{bmatrix}4\\1\end{bmatrix},\begin{bmatrix}-2\\-3\end{bmatrix},\begin{bmatrix}3\\-3\end{bmatrix}\right\}.$$

Compute the centroid, and then find the 1-dimensional affine subspace that best approximates this collection of points. What is the total squared distance of the points to the subspace?

Exercise 11.12.6 *Consider the following collection of points in* $\mathbb{R}^3$*:*

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

Find the 1- and 2-dimensional affine subspaces that best approximate this collection of points. What is the total squared distance of the points to each subspace?

A. Complex numbers

Throughout history, mankind has invented more and more complicated number systems in an effort to make algebra easier.

- In the beginning, there were the **natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$. However, after a while, it became a problem that certain equations, such as $x + 5 = 3$, do not have a solution in the natural numbers.

- To solve this problem, *zero* and *negative numbers* were invented, resulting in the set of **integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. In the integers, the equation $x + 5 = 3$ has a solution, namely $x = -2$. But some other equations, such as $2x = 1$, still do not have a solution in the integers.

- To solve this problem, the **rational numbers** $\mathbb{Q}$ were invented. In the rational numbers, the equation $2x = 1$ has a solution, namely $x = \frac{1}{2}$. However, some other equations, such as $x^2 = 2$, still do not have a solution in the rational numbers.

- To solve this problem, the **real numbers** $\mathbb{R}$ were invented. In the real numbers, the equation $x^2 = 2$ has a solution, namely $x = \sqrt{2}$. However, some other equations, such as $x^2 = -1$, still don't have a solution in the real numbers.

- To solve this problem, the **complex numbers** were invented.

The purpose of this section is to summarize the most important facts about the complex numbers. We will also see that the above process does not continue. In the complex numbers, all non-trivial polynomial equations have a solution, and therefore, no additional numbers are "missing". This property is called the **fundamental theorem of algebra**. Gauss is usually credited with giving a proof of this theorem in 1797 but many others worked on it and the first completely correct proof was due to Argand in 1806.

A.1 The complex numbers

Outcomes

A. *Add, subtract, multiply, and divide complex numbers.*

B. *Find the conjugate and the magnitude of a complex number.*

C. *Apply algebraic properties of the complex numbers to simplify equations.*

Definition A.1: The complex numbers

Let i be an imaginary number such that $i^2 = -1$. A ***complex number*** *is a number of the form*

$$z = a + bi,$$

where a and b are real numbers. The set of all complex numbers is denoted $\mathbb{C}$.

The form $z = a + bi$ is called the **standard form** or **Cartesian form** of the complex number z. We refer to a as the **real part** and to b as the **imaginary part** of z.

Addition, **subtraction**, and **multiplication** of complex numbers are defined in the obvious way, keeping in mind that $i^2 = -1$. Namely, we have

$$\begin{aligned} (a+bi)+(c+di) &= (a+c)+(b+d)i, \\ (a+bi)-(c+di) &= (a-c)+(b-d)i, \\ (a+bi)(c+di) &= ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i. \end{aligned}$$

Example A.2: Addition, subtraction, and multiplication of complex numbers

- $(3+5i)+(2-3i) = (3+2)+(5-3)i = 5+2i$.
- $(3+5i)-(2-3i) = (3-2)+(5+3)i = 1+8i$.
- $(3+5i)(2-3i) = 6-9i+10i-15i^2 = 21+i$.

Division of complex numbers is more complicated. We first note that it is easy to divide a complex number by a *real* number. Namely,

$$\frac{a+bi}{r} = \frac{a}{r} + \frac{b}{r}i.$$

But how can we divide by a complex number? We use the following trick. Let $z = a + bi$ be a complex number, and consider the product $(a+bi)(a-bi)$. It is equal to

$$(a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2.$$

Therefore, $(a+bi)(a-bi)$ is always a *real* number, and therefore easy to divide by. Therefore, we can compute the **multiplicative inverse** of a complex number $z = a + bi$ as follows:

$$z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi}\,\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}.$$

Example A.3: Inverse of a complex number

$$\frac{1}{2+5i} = \frac{2-5i}{2^2+5^2} = \frac{2}{29} - \frac{5}{29}i.$$

You should verify that this is indeed the inverse, by multiplying $2+5i$ by $\frac{2}{29}-\frac{5}{29}i$ and checking that the answer is indeed 1.

Division of complex numbers can then be defined in terms of the multiplicative inverse, i.e., $\frac{z}{w}=zw^{-1}$.

Example A.4: Division of complex numbers

$$\frac{5+7i}{3-4i}=\frac{(5+7i)(3+4i)}{(3-4i)(3+4i)}=\frac{-13+41i}{3^2+4^2}=-\frac{13}{25}+\frac{41}{25}i.$$

As a special case of division, note that

$$i^{-1}=\frac{1}{i}=\frac{1(-i)}{i(-i)}=-i.$$

The complex numbers form a *field*, i.e., they satisfy the nine field axioms. See Section 1.8 for the definition of a field.

Proposition A.5: The complex numbers form a field

The complex numbers, with the operations of addition, subtraction, multiplication, and division, form a field. *Specifically, this means that they satisfy the following properties:*

(A1) Commutative law of addition: $z+w=w+z$;

(A2) Associative law of addition: $(z+w)+u=z+(w+u)$;

(A3) Unit law of addition: $0+z=z$;

(A4) Additive inverse: $z+(-z)=0$;

(M1) Commutative law of multiplication: $zw=wz$;

(M2) Associative law of multiplication: $(zw)u=z(wu)$;

(M3) Unit law of multiplication: $1z=z$;

(M4) Multiplicative inverse: when z *is non-zero:* $zz^{-1}=1$;

(D) Distributive law: $z(w+u)=zw+zu$.

Another useful operation on complex numbers is the complex conjugate. Let $z=a+bi$ be a complex number. Then the **conjugate** of z, written $\bar{z}$, is given by

$$\bar{z}=a-bi.$$

Note that if $z=a+bi$ is a complex number, then $z\bar{z}=(a+bi)(a-bi)=a^2+b^2$. Therefore, $z\bar{z}$ is always a real number and $z\bar{z}\geq 0$. We define the **magnitude** of z to be

$$|z|=\sqrt{z\bar{z}}=\sqrt{a^2+b^2}.$$

The magnitude is also sometimes called the **absolute value** or the **modulus** of the complex number.

Example A.6: Conjugate and magnitude of a complex number

- $\overline{3+5i} = 3-5i$.
- $\overline{i} = -i$.
- $\overline{7} = 7$.
- $|3+5i| = \sqrt{3^2+5^2} = \sqrt{34}$.
- $|i| = 1$.
- $|-6| = 6$.

Note that for a real number a, we have $\overline{a} = a$. Also, when a is real, the magnitude $|a| = \sqrt{a^2}$ is just the usual absolute value of real numbers. The following two propositions list some basic properties of the conjugate and of the magnitude.

Proposition A.7: Properties of the conjugate

Let z and w be complex numbers. Then, the following properties of the conjugate hold.

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
- $\overline{(zw)} = \overline{z}\,\overline{w}$.
- $\overline{z^{-1}} = \overline{z}^{-1}$.
- $\overline{z/w} = \overline{z}/\overline{w}$.
- $\overline{\overline{z}} = z$.
- *z is real if and only if $\overline{z} = z$.*

Proposition A.8: Properties of the magnitude

Let z,w be complex numbers. The following properties hold.

- $|z| \geq 0$, *and* $|z| = 0$ *if and only if* $z = 0$.
- $|zw| = |z|\,|w|$.
- $|\overline{z}| = |z|$.
- $|z/w| = |z|\,/\,|w|$.
- ***Triangle inequality***: $|z+w| \leq |z| + |w|$.

Exercises

Exercise A.1.1 *Let $z = 2+7i$ and let $w = 3-8i$. Compute $z+w$, $z-2w$, zw, and $\frac{w}{z}$.*

Exercise A.1.2 *Let $z = 1-4i$. Compute $\overline{z}$, z^{-1}, and $|z|$.*

Exercise A.1.3 *Let $z = 3+5i$ and $w = 2-i$. Compute $\overline{zw}$, $|zw|$, and $z^{-1}w$.*

Exercise A.1.4 *Use the properties of complex numbers to prove that if z is a complex number, then there exists a complex number w with $|w| = 1$ and $wz = |z|$.*

Exercise A.1.5 *I claim that $1 = -1$. Here is why.*

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

What is wrong with this argument?

A.2 Geometric interpretation

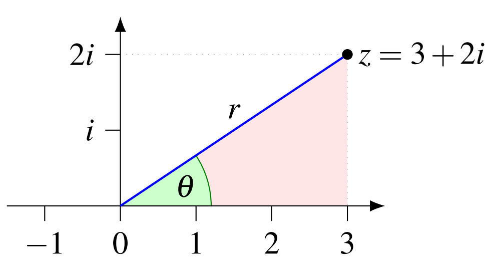
Outcomes

A. *View complex numbers as points in the plane.*

B. *Understand the geometric meaning of addition, subtraction, multiplication, and the complex conjugate.*

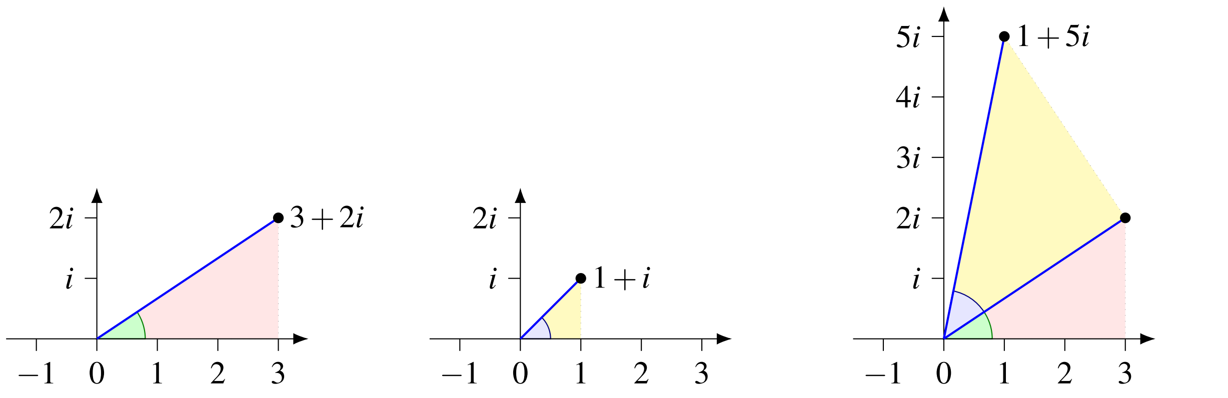
C. *Understand the geometric meaning of the magnitude and argument of a complex number.*

Just as a real number can be considered as a point on the line, a complex number $z = a+bi$ can be considered as a point (a,b) in the plane whose x coordinate is a and whose y coordinate is b. For example, in the following picture, the complex number $z = 3+2i$ can be represented as the point in the plane with coordinates $(3,2)$.



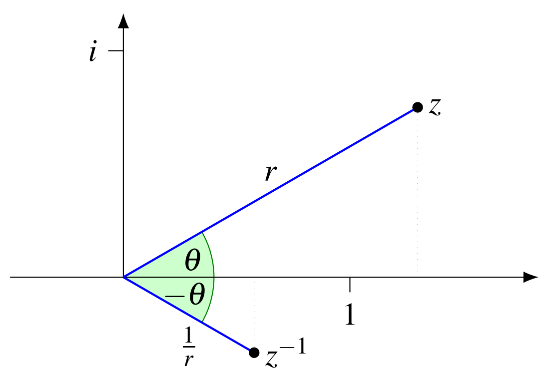
The **magnitude** $r = |z|$ of a complex number is its distance from the origin. We also define the **argument** of z to be the angle θ between the x-axis and the line from the origin to z, counted positively in the counterclockwise direction. The magnitude r and argument θ are shown in the above picture.

Addition of complex numbers is like vector addition. The effect of multiplying two complex numbers is to multiply their magnitudes and add their arguments. For example, the following picture illustrates the multiplication $(3+2i)(1+i) = 1+5i$.

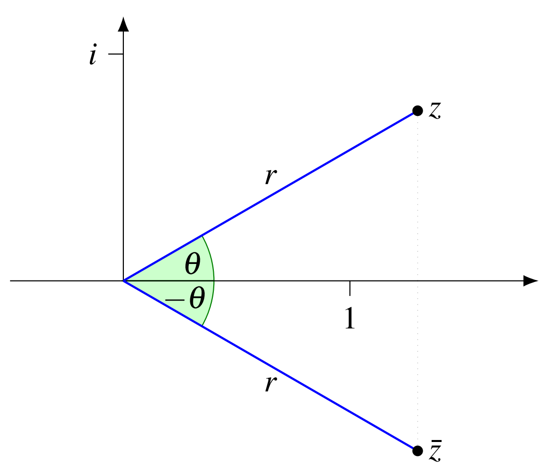


The number $3+2i$ | The number $1+i$ | The product $(3+2i)(1+i) = 1+5i$

To take the multiplicative inverse of a complex number, we take the reciprocal of the magnitude and negate the argument.



The effect of taking the complex conjugate is to reflect the given complex number about the x axis (or equivalently, keep the magnitude unchanged and negate the argument).



Exercises

Exercise A.2.1 *Draw the complex numbers $z = 2+i$ and $w = -2+3i$ as points in the plane. Then use the geometric interpretation to find $z+w$, $z-w$, zw, z^{-1}, $\bar{z}$, and $|z|$.*

Exercise A.2.2 *Use the geometric interpretation to find a complex number z such that $z^2 = i$. Can you find two such numbers?*

Exercise A.2.3 *Use the geometric interpretation to find 3 different complex numbers z such that $z^3 = -1$. Hint: these numbers will lie on the unit circle.*

A.3 The fundamental theorem of algebra

Outcomes

A. *Find the complex roots of a quadratic polynomial.*

B. *In special cases, find the complex roots of a polynomial of degree 3 or more.*

C. *Factor a polynomial into linear factors.*

The complex numbers were invented so that equations such as $z^2+1=0$ would have solutions. In fact, this equation has two complex solutions, namely $z = i$ and $z = -i$. However, something much more general (and surprising) is true: *every* non-trivial polynomial equation has a solution in the complex numbers. To understand this statement, recall that a **polynomial** is an expression of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0.$$

The constants $a_0, \ldots, a_n$ are called the **coefficients** of the polynomial. If a_n is the largest non-zero coefficient, we say that the polynomial has **degree** n. A polynomial of degree 0 is of the form $p(z) = a_0$, and is also called a **constant polynomial**. Recall that a **root** of a polynomial is a number z such that $p(z) = 0$. The fundamental theorem of algebra is the following:

Theorem A.9: Fundamental theorem of algebra

Every non-constant polynomial $p(z)$ with real or complex coefficients has a complex root.

The proof of this theorem is beyond the scope of this book. Note that the theorem does not say that the roots are always easy to find. To find the roots of a polynomial of degree 2, we can use the quadratic formula. However, if the degree is greater than 2, we may sometimes have to use fancier methods, such as Newton's method from calculus, or even a computer algebra system, to locate the roots. We give some examples.

Example A.10: Roots of a quadratic polynomial

Find the roots of the polynomial $p(z) = z^2 - 2z + 2$.

Solution. The quadratic formula gives

$$z = \frac{2 \pm \sqrt{-4}}{2}.$$

Of course, in the real numbers, the square root of -4 does not exist, so $p(z)$ has no roots in the real numbers. However, in the complex numbers, the square root of -4 exists and is equal to $\pm 2i$. Thus, the roots of $p(z)$ are:

$$z = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Indeed, we can double-check that $1+i$ and $1-i$ are in fact roots:

$$\begin{aligned} p(1+i) &= (1+i)^2 - 2(1+i) + 2 = (1+2i+(-1)) - 2 - 2i + 2 = 0, \\ p(1-i) &= (1-i)^2 - 2(1-i) + 2 = (1-2i+(-1)) - 2 + 2i + 2 = 0. \end{aligned}$$

♠

Example A.11: Roots of a cubic polynomial

Find the roots of the polynomial $p(z) = z^3 - 4z^2 + 9z - 10$.

Solution. By the intermediate value theorem of calculus, we know that a cubic polynomial with real coefficients always has at least one real root. This is because $p(z)$ goes to $-\infty$ when $z \to -\infty$ and to ∞ when $z \to \infty$. By trial and error, we find that $z = 2$ is a root of this polynomial. We can therefore factor out $(z-2)$ from this polynomial:

$$p(z) = z^3 - 4z^2 + 9z - 10 = (z-2)(z^2 - 2z + 5).$$

Now we can use the quadratic formula to find the roots of $z^2 - 2z + 5$. We find

$$z = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Thus, the three complex roots of $p(z)$ are $z = 2$, $z = 1 + 2i$, and $z = 1 - 2i$. ♠

The following proposition is an important and useful consequence of the fundamental theorem of algebra:

Proposition A.12: Factoring a polynomial

Let $p(z)$ be a polynomial of degree n with real or complex coefficients. Then $p(z)$ can be factored into n linear factors over the complex numbers, i.e., $p(z)$ can be written in the form

$$p(z) = a(z - b_1)(z - b_2)\cdots(z - b_n),$$

where $b_1, \ldots, b_n$ are (not necessarily distinct) roots of $p(z)$.

Proof. If $n = 0$, then $p(z) = a$ and there is nothing to show. Otherwise, by the fundamental theorem of algebra, $p(z)$ has at least one complex root, say b_1. From calculus, we know that we can factor out $(z - b_1)$ from $p(z)$, i.e., we can find a polynomial $q(z)$ of degree $n - 1$ such that

$$p(z) = (z - b_1)q(z),$$

We can repeatedly apply the same procedure to $q(z)$ until $p(z)$ has been factored into linear factors. ♠

Example A.13: Factoring a polynomial

Factor $p(z) = z^3 - 4z^2 + 9z - 10$ *into linear factors.*

Solution. From Example A.11, we know that $p(z)$ has three distinct roots $b_1 = 2$, $b_2 = 1 + 2i$, and $b_3 = 1 - 2i$. We can therefore write

$$p(z) = a(z - b_1)(z - b_2)(z - b_3).$$

Since the leading term is z^3, we find that $a = 1$. Therefore

$$p(z) = (z - 2)(z - 1 - 2i)(z - 1 + 2i).$$

♠

Exercises

Exercise A.3.1 *Find the roots of the polynomial* $p(z) = z^2 - 6z + 13$.

Exercise A.3.2 *Find the roots of* $p(z) = z^3 + 5z^2 + 4z - 10$. *Hint: one of the roots is* $z = 1$.

Exercise A.3.3 *Factor* $z^3 - 3z + 2$ *into a product of linear factors.*

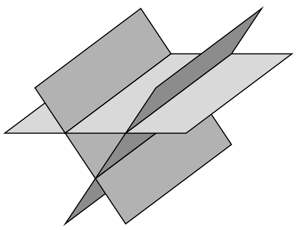
Exercise A.3.4 *Let* $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *be a polynomial with real coefficients, i.e., such that all the* a_k *are real numbers. Suppose that* z *is a root of* p*. Show that* $\bar{z}$ *is also a root of* p.

B. Answers to selected exercises

1.1.1 $\begin{array}{l} x+3y=1 \\ 4x-y=3 \end{array}$, Solution is: $\left[x=\frac{10}{13}, y=\frac{1}{13}\right]$.

1.1.2 $\begin{array}{l} 3x+y=3 \\ x+2y=1 \end{array}$, Solution is: $[x=1, y=0]$

1.1.4



1.2.1 All of them except (c) and (e).

1.2.2 (a) is a solution to equations 1 and 2, (b) is a solution to equations 2 and 3, (c) is a solution to equations 1, 2, and 3, (d) is a solution to 1 and 3, and (e) is a solution to equations 1, 2, and 3. Only (c) and (e) are solutions to the system of equations.

1.3.1 $(x,y)=(1,0)$.

1.3.2 $\begin{array}{l} x+3y=1 \\ 4x-y=3 \end{array}$, Solution is: $(x,y)=\left(\frac{10}{13}, \frac{1}{13}\right)$

1.3.3 $\begin{array}{r} x+2y=1 \\ 2x-y=1 \\ 4x+3y=3 \end{array}$, Solution is: $(x,y)=\left(\frac{3}{5}, \frac{1}{5}\right)$

1.3.4 No solution exists. $\begin{array}{rcrcrcl} x & + & y & - & 3z & = & 2 \\ 2x & + & y & + & z & = & 1 \\ 3x & + & 2y & - & 2z & = & 0 \end{array}$, after elementary operations: $\begin{array}{rcrcl} x & & & + 4z & = 0 \\ & & y & - 7z & = 0 \\ & & & 0 & = 1 \end{array}$. Thus one of the equations says $0=1$ in an equivalent system of equations.

1.3.5 $(x,y,z)=(4,1,1)$.

1.3.6 $\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 5 \\ -x & - & y & + & z & = & -2 \\ x & - & 3y & & & = & -3. \end{array}$ As an augmented matrix: $\left[\begin{array}{rrr|r} 1 & 2 & -3 & 5 \\ -1 & -1 & 1 & -2 \\ 1 & -3 & 0 & -3 \end{array}\right]$.

1.3.7 $\begin{array}{c} 4g - I = 150 \\ 4I - 17g = -660 \\ 4g + s = 290 \\ g + I + s - b = 0 \end{array}$, Solution is : $\{g = 60, I = 90, b = 200, s = 50\}$

1.4.1 The solution exists but is not unique.

1.4.2 A solution exists and is unique.

1.4.4 There might be a solution. If so, there are infinitely many.

1.4.5 No. Consider $x + y + z = 2$ and $x + y + z = 1$.

1.4.6 These can have a solution. For example, $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 3$ even has an infinite set of solutions.

1.4.7 $h = 4$.

1.4.8 Any h will work.

1.4.9 Any h will work.

1.4.10 If $h \neq 2$ there will be a unique solution for any k. If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

1.4.11 If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

1.4.12 There is no solution. The system is inconsistent. You can see this from the augmented matrix. $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{array}\right]$, echelon form: $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -3 & 0 & 2 & -1 \\ 0 & 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

1.4.13 The solution is: $x = \frac{1}{3} - \frac{1}{3}t$, $y = \frac{2}{3} + \frac{2}{3}t$, $z = \frac{1}{3}$, $w = t$.

1.4.14 (a) Yes. (b) No. (c) Yes.

1.4.16 (b) The echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, the solution is of the form $z = t$, $y = \frac{3}{4} + t\left(\frac{1}{4}\right)$, $x = \frac{1}{2} - \frac{1}{2}t$, where $t \in \mathbb{R}$.

(c) The echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array}\right]$ and so the solution is $z = t$, $y = -1 + 4t$, $x = 2 - 4t$.

(d) The echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array}\right]$ and so $x_5 = t$, $x_4 = 1-6t$, $x_3 = -1+7t$, $x_2 = 4t$, $x_1 = 3-9t$.

(e) The echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, let $x_5 = t$, $x_3 = s$. Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t$, $x_2 = \frac{3}{2} - t\frac{1}{2}$, $x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

1.4.17 Solution is: $[x = 1-2t, z = 1, y = t]$.

1.4.18 Solution is: $[x = 2-4t, y = -8t, z = t]$.

1.4.19 Solution is: $[x = -1, y = 2, z = -1]$.

1.4.20 Solution is: $[x = 2, y = 4, z = 5]$.

1.4.21 Solution is: $[x = 1, y = 2, z = -5]$.

1.4.22 No. Consider $x+y+z = 2$ and $x+y+z = 1$.

1.4.23 No. This would lead to $0 = 1$.

1.4.24 Yes. It has a unique solution.

1.4.25 The last column must not be a pivot column. The remaining columns must each be pivot columns.

1.4.26 You need $\begin{array}{c} \frac{1}{4}(20+30+w+x) - y = 0 \\ \frac{1}{4}(y+30+0+z) - w = 0 \\ \frac{1}{4}(20+y+z+10) - x = 0 \\ \frac{1}{4}(x+w+0+10) - z = 0 \end{array}$. Solution is: $[x = 15, y = 20, z = 10, w = 15]$.

1.4.28 The rank is the number of pivot entries in the echelon form. There is at most one pivot entry in each row and column. Therefore, the rank cannot be larger than the number of rows or the number of columns; in other words, the rank is at most $\min(m,n)$.

1.4.29 (a) The echelon form has 4 non-zero rows and 6 columns, so there are 2 free variables, and the system has infinitely many solutions.

(b) Such a system of equations of equations does not exist. If you add in another column, the rank does not get smaller.

(c) Such a system of equations does not exist, because the rank cannot equal 4 if there are only two columns.

(d) The echelon form has 4 non-zero rows on the left-hand side, but 5 non-zero rows if we also include the right-hand side. Therefore the system is inconsistent, i.e., it has no solutions.

(e) The echelon form has 2 non-zero rows, so there are 2 pivot variables and no free variables. Therefore, the system has a unique solution.

1.4.30 These are not legitimate row operations. They do not preserve the solution set of the system.

1.5.1 (a) This one is not.

(b) This one is.

(c) This one is.

1.5.3 Solution is: $[x=-1, y=-5, z=4]$

1.5.4 Solution is: $[x=2t+1, y=4t, z=t]$

1.5.5 Solution is: $[x=1, y=5, z=3]$

1.5.6 Solution is: $[x=4, y=-4, z=-2]$

1.5.7 The rank of the coefficient matrix is 3, so both systems have a unique solution. The solution of the first system is $(x,y,z)=(1,0,1)$, and the solution of the second system is $(x,y,z)=(1,1,2)$.

1.6.2 The systems (b), (c), and (d) have non-trivial solutions, because the rank is less than the number of variables.

1.6.3 The general solution is $(x,y,z)=(1,2,4)+s(1,0,1)+t(0,1,-1)$, or equivalently, $(x,y,z)=(1+s, 2+t, 4+s-t)$, where s and t are parameters.

1.8.6 (a) Reduced echelon form: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & i & 1 \end{array}\right]$. General solution: $x=1-t$, $y=1-it$, $z=t$.

(b) Solution: $x=1$, $y=1+i$, $z=i$.

1.11.1 The other three equations are

$$\begin{array}{rcl} 4I_1+I_1-I_4+2I_1-2I_2 & = & -10 \\ 6I_3-6I_4+I_3+I_3+5I_3-5I_2 & = & -20 \\ 2I_4+3I_4+6I_4-6I_3+I_4-I_1 & = & 0. \end{array}$$

Then the system is

$$\begin{gathered}2I_2-2I_1+5I_2-5I_3+3I_2=5\\4I_1+I_1-I_4+2I_1-2I_2=-10\\6I_3-6I_4+I_3+I_3+5I_3-5I_2=-20\\2I_4+3I_4+6I_4-6I_3+I_4-I_1=0.\end{gathered}$$

The solution is:

$$\begin{aligned}I_1&=-\frac{750}{373}\\I_2&=-\frac{1421}{1119}\\I_3&=-\frac{3061}{1119}\\I_4&=-\frac{1718}{1119}.\end{aligned}$$

1.11.2 We have

$$\begin{aligned}2I_1+5I_1+3I_1-5I_2&=10\\I_2-I_3+3I_2+7I_2+5I_2-5I_1&=-12\\2I_3+4I_3+4I_3+I_3-I_2&=0.\end{aligned}$$

Simplifying this yields

$$\begin{aligned}10I_1-5I_2&=10\\-5I_1+16I_2-I_3&=-12\\-I_2+11I_3&=0.\end{aligned}$$

The solution is given by

$$I_1=\frac{218}{295},\quad I_2=-\frac{154}{295},\quad I_3=-\frac{14}{295}.$$

2.1.2

$$\overrightarrow{PQ}=\overrightarrow{0Q}-\overrightarrow{0P}=\begin{bmatrix}5-2\\-2-0\\1-(-4)\end{bmatrix}=\begin{bmatrix}3\\-2\\5\end{bmatrix}$$

$$\overrightarrow{QP}=\overrightarrow{0P}-\overrightarrow{0Q}=\begin{bmatrix}2-5\\0-(-2)\\(-4)-1\end{bmatrix}=\begin{bmatrix}-3\\2\\-5\end{bmatrix}$$

2.1.3 We need $5x-3y=2x-2y$ and $4=2y$. The unique solution is $x=\frac{2}{3}$ and $y=2$.

2.2.1 $\begin{bmatrix}1\\9\\0\end{bmatrix}$.

2.2.2 (a) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = (\mathbf{v}+\mathbf{u})+\mathbf{w} = \mathbf{v}+(\mathbf{u}+\mathbf{w}) = \mathbf{v}+(\mathbf{w}+\mathbf{u}) = (\mathbf{v}+\mathbf{w})+\mathbf{u}$. Here we have used the commutative law, the associative law, the commutative law, and the associative law, in that order.

(b) $(\mathbf{u}+\mathbf{0})+(\mathbf{v}+(-\mathbf{u})) = \mathbf{u}+(\mathbf{v}+(-\mathbf{u})) = \mathbf{u}+((-\mathbf{u})+\mathbf{v}) = (\mathbf{u}+(-\mathbf{u}))+\mathbf{v} = \mathbf{0}+\mathbf{v} = \mathbf{v}+\mathbf{0} = \mathbf{v}$. Here we have used the additive unit, the commutative law, the associative law, the additive inverse, the commutative law, and the additive unit, in that order.

2.3.1

2.3.2 $\begin{bmatrix} -55 \\ 13 \\ -21 \\ 39 \end{bmatrix}$.

2.3.3 (a) $(k+\ell)(\mathbf{u}+\mathbf{v}) = (k+\ell)\mathbf{u}+(k+\ell)\mathbf{v} = k\mathbf{u}+k\mathbf{v}+\ell\mathbf{u}+\ell\mathbf{v}$. Here we used the distributive law over vector addition in the first step, and the distributive law over scalar addition in the second step.

(b) We have $0\mathbf{u} = (0+0)\mathbf{u} = 0\mathbf{u}+0\mathbf{u}$ by properties of scalars and by the distributive law, respectively. Adding $-(0\mathbf{u})$ to both sides of the equation, and using the additive unit law and associativity, we have $\mathbf{0} = 0\mathbf{u}$.

(c) We have $(-1)\mathbf{u} = (-1)\mathbf{u}+\mathbf{0} = (-1)\mathbf{u}+(\mathbf{u}+(-\mathbf{u})) = ((-1)\mathbf{u}+\mathbf{u})+(-\mathbf{u}) = ((-1)\mathbf{u}+1\mathbf{u})+(-\mathbf{u}) = ((-1)+1)\mathbf{u}+(-\mathbf{u}) = 0\mathbf{u}+(-\mathbf{u}) = \mathbf{0}+(-\mathbf{u}) = -\mathbf{u}$. Here, we have used the additive unit law, the additive inverse law, the associative law, the rule for multiplication by 1, the distributive law, properties of scalars, the property of part (b), and the additive unit law, respectively.

2.4.2

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

2.4.3 The system

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = a_1\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} + a_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

has no solution.

2.5.1 $\sqrt{17}$.

2.5.2 $\sqrt{27}$.

2.5.3 The form the plane $x+y+z=0$.

2.5.4 They form a sphere of radius 1, given by the equation $x^2+y^2+z^2=1$.

2.5.5 $\|\mathbf{u}\| = \sqrt{13}$, $\|\mathbf{v}\| = \sqrt{30}$, $\|\mathbf{w}\| = \sqrt{22}$.

2.5.7 This follows from the last property of Proposition 2.20: $\|-\mathbf{u}\| = \|(-1)\mathbf{u}\| = |-1|\,\|\mathbf{u}\| = \|\mathbf{u}\|$.

2.5.8 Only $\mathbf{w}$ is a unit vector.

2.5.9

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{17}}\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \quad \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{6}\begin{bmatrix} 5 \\ -3 \\ 1 \\ -1 \end{bmatrix}.$$

2.6.1 $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 17.$

2.6.4 $\cos\theta = \dfrac{\begin{bmatrix} 3 & -1 & -1 \end{bmatrix}^T \bullet \begin{bmatrix} 1 & 4 & 2 \end{bmatrix}^T}{\sqrt{9+1+1}\sqrt{1+16+4}} = \dfrac{-3}{\sqrt{11}\sqrt{21}}.$

2.6.5 $\cos\theta = \dfrac{-10}{\sqrt{1+4+1}\sqrt{1+4+49}}.$

2.6.6 This formula says that $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$ where θ is the included angle between the two vectors. Thus

$$\|\mathbf{u} \bullet \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\|\cos\theta\| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

and equality holds if and only if $\theta = 0$ or π. This means that the two vectors either point in the same direction or opposite directions. Hence one is a multiple of the other.

2.6.7 This triangle has sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$. The direction of the sides does not matter, since scalar multiplication preserves orthogonality.

$$\begin{aligned} \overrightarrow{PQ} &= (5,-2,1)-(2,0,3) = (3,-2,4), \\ \overrightarrow{QR} &= (7,5,3)-(5,-2,1) = (2,7,2), \\ \overrightarrow{RP} &= (2,0,-3)-(7,5,3) = (-5,-5,-6). \end{aligned}$$

If any pair of these is orthogonal then we are done:

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (3,-2,4)\cdot(2,7,2) = 6-14+8 = 0.$$

The triangle has a right angle at B.

2.6.8 $\dfrac{\mathbf{v}\bullet\mathbf{w}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v} = \dfrac{-5}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-\frac{5}{14}\\-\frac{5}{7}\\-\frac{15}{14}\end{bmatrix}.$

2.6.9 $\dfrac{\mathbf{v}\bullet\mathbf{w}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v} = \dfrac{-5}{10}\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\0\\-\frac{3}{2}\end{bmatrix}.$

2.6.10 $\dfrac{\mathbf{v}\bullet\mathbf{w}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v} = \dfrac{\begin{bmatrix}1 & 2 & 3 & 0\end{bmatrix}^T\bullet\begin{bmatrix}1 & 2 & -2 & 1\end{bmatrix}^T}{1+4+9}\begin{bmatrix}1\\2\\3\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{14}\\-\frac{2}{14}\\-\frac{3}{14}\\0\end{bmatrix}.$

2.6.13 No, it does not. The vector $\mathbf{0}$ has no direction. The formula for $\text{proj}_{\mathbf{0}}(\mathbf{w})$ doesn't make sense either.

2.6.14

$$\mathbf{a} = \begin{bmatrix}-3/2\\3/2\\-3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}9/2\\1/2\\-2\end{bmatrix}.$$

2.6.15

$$\left(\mathbf{u}-\frac{\mathbf{v}\bullet\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\bullet\left(\mathbf{u}-\frac{\mathbf{v}\bullet\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}\right) = \|\mathbf{u}\|^2-2(\mathbf{v}\bullet\mathbf{u})^2\frac{1}{\|\mathbf{v}\|^2}+(\mathbf{v}\bullet\mathbf{u})^2\frac{1}{\|\mathbf{v}\|^2}\geq 0,$$

and so

$$\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 \geq (\mathbf{v}\bullet\mathbf{u})^2.$$

We get equality exactly when $\mathbf{u} = \text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{v}\bullet\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}$, or in other words, when $\mathbf{u}$ is a multiple of $\mathbf{v}$.

2.6.16

$$\begin{aligned}\mathbf{w}-\text{proj}_{\mathbf{v}}(\mathbf{w})+\mathbf{u}-\text{proj}_{\mathbf{v}}(\mathbf{u}) &= \mathbf{w}+\mathbf{u}-(\text{proj}_{\mathbf{v}}(\mathbf{w})+\text{proj}_{\mathbf{v}}(\mathbf{u}))\\ &= \mathbf{w}+\mathbf{u}-\text{proj}_{\mathbf{v}}(\mathbf{w}+\mathbf{u}).\end{aligned}$$

This follows because

$$\begin{aligned}\text{proj}_{\mathbf{v}}(\mathbf{w})+\text{proj}_{\mathbf{v}}(\mathbf{u}) &= \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v}+\frac{\mathbf{v}\bullet\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v}\\ &= \frac{\mathbf{v}\bullet(\mathbf{w}+\mathbf{u})}{\|\mathbf{v}\|^2}\mathbf{v}\\ &= \text{proj}_{\mathbf{v}}(\mathbf{w}+\mathbf{u}).\end{aligned}$$

2.6.17 $\mathbf{u}\bullet(\mathbf{v}-\text{proj}_{\mathbf{u}}(\mathbf{v})) = \mathbf{u}\bullet\mathbf{v}-\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\bullet\mathbf{u} = \mathbf{u}\bullet\mathbf{v}-\mathbf{u}\bullet\mathbf{v} = 0$. Therefore, we can write $\mathbf{v} = (\mathbf{v}-\text{proj}_{\mathbf{u}}(\mathbf{v}))+\text{proj}_{\mathbf{u}}(\mathbf{v})$. The first is orthogonal to $\mathbf{u}$ and the second is a multiple of $\mathbf{u}$ so is parallel to $\mathbf{u}$.

2.7.1 (a) Left-handed. (b) Right-handed. (c) Right-handed. (d) Left-handed.

2.7.2 If $\mathbf{a} \neq \mathbf{0}$, then the condition says that $\|\mathbf{a} \times \mathbf{u}\| = \|\mathbf{a}\| \sin\theta = 0$ for all angles θ. Hence $\mathbf{a} = \mathbf{0}$.

2.7.3 $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix}$. The area of the parallelogram is $8\sqrt{3}$.

2.7.4 $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -2 \end{bmatrix}$. The area is of the parallelogram is $\sqrt{36+121+4} = \sqrt{161}$.

2.7.5 Let $P = (-2,3,1)$, $Q = (2,1,1)$, $R = (1,2,-1)$, and $S = (5,0,-1)$. We have $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$. The area of the parallelogram is $\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \sqrt{5^2+8^2+2^2} = \sqrt{93}$.

2.7.6 Let $P = (1,0,3)$, $Q = (4,1,0)$, and $R = (-3,1,1)$. $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \times \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \\ 7 \end{bmatrix}$. The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{1+18^2+7^2} = \frac{1}{2}\sqrt{374}$.

2.7.7 Let $P = (1,2,3)$, $Q = (2,3,4)$, and $R = (3,4,5)$. $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The area of the triangle is 0. It means the three points are on a line.

2.7.8 $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$. However, $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{0}$ and so the cross product is not associative. The expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ has no meaning.

2.7.10 Let $\mathbf{u} = [u_1, u_2, u_3]^T$, $\mathbf{v} = [v_1, v_2, v_3]^T$, and $\mathbf{w} = [w_1, w_2, w_3]^T$. Then

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} = \begin{bmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{bmatrix} \\ &= \begin{bmatrix} u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3 \\ u_3v_2w_3 - u_3v_3w_2 - u_1v_1w_2 + u_1v_2w_1 \\ u_1v_3w_1 - u_1v_1w_3 - u_2v_2w_3 + u_2v_3w_2 \end{bmatrix} \end{aligned}$$

and

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = (u_1w_1 + u_2w_2 + u_3w_3)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - (u_1v_1 + u_2v_2 + u_3v_3)\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1w_1v_1 + u_2w_2v_1 + u_3w_3v_1 - u_1v_1w_1 - u_2v_2w_1 - u_3v_3w_1 \\ u_1w_1v_2 + u_2w_2v_2 + u_3w_3v_2 - u_1v_1w_2 - u_2v_2w_2 - u_3v_3w_2 \\ u_1w_1v_3 + u_2w_2v_3 + u_3w_3v_3 - u_1v_1w_3 - u_2v_2w_3 - u_3v_3w_3 \end{bmatrix}.$$

We can see by careful inspection that these are equal.

2.7.11

$$\begin{aligned} &\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \;+\; \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) \;+\; \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) \\ &= ((\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}) \;+\; ((\mathbf{v}\cdot\mathbf{u})\,\mathbf{w} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}) \;+\; ((\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}) \\ &= ((\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}) \;+\; ((\mathbf{v}\cdot\mathbf{u})\,\mathbf{w} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}) \;+\; ((\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}) \\ &= \mathbf{0}. \end{aligned}$$

2.7.12 $\left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} -7 \\ -2 \\ 2 \end{bmatrix}\right) \cdot \begin{bmatrix} -5 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -23 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -6 \\ 3 \end{bmatrix} = 113.$

2.7.13 (a) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = [-2,1,2]^T \cdot [0,0,1]^T = 2$, so the box product is positive. This means that the system of vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ is right-handed.

(b) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = [-2,-1,1]^T \cdot [1,1,2]^T = -1$, so the box product is negative and the system is left-handed.

(c) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = [-1,-1,1]^T \cdot [3,1,4]^T = 0$, so the box product is zero. This means that the volume of the parallelepiped spanned by $\mathbf{u},\mathbf{v},\mathbf{w}$ is empty, i.e., the vectors are coplanar.

(d) We have $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = [0,0,2]^T \cdot [2,0,-1]^T = -2$, so the box product is negative and the system is left-handed.

2.7.14 Yes. It will involve the sum of a product of integers and so it will be an integer.

2.7.15 It means that if you place them so that they all have their tails at the same point, the three will lie in the same plane.

2.7.17 First, we have

$$(\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{z}) \;=\; ((\mathbf{v}\times\mathbf{w})\cdot\mathbf{z})\,\mathbf{w} - ((\mathbf{v}\times\mathbf{w})\cdot\mathbf{w})\,\mathbf{z}.$$

Since $\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{w}$, their dot product $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{w}$ is zero, and therefore

$$(\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{z}) \;=\; ((\mathbf{v}\times\mathbf{w})\cdot\mathbf{z})\,\mathbf{w}.$$

But then

$$(\mathbf{u}\times\mathbf{v})\cdot((\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{z})) \;=\; (\mathbf{u}\times\mathbf{v})\cdot(((\mathbf{v}\times\mathbf{w})\cdot\mathbf{z})\,\mathbf{w}) \;=\; ((\mathbf{v}\times\mathbf{w})\cdot\mathbf{z})\,((\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}).$$

2.7.18 We have

$$\begin{aligned} \|\mathbf{u}\times\mathbf{v}\|^2 &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\sin^2\theta \\ &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\,(1-\cos^2\theta) \\ &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\cos^2\theta \\ &= \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2, \end{aligned}$$

which implies the expression equals 0.

3.1.6 We have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (2-s) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

3.1.15 From trigonometry, we have the following properties of the cosine function:

- $\cos\theta$ is positive for $0 \leq \theta \leq \frac{\pi}{2}$, and negative for $\frac{\pi}{2} \leq \theta \leq \pi$.
- $\cos(\pi - \theta) = -\cos\theta$.

By the method in Example 3.10, we calculated θ such that

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

If $0 \leq \theta \leq \frac{\pi}{2}$, the answer is θ. If $\frac{\pi}{2} \leq \theta \leq \pi$, the answer is $\phi = \pi - \theta$. But in the last case, the dot product is negative and we have

$$\cos\phi = \cos(\pi - \theta) = -\cos\theta = \frac{-\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

So in either case we get the correct answer by taking the absolute value.

3.2.2 We have

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + (1 - r_1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (r_1 + r_2) \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + r_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + r_2 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

3.2.8 The general solution of $ax + by + cz + dw = e$ involves at least three parameters. But the vector equation of a plane only has two parameters, therefore $ax + by + cz + dw = e$ does not describe a plane. (It describes a 3-dimensional so-called *hyperplane* inside $\mathbb{R}^4$).

3.2.13 (a) If the dot product is negative, ϕ will be greater than $\frac{\pi}{2}$, and therefore θ will end up being negative. We can fix this, similarly to Exercise 3.1.15, by taking the absolute value of the dot product, i.e., by solving

$$\cos\phi = \frac{|\mathbf{n} \cdot \mathbf{d}|}{\|\mathbf{n}\| \, \|\mathbf{d}\|} \tag{2.1}$$

(b) From trigonometry, we know that $\sin\theta = \sin(\frac{\pi}{2} - \phi) = \cos\phi$. Together with (2.1), this gives the desired formula.

3.2.15 $\mathbf{x} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

4.1.1 Yes, a 1×1-matrix is both a row vector and a column vector. However, a column vector of dimension 2 or greater can never be equal to a row vector, because one is an $n \times 1$-matrix and the other is a $1 \times n$-matrix.

4.1.2 $x = 2$, $y = -1$, $z = 2$.

4.1.3 2×3, 3×3, 4×2.

4.1.4 7.

4.2.3 The equation simplifies to $X = B + B$, so $X = \begin{bmatrix} 0 & 6 & 0 \\ 2 & -2 & 2 \end{bmatrix}$.

4.2.4 Suppose $A + B = 0$. Then we have $-A = (-A) + 0 = (-A) + (A + B) = ((-A) + A) + B = (A + (-A)) + B = 0 + B = B + 0 = B$. Here, we have used the additive unit law, the assumption $A + B = 0$, associativity, commutativity, the additive inverse law, commutativity, and the additive unit law, in that order.

4.2.5 Suppose $A + B = A$. Then $B = B + 0 = 0 + B = (A + (-A)) + B = ((-A) + A) + B = (-A) + (A + B) = (-A) + A = A + (-A) = 0$. Here, we have used the additive unit law, commutativity, the additive inverse law, commutativity, associativity, the assumption $A + B = A$, commutativity, and the additive inverse law, in that order.

4.3.2 By definition of scalar multiplication, this is equivalent to the matrix equation

$$\begin{bmatrix} x + 0y + 0z + w & 2x + y + 2z + 0w \\ 3x + y - z + 0w & 5x + y + 2z + w \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}.$$

By definition of equality of matrices, this is equivalent to the system of four scalar equations $x + 0y + 0z + w = 1$, $2x + y + 2z + 0w = 2$, $3x + y - z + 0w = 6$, $5x + y + 2z + w = 5$. Solving the system of equations, we find $x = 1$, $y = 2$, $z = -1$, $w = 0$.

4.3.3 $0A = (0 + 0)A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $0 = 0A$.

4.4.1

$$(a) \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad (b) \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad (c) \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad (d) \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (e) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

4.4.3 By columns:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}.$$

This is exactly the same as the "by rows" formula.

4.4.4 $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{bmatrix}$

4.4.5 $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$

4.4.6 (a) $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$.

(b) $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$.

(c) Not possible.

(d) $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$.

(e) Not possible.

(f) Not possible.

4.4.7 (a) $\begin{bmatrix} -3 & -6 \\ -9 & -6 \\ -3 & 3 \end{bmatrix}$.

(b) Not possible.

(c) $\begin{bmatrix} 11 & 2 \\ 13 & 6 \\ -4 & 2 \end{bmatrix}$.

(d) Not possible.

(e) $\begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}$.

(f) Not possible.

(g) Not possible.

(h) $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

4.4.8 $\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -x-z & -w-y \\ 3x+3z & 3w+3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Solution is: $w = -y$, $x = -z$. So the matrices are of the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

4.4.9

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 7 & 2k+2 \\ 15 & 4k+6 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k+3 & 4k+6 \end{bmatrix}.$$

Thus you must have $3k+3=15$ and $2k+2=10$. Solution is: $k=4$.

4.4.10

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} = \begin{bmatrix} 3 & 2k+2 \\ 7 & 4k+6 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k+1 & 4k+2 \end{bmatrix}.$$

However, $7 \neq 3$ and so there is no possible choice of k which will make these matrices commute.

4.4.12 $\begin{bmatrix} -71 & 168 \\ -112 & 265 \end{bmatrix}$.

4.4.13 $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

4.4.14 $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

4.4.15 $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.4.16 $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4.4.18 (a) Not necessarily true.

(b) Not necessarily true.

(c) Not necessarily true.

(d) Necessarily true.

(e) Necessarily true.

(f) Not necessarily true.

(g) Not necessarily true.

4.5.1 (a) Yes. (b) No. (c) Yes.

4.5.2 Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1}.

4.5.3 No. For example, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$. Then $AB = AC$ but $B \neq C$.

4.5.4 $A^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$. D^{-1} does not exist because the reduced echelon form of D is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, which has a row of zeros. E^{-1} does not exist because E is not a square matrix.

4.5.5 $A^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{bmatrix}$. C^{-1} does not exist because the reduced echelon form of C is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$, which has a row of zeros. $D^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{bmatrix}$.

4.5.6 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

4.5.8 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix}$.

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 5 \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3c-2a \\ \frac{1}{3}b - \frac{2}{3}c \\ a-c \end{bmatrix}.$$

4.5.9 Multiply both sides of $AX = B$ on the left by A^{-1}.

4.5.10 Multiply on both sides on the left by A^{-1}. Thus $0 = A^{-1}0 = A^{-1}(AX) = (A^{-1}A)X = IX = X$.

4.5.11 $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$ and $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

4.5.12 It is not possible, because in that case, we would have $A = IA = (AB)A = A(BA) = A0 = 0$, and therefore $AB = 0$, contradicting $AB = I$.

4.5.14 This follows from Problem 4.5.11 by taking $A = B$.

4.5.15 $A^{-1}A = AA^{-1} = I$ and so A is an inverse of A^{-1}. Since $(A^{-1})^{-1}$, by definition, is also an inverse of A^{-1}, we have $(A^{-1})^{-1} = A$ by uniqueness.

4.5.16 (a) B is a right inverse of A, (b) B is a left inverse of A, (c) B is both a right inverse and left inverse of A, (d) B is neither a right inverse nor a left inverse of A.

4.5.17 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$

4.5.18 The system can be written in matrix form as $A\mathbf{v} = \mathbf{b}$, where

$$A = \begin{bmatrix} 8 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 16 \\ -1 & 2 & 8 \end{bmatrix}.$$

Therefore,

$$\mathbf{v} = A^{-1}\mathbf{b} = \begin{bmatrix} -12 \\ 28 \\ 13 \end{bmatrix}.$$

4.5.19 To solve for A, we invert both sides of the equation $(A+B)^{-1} = CB^{-1}$ and use matrix algebra to get $A + B = (CB^{-1})^{-1} = (B^{-1})^{-1}C^{-1} = BC^{-1}$. Therefore, $A = BC^{-1} - B$.

To solve for B, we note that $A = BC^{-1} - B = B(C^{-1} - I)$. Multiplying both sides of the equation on the right by the inverse of $C^{-1} - I$, we get $B = A(C^{-1} - I)^{-1}$.

To solve for C, we take the original equation $(A+B)^{-1} = CB^{-1}$ and right-multiply both sides of the equation B. This yields $C = (A+B)^{-1}B$.

4.5.20 The matrix A is right invertible. Two possible right inverses are

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

The matrices B and C are not right invertible. The matrix D is right invertible with inverse

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Since D is square, its right inverse is actually an inverse, and therefore unique.

4.6.1 (a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, (b) $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, (c) $E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

4.7.1 $X^T Y = \begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}$, $XY^T = 1$.

4.7.2 (a) $\begin{bmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{bmatrix}$.

(c) $\begin{bmatrix} -7 & 1 & 5 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$.

(e) $\begin{bmatrix} 13 & -16 & 1 \\ -16 & 29 & -8 \\ 1 & -8 & 5 \end{bmatrix}$.

(f) $\begin{bmatrix} 5 & 7 & -1 \\ 5 & 15 & 5 \end{bmatrix}$.

(g) Not possible because B is a 2×3-matrix and E is a 2×1-matrix, cannot multiply BE.

4.7.3 A is antisymmetric, B is symmetric, C is neither, and D is both.

4.7.4 We have $A = A^T = -A$. Therefore, each entry a_{ij} of A is equal to its own negation. This implies that $a_{ij} = 0$, and therefore A is the zero matrix.

4.7.5 $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

4.7.6 If A is antisymmetric then $A = -A^T$. It follows that $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

4.7.7 This follows from properties 2 and 3 of Proposition 4.67.

4.7.8 $(A\mathbf{u}) \bullet \mathbf{v} = (A\mathbf{u})^T \mathbf{v} = (\mathbf{u}^T A^T)\mathbf{v} = \mathbf{u}^T (A^T \mathbf{v}) = \mathbf{u} \bullet (A^T \mathbf{v})$.

4.7.9 We need to show that $(A^{-1})^T$ is the inverse of A^T. From properties of the transpose,

$$\begin{aligned} A^T (A^{-1})^T &= (A^{-1}A)^T = I^T = I, \\ (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I. \end{aligned}$$

Hence A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

4.7.10 We have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, and therefore A^{-1} is equal to its own transpose, hence symmetric.

4.9.1 Ciphertext: "`ZRUJPZVAEJTWOXGJZV`".

4.9.2 The decryption matrix is

$$A^{-1} = \begin{bmatrix} 16 & 22 & 5 \\ 22 & 26 & 17 \\ 5 & 17 & 2 \end{bmatrix}.$$

Plaintext: "Spies are at the gate".

4.9.3 The first two plaintext blocks are $(8,5),(12,12)$ and the first two ciphertext blocks are $(20,7),(22,24)$. Eve solves the equation

$$A^{-1}\begin{bmatrix} 20 & 22 \\ 7 & 24 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 5 & 12 \end{bmatrix}$$

to find the secret decryption matrix

$$A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

The plaintext is "Hello, password is kiwifruit".

4.9.4 Ciphertext: "`VXUAJYY,AFJE`".

4.9.5 The inverse of the diffusion matrix is

$$A^{-1} = \begin{bmatrix} 11 & 2 & 21 \\ 21 & 11 & 2 \\ 2 & 21 & 11 \end{bmatrix}.$$

The plaintext is: "Eat more fruit".

5.1.1 (a) $\mathbf{x}$ is not in the span. (b) $\mathbf{y} = 7\mathbf{u}_1 - 5\mathbf{u}_2$. (c) $\mathbf{z} = -3\mathbf{u}_1 + 4\mathbf{u}_2$.

5.1.2 It is the plane $2x + 3y - z = 0$.

5.1.3 It is the hyperplane given by the equation $x - 2y - 3z + 2w = 0$.

5.1.5 $0\mathbf{u}_1 + \ldots + 0\mathbf{u}_k = \mathbf{0}$.

5.1.6 (a) $\mathbf{x} = 3\mathbf{u}_2$. (b) $\mathbf{y}$ is not in the span. (c) $\mathbf{z} = 2\mathbf{u}_1 + 1\mathbf{u}_2$.

5.2.1 $\mathbf{u}_2$ and $\mathbf{u}_4$ are redundant. We have $\mathbf{u}_2 = 2\mathbf{u}_1$ and $\mathbf{u}_4 = 3\mathbf{u}_3 - 2\mathbf{u}_1$.

5.2.3 The vectors are linearly dependent. We have $\mathbf{u}_2 + 3\mathbf{u}_3 + 2\mathbf{u}_4 = \mathbf{0}$.

5.2.5 The vectors are linearly dependent. We have $\mathbf{w} = 3\mathbf{u} - 2\mathbf{v}$.

5.2.6 $\begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & 3 & 3 & -3 \\ 3 & 5 & 8 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 4 & 5 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Linearly independent subset: $\{\mathbf{u}_1, \mathbf{u}_2\}$.

5.2.10

$$\begin{bmatrix} 1 & 2 & 2 & 5 & 12 \\ 1 & 2 & 7 & 7 & 17 \\ -2 & -4 & -4 & -10 & -24 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 2 & 5 & 12 \\ 0 & 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Linearly independent subset: $\{\mathbf{u}_1, \mathbf{u}_3\}$. Since the rank is 2, this is the smallest possible.

5.2.12 We write the vectors as the columns of a matrix and reduce to reduced echelon form over $\mathbb{Z}_3$:

$$\begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 \end{bmatrix} \quad \simeq \ldots \simeq \quad \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\mathbf{u}_3$ and $\mathbf{u}_4$ are redundant. We have $\mathbf{u}_3 = 2\mathbf{u}_1 + 1\mathbf{u}_2$ and $\mathbf{u}_4 = 2\mathbf{u}_1 + 2\mathbf{u}_2$.

5.2.13 From $a(\mathbf{u}+\mathbf{v}) + b(2\mathbf{u}+\mathbf{w}) + c(\mathbf{w}-2\mathbf{v}) = \mathbf{0}$ we get $(a+2b)\mathbf{u} + (a-2c)\mathbf{v} + (b+c)\mathbf{w} = \mathbf{0}$. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, this last system has only the trivial solution, so $a+2b = 0$, $a-2c = 0$, and $b+c = 0$. However, these three equations have non-trivial solutions, for example $(a,b,c) = (2,-1,1)$. So the vectors $\mathbf{u}+\mathbf{v}$, $2\mathbf{u}+\mathbf{w}$, and $\mathbf{w}-2\mathbf{v}$ are linearly dependent.

5.2.14 From $a(\mathbf{u}+\mathbf{v}) + b(\mathbf{u}+\mathbf{w}) + c(\mathbf{w}+\mathbf{v}) = \mathbf{0}$ we get $(a+b)\mathbf{u} + (a+c)\mathbf{v} + (b+c)\mathbf{w} = \mathbf{0}$. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, this last system has only the trivial solution, so $a+b = 0$, $a+c = 0$, and $b+c = 0$. Solving, we find the unique solution $(a,b,c) = (0,0,0)$. So the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{w}$, and $\mathbf{w}+\mathbf{v}$ are linearly independent.

5.2.15 Assume $a_1\mathbf{z}_1 + \ldots + a_k\mathbf{z}_k = \mathbf{0}$. Multiplying both sides of the equation by A, we get

$$a_1 A\mathbf{z}_1 + \ldots + a_k A\mathbf{z}_k = a_1\mathbf{w}_1 + \ldots + a_k\mathbf{w}_k = \mathbf{0}.$$

Since the $\mathbf{w}_i$ are linearly independent, it follows that each $a_i = 0$. Therefore the $\mathbf{z}_i$ are linearly independent as well.

5.3.1 (a) No. We have $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in V_1$ but $10 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin V_1$.

(b) This is not a subspace. The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in V_2. However, $(-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not.

(c) This is a subspace. It contains the zero vector and is closed with respect to vector addition and scalar multiplication.

(d) This is not a subspace. The vector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is in V_4. However $(-1)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is not.

(e) This is a subspace. It contains the zero vector and is closed with respect to vector addition and scalar multiplication.

5.3.2 Yes, this is a subspace because it contains the zero vector and is closed with respect to vector addition and scalar multiplication. For example, if $\mathbf{u},\mathbf{v} \in M$, then $\mathbf{w} \bullet \mathbf{u} = 0$ and $\mathbf{w} \bullet \mathbf{v} = 0$, therefore $\mathbf{w} \bullet (\mathbf{u}+\mathbf{v}) = 0$, therefore $\mathbf{u}+\mathbf{v} \in M$.

5.3.3 Yes, this is a subspace.

5.3.4 (a) V_1 is a subspace.

(b) V_2 is not a subspace: not closed under addition. For example,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in V_2, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in V_2, \quad \text{but} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin V_2.$$

(c) V_3 is a subspace.

(d) V_4 is not a subspace. For example, it does not contain the zero vector.

5.3.5 Because $\mathbf{0} \in V$ and $\mathbf{0} \in W$, we have $\mathbf{0} \in V \cap W$. To show that $V \cap W$ is closed under addition, assume $\mathbf{u},\mathbf{v} \in V \cap W$. Then $\mathbf{u},\mathbf{v} \in V$, and since V is a subspace we have $\mathbf{u}+\mathbf{v} \in V$. Similarly $\mathbf{u},\mathbf{v} \in W$, and since W is a subspace we have $\mathbf{u}+\mathbf{v} \in W$. It follows that $\mathbf{u}+\mathbf{v}$ is in both V and W, and therefore in $V \cap W$. To show that $V \cap W$ is closed under scalar multiplication, assume $\mathbf{u} \in V \cap W$ and $k \in \mathbb{R}$. Then $\mathbf{u} \in V$, and therefore $k\mathbf{u} \in V$. Similarly $k\mathbf{u} \in W$, and therefore also $k\mathbf{u} \in V \cap W$.

5.4.11 (a) Basis $\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$, dimension 2.

(b) Basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$, dimension 1.

(c) Basis $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$, dimension 2.

5.4.14 (a) No. For example, the vectors $\{\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}\}$ are linearly dependent.

(b) No. For example, the vectors $\{\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}\}$ are linearly dependent.

(c) Yes, by Proposition 5.48.

(d) Correct, by Proposition 5.48.

(e) Yes, by Proposition 5.49.

(f) No, in fact no such set is a basis of $\mathbb{R}^5$, since every basis of $\mathbb{R}^5$ consists of 5 vectors by Theorem 5.38.

(g) No, as noted in the previous answer.

(h) No, no basis of $\mathbb{R}^5$ can have 6 elements.

(i) Yes, because a linearly independent set of vectors is a basis of the subspace it spans.

5.4.15 No. As a 5-dimensional space, $\mathbb{R}^5$ is spanned by 5 vectors. By the Exchange Lemma, any linearly independent set can have size at most 5.

5.5.3 Rank 2, nullity 1, basis of column space $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$, basis of row space $\{[1,2,0],[0,0,1]\}$, basis of null space $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

5.5.4 Clearly $\mathbf{0} \in \text{null}(A)$, since $A\mathbf{0} = \mathbf{0}$. If $\mathbf{x}, \mathbf{y} \in \text{null}(A)$, then $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, and therefore $\mathbf{x}+\mathbf{y} \in \text{null}(A)$. Similarly, if $\mathbf{x} \in \text{null}(A)$ and k is a scalar, then $A(k\mathbf{x}) = k(A\mathbf{x}) = k\mathbf{0} = \mathbf{0}$, so $k\mathbf{x} \in \text{null}(A)$. So $\text{null}(A)$ contains $\mathbf{0}$, and is closed under addition and scalar multiplication. It is therefore a subspace of $\mathbb{R}^n$.

5.5.5 Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Since $\text{col}(A)$ is the span of the columns of A, we have $\mathbf{v} \in \text{col}(A)$ if and only if there exists scalars $u_1, \ldots, u_n$ such that

$$\mathbf{v} = u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \ldots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

But this equation is equivalent to

$$\mathbf{v} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\mathbf{u}.$$

Therefore, $\mathbf{v} \in \text{col}(A)$ if and only if $\mathbf{v}$ is of the form $A\mathbf{u}$, for some $\mathbf{u} \in \mathbb{R}^n$. In other words, $\text{col}(A) = \{A\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n\}$.

5.5.6 The row space of A is the same as the column space of A^T (except that it uses row vectors instead of column vectors). Therefore, $\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A^T)) = \text{rank}(A^T)$.

5.5.7 From the theory of elementary matrices, we know that B can be written as a product of elementary matrices $B = E_1 \cdots E_k$. It follows that $BA = E_1 \cdots E_k A$. Since each elementary matrix corresponds to an elementary row operation, it follows that BA and A are row equivalent. Therefore, BA and A have the same row space. It follows that $\text{rank}(BA) = \dim(\text{row}(BA)) = \dim(\text{row}(A)) = \text{rank}(A)$. This proves the first claim. To show the claim about AC, first note that by the above argument, $\text{rank}(C^T A^T) = \text{rank}(A^T)$, because C^T is invertible. Then $\text{rank}(AC) = \text{rank}(A)$ follows by taking the transpose.

5.5.8 Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis of $\text{col}(B) \cap \text{null}(A)$. Then for each i, we have $\mathbf{w}_i \in \text{col}(B)$, and therefore by Exercise 5.5.5, we can find $\mathbf{z}_i$ such that $\mathbf{w}_i = B\mathbf{z}_i$. Also let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a basis for $\text{null}(B)$. Now assume $\mathbf{x} \in \text{null}(AB)$. Then $AB\mathbf{x} = \mathbf{0}$, and therefore $B\mathbf{x} \in \text{null}(A) \cap \text{col}(B)$. We therefore have

$$B\mathbf{x} = c_1\mathbf{w}_1 + \ldots + c_k\mathbf{w}_k = B(c_1\mathbf{z}_1 + \ldots + c_k\mathbf{z}_k).$$

This implies

$$\mathbf{x} - (c_1\mathbf{z}_1 + \ldots + c_k\mathbf{z}_k) \in \text{null}(B)$$

and so it is of the form

$$\mathbf{x} - (c_1\mathbf{z}_1 + \ldots + c_k\mathbf{z}_k) = d_1\mathbf{u}_1 + \ldots + d_r\mathbf{u}_r.$$

It follows that

$$\mathbf{x} \in \text{span}\{\mathbf{z}_1, \ldots, \mathbf{z}_k, \mathbf{u}_1, \ldots, \mathbf{u}_r\}.$$

Since we have shown that every element of $\text{null}(AB)$ is in the span of these $k + r$ vectors, it follows that

$$\begin{aligned} \dim(\text{null}(AB)) &\leq k + r \\ &= \dim(\text{col}(B) \cap \text{null}(A)) + \dim(\text{null}(B)) \\ &\leq \dim(\text{null}(A)) + \dim(\text{null}(B)), \end{aligned}$$

and therefore $\text{nullity}(AB) \leq \text{nullity}(A) + \text{nullity}(B)$.

6.1.1 T_1 and T_3 are linear, and T_2 is not. The transformation T_3 is called the **zero transformation**.

6.1.3 We have $T(a\mathbf{v} + b\mathbf{w}) = A(a\mathbf{v} + b\mathbf{w}) = a(A\mathbf{v}) + b(A\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$ by properties of matrix multiplication. Therefore, T is linear by Proposition 6.3.

6.1.4 We have

$$\begin{aligned} T(a\mathbf{v} + b\mathbf{w}) &= a\mathbf{v} + b\mathbf{w} - \frac{\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w})}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= a\mathbf{v} - a\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} + b\mathbf{w} - b\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= aT(\mathbf{v}) + bT(\mathbf{w}). \end{aligned}$$

Therefore, T is linear.

6.1.5 If T were a linear transformation, it should satisfy $T(\mathbf{0}) = \mathbf{0}$, but it does not. Also $T(\mathbf{v}+\mathbf{w}) \neq T\mathbf{v}+T\mathbf{w}$.

6.2.1 (a) The matrix of T is the elementary matrix that is like the identity matrix, except that the (j,j)-entry is b.

(b) The matrix of T is the elementary matrix that is like the identity matrix, except that the (i,j)-entry is b.

(c) The matrix of T is the elementary matrix that switches the i^{th} and the j^{th} rows.

6.2.2 Since $\mathbf{u}_1,\ldots,\mathbf{u}_n$ is a basis of $\mathbb{R}^n$, we know that A is invertible by Proposition 5.31. For each $i = 1,\ldots,n$, since $A\mathbf{e}_i = \mathbf{u}_i$, we have $A^{-1}\mathbf{u}_i = \mathbf{e}_i$, and therefore

$$BA^{-1}\mathbf{u}_i = B\mathbf{e}_i = \mathbf{v}_i = T(\mathbf{u}_i).$$

Now let $\mathbf{w}$ be some arbitrary vector in $\mathbb{R}^n$. Since $\mathbf{u}_1,\ldots,\mathbf{u}_n$ is a basis, there exists $a_1,\ldots,a_n$ such that $\mathbf{w} = a_1\mathbf{u}_1+\ldots+a_n\mathbf{u}_n$. Then

$$\begin{aligned} BA^{-1}\mathbf{w} &= BA^{-1}(a_1\mathbf{u}_1+\ldots+a_n\mathbf{u}_n) \\ &= a_1(BA^{-1}\mathbf{u}_1)+\ldots+a_n(BA^{-1}\mathbf{u}_n) \\ &= a_1T(\mathbf{u}_1)+\ldots+a_nT(\mathbf{u}_n) \\ &= T(a_1\mathbf{u}_1+\ldots+a_n\mathbf{u}_n) \;=\; T(\mathbf{w}). \end{aligned}$$

Therefore, BA^{-1} is the matrix of T.

6.2.3

$$\begin{bmatrix} 5 & 1 & 5 \\ 1 & 1 & 3 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -1 \\ -6 & 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 1 & 5 \\ 1 & 1 & 3 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 17 & 11 \\ 17 & 7 & 5 \\ 11 & 14 & 6 \end{bmatrix}.$$

6.2.4

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -8 & 6 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \\ 5 & 3 & 1 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 52 & 21 & 9 \\ 44 & 23 & 8 \\ 5 & 4 & 1 \end{bmatrix}.$$

6.2.6 Recall that the desired matrix has i^{th} column equal to $\operatorname{proj}_{\mathbf{u}}(\mathbf{e}_i) = \dfrac{\mathbf{u}\bullet\mathbf{e}_i}{\|\mathbf{u}\|^2}\mathbf{u}$. Therefore, the matrix is

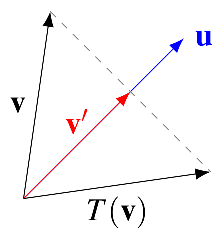
$$\frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}.$$

6.2.7

$$\frac{1}{35}\begin{bmatrix} 1 & 5 & 3 \\ 5 & 25 & 15 \\ 3 & 15 & 9 \end{bmatrix}.$$

6.3.1 $\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$

6.3.8 First, we compute $\mathbf{v}'$, the projection of $\mathbf{v}$ onto $\mathbf{u}$.



We have

$$\mathbf{v}' = \text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|^2}\,\mathbf{u}.$$

But since $\mathbf{u}$ is a unit vector, this simplifies to $\mathbf{v}' = (\mathbf{u} \bullet \mathbf{v})\mathbf{u}$. From the above picture, we see that $T(\mathbf{v}) = \mathbf{v} + 2(\mathbf{v}' - \mathbf{v}) = 2\mathbf{v}' - \mathbf{v} = 2(\mathbf{u} \bullet \mathbf{v})\mathbf{u} - \mathbf{v}$. To get the matrix of this linear transformation, we must compute the image of the standard basis vectors:

$$T(\mathbf{e}_1) = 2(\mathbf{u} \bullet \mathbf{e}_1)\mathbf{u} - \mathbf{e}_1 = 2a \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a^2 - 1 \\ 2ab \end{bmatrix}$$

$$T(\mathbf{e}_2) = 2(\mathbf{u} \bullet \mathbf{e}_2)\mathbf{u} - \mathbf{e}_2 = 2b \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2ab \\ 2b^2 - 1 \end{bmatrix}.$$

Therefore, the matrix is

$$A = \begin{bmatrix} 2a^2 - 1 & 2ab \\ 2ab & 2b^2 - 1 \end{bmatrix}.$$

6.4.2

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

6.4.3

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

6.4.4

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

6.4.5

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6.4.6 (a) $T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}$. (b) $T(-\mathbf{v}) = T((-1)\mathbf{v}) = (-1)T(\mathbf{v}) = -T(\mathbf{v})$. (c) $T(a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k) = T(a_1\mathbf{v}_1) + \ldots + T(a_k\mathbf{v}_k) = a_1T(\mathbf{v}_1) + \ldots + a_kT(\mathbf{v}_k)$.

6.4.7 The matrix of $S \circ T$ is given by

$$BA = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix}.$$

Now,

$$(S \circ T)(\mathbf{v}) = BA\mathbf{v} = \begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

6.4.8 We have

$$(S \circ T)(\mathbf{v}) = S(T(\mathbf{v})) = B(T(\mathbf{v})) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}.$$

6.4.9 The inverse of a reflection is a reflection, namely, itself. (For example, reflecting twice about the x-axis returns each vector to its original position). The inverse of a rotation is a rotation by the same angle in the opposite direction. The inverse of a shearing is a shearing in the opposite direction. The inverse of a scaling by factor a is a scaling by factor $1/a$.

6.4.10 The matrix of T^{-1} is A^{-1}.

$$\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

6.4.11 The matrix of T is $A = \begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$. The matrix of T^{-1} is $A^{-1} = \begin{bmatrix} 1 & -3/2 \\ 1 & -2 \end{bmatrix}$.

7.2.3 (a) -14. (b) 24. (c) 0.

7.2.4

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6.$$

7.2.5

$$\begin{vmatrix} 2 & 3 & 1 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{vmatrix} = 2.$$

7.2.6

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 4 & 0 & 2 \end{vmatrix} = 6.$$

7.2.7

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -4.$$

7.3.1 (a) $\det(A) = 2$. (b) $\det(B) = -40$. (c) $\det(C) = -24$.

7.4.1 (a)

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix} = -6.$$

(b)

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix} = -32.$$

(c) One can row reduce this, using only row operations of the third kind, to

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 0 & 2 & \frac{9}{5} \\ 0 & 0 & 0 & -\frac{63}{10} \end{bmatrix}.$$

Therefore, the determinant is 63.

(d) One can row reduce this, using only row operations of the third kind, to

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & -10 & -5 & -3 \\ 0 & 0 & 2 & \frac{19}{5} \\ 0 & 0 & 0 & -\frac{211}{20} \end{bmatrix}.$$

Thus the determinant is 211.

7.5.1 (a) The transpose was taken and $\det(B) = \det(A)$.

(b) Two rows were switched and $\det(B) = -\det(A)$.

(c) The first row was added to the second row and $\det(B) = \det(A)$.

(d) The second row was multiplied by 2 and $\det(B) = 2\det(A)$.

(e) Two columns were switched and $\det(B) = -\det(A)$.

7.5.2 By assumption, we can obtain a row of zeros by doing row operations. Row operations do not change whether the determinant is zero, so the determinant must have been zero all along.

7.5.3 If $\det(A) \neq 0$, then A^{-1} exists. Therefore $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

7.5.4 The matrix kI has k down the main diagonal and has determinant equal to k^n by Theorem 7.14. Using Theorem 7.23, it follows that $\det(kA) = \det(kIA) = \det(kI)\det(A) = k^n \det(A)$.

7.5.5

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix}\right) = -8,$$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)\det\left(\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix}\right) = -2 \times 4 = -8.$$

7.5.6 This is not true at all. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\det(A) = 1$, $\det(B) = 1$, and $\det(A+B) = 0$.

7.5.7 Since $A^k = 0$, we have $\det(A)^k = \det(A^k) = \det(0) = 0$. Therefore, it must be the case that $\det(A) = 0$.

7.5.8 If A is orthogonal, we have $\det(A)^2 = \det(A^T)\det(A) = \det(A^TA) = \det(I) = 1$. Therefore the only possible values for $\det(A)$ are ± 1.

7.5.9 $\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P) = \frac{1}{\det(P)}\det(B)\det(P) = \det(B)$.

7.5.10 The determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ b & b^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ a & a^2 \end{vmatrix} = ab^2 - a^2b - b^2 + b + a^2 - a = (a-1)(b-1)(b-a).$$

Therefore the determinant is 0 if $a = 1$, $b = 1$, or $a = b$. In all other cases, the determinant is non-zero and the matrix is invertible.

7.5.11 This follows because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is non-zero, then each determinant in the product is non-zero. Therefore, each of these matrices is invertible.

7.5.12 The given condition is what it takes for the determinant to be non-zero. Recall that the determinant of an upper triangular matrix is just the product of the entries on the main diagonal. The inverse will also be upper triangular; this can be seen by noting that every invertible upper triangular can be written as a product of upper triangular elementary matrices; the inverse of each such elementary matrix is upper triangular, and therefore so is their product. The analogous statement about lower triangular matrices is also true.

7.5.13 (a) False. Consider $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(b) True.

(c) False.

(d) False.

(e) True.

(f) True.

(g) True.

(h) True.

(i) True.

(j) True.

7.6.2 (a) $\det(A) = -13$ and so A is invertible. This inverse is

$$\frac{1}{-13}\begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T = \frac{1}{-13}\begin{bmatrix} -1 & 3 & -6 \\ 3 & -9 & 5 \\ -4 & -1 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{9}{13} & \frac{1}{13} \\ \frac{6}{13} & -\frac{5}{13} & -\frac{2}{13} \end{bmatrix}.$$

(b) $\det(B) = 7$, so B is invertible. The inverse is $\frac{1}{7}\begin{bmatrix} 1 & 3 & -6 \\ -2 & 1 & 5 \\ 2 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{6}{7} & \frac{5}{7} & \frac{2}{7} \end{bmatrix}$.

(c) $\det(C) = 3$, so C is invertible. The inverse is $\begin{bmatrix} 1 & 0 & -3 \\ -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$.

(d) $\det(D) = 0$, so D is not invertible.

(e) $\det(E) = 2$, and so E is invertible. The inverse is $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{9}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$.

7.6.3 (a) $\det(A) = 1$, so A is invertible. The inverse is $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(b) $\det(B) = -15$, so B is invertible. The inverse is $B^{-1} = \frac{1}{15}\begin{bmatrix} -1 & -1 & 4 \\ -4 & 11 & 1 \\ 8 & -7 & -2 \end{bmatrix}$.

(c) $\det(C) = 0$, so C is not invertible.

7.6.4 We have

$$\det(A) = 36, \quad \operatorname{cof}(A) = \begin{bmatrix} 6 & 12 & 6 \\ 12 & 6 & -12 \\ -6 & 6 & 6 \end{bmatrix}, \quad \operatorname{adj}(A) = \begin{bmatrix} 6 & 12 & -6 \\ 12 & 6 & 6 \\ 6 & -12 & 6 \end{bmatrix},$$

and therefore

$$A^{-1} = \frac{1}{36}\begin{bmatrix} 6 & 12 & -6 \\ 12 & 6 & 6 \\ 6 & -12 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

7.6.5 We have

$$\det(A) = 2, \quad \operatorname{cof}(A) = \begin{bmatrix} -9 & 11 & 8 \\ -2 & 2 & 2 \\ 5 & -5 & -4 \end{bmatrix}, \quad \operatorname{adj}(A) = \begin{bmatrix} -9 & -2 & 5 \\ 11 & 2 & -5 \\ 8 & 2 & -4 \end{bmatrix},$$

and therefore

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -9 & -2 & 5 \\ 11 & 2 & -5 \\ 8 & 2 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & -1 & \frac{5}{2} \\ \frac{11}{2} & 1 & -\frac{5}{2} \\ 4 & 1 & -2 \end{bmatrix}.$$

7.6.6 No. It has non-zero determinant $\det(A) = \cos^2 t + \sin^2 t = 1$ for all t, so it is invertible for all t.

7.6.7 $\det(A) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{vmatrix} = t^3 + 2$, and so A has no inverse when $t = -\sqrt[3]{2}$.

7.6.8 Since the matrix A has two identical rows, we have $\det(A) = 0$ for all t. So this matrix is non-invertible for all t.

7.6.9

$$\det \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix} = 5e^{-t} \neq 0$$

and so this matrix is always invertible.

7.6.10

$$\det\begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & \cos t - \sin t & \cos t + \sin t \end{bmatrix} = e^t.$$

Hence the inverse is

$$e^{-t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t\cos t + e^t\sin t & -e^t(\cos t - \sin t) \\ 0 & -e^t\sin t & e^t\cos t \end{bmatrix}^T = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t + \sin t & -\sin t \\ 0 & \sin t - \cos t & \cos t \end{bmatrix}.$$

7.6.11

$$\begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{-t} \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t & -\sin t & \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t & \cos t & -\frac{1}{2}\cos t - \frac{1}{2}\sin t \end{bmatrix}.$$

7.7.1 False. Cramer's rule only works when the coefficient matrix is invertible. In these cases, the solution is always unique.

7.7.2 Solution is: $(x,y) = (1,0)$.

7.7.3 Solution is: $(x,y,z) = (1,1,0)$. For example,

$$y = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix}} = 1$$

7.7.4 The solution is $(x,y,z) = (1,-1,2)$.

7.7.5 By Cramer's rule, we have

$$y = \frac{\begin{vmatrix} 1 & t & 1 \\ 1 & s & t^2 \\ 1 & 1 & s^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & s & s^2 \end{vmatrix}} = \frac{s^3+t^3+1-2-ts^2-t^2}{ts^2+t^2+s-t-st^2-s^2}.$$

8.1.7 If $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ, then $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}$. Therefore, $\mathbf{v}$ is an eigenvector of A^2 with eigenvalue λ^2.

8.1.8 We have $\lambda A^{-1}\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}A\mathbf{v} = \mathbf{v}$. Since $\mathbf{v} \neq 0$, this implies $\lambda \neq 0$. Moreover, it implies $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Thus, λ^{-1} is an eigenvalue of A^{-1}.

8.1.9 Say $A\mathbf{v} = \lambda\mathbf{v}$. Then $cA\mathbf{v} = c\lambda\mathbf{v}$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A.

8.1.10 Suppose $\mathbf{v}$ is an eigenvector of B, i.e., $B\mathbf{v} = \lambda\mathbf{v}$. Then $BA\mathbf{v} = AB\mathbf{v} = A\lambda\mathbf{v} = \lambda A\mathbf{v}$, and therefore $A\mathbf{v}$ is an eigenvector of B.

8.1.11 Let $\mathbf{v}$ be the eigenvector. Then $A^m\mathbf{v} = \lambda^m\mathbf{v}$ and $A^m\mathbf{v} = A\mathbf{v} = \lambda\mathbf{v}$. Therefore $\lambda^m = \lambda$. Hence if $\lambda \neq 0$, we must have $\lambda^{m-1} = 1$, which implies that $\lambda = \pm 1$.

8.1.12 The formula follows from properties of matrix multiplication. However, this vector might not be an eigenvector because it might equal 0 and eigenvectors cannot equal 0.

8.2.8 Yes. $\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$ works.

8.3.2 A rotation by 60° cannot map any non-zero vector to a scalar multiple of itself. The only rotations of $\mathbb{R}^2$ that have real eigenvalues are rotations by 180° and 0°. The former has eigenvalue -1, and the latter has eigenvalue 1 (with all vectors being eigenvectors).

8.3.3 The matrix of T is $\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]$. The eigenvectors and eigenvalues are:

$$\left\{\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right\} \text{ for eigenvalue } -1, \quad \left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right\} \text{ for eigenvalue } 1.$$

8.3.4 The matrix of T is $\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$. The eigenvectors and eigenvalues are:

$$\left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right\} \text{ for eigenvalue } -1, \quad \left\{\left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right\} \text{ for eigenvalue } 1.$$

8.3.5 The matrix of T is $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$ The eigenvectors and eigenvalues are:

$$\left\{\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right\} \text{ for eigenvalue } -1, \quad \left\{\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]\right\} \text{ for eigenvalue } 1.$$

8.4.2 The eigenvectors and eigenvalues are:

$$\left\{\left[\begin{array}{c} 2 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} -2 \\ 1 \\ 0 \end{array}\right]\right\} \text{ for eigenvalue } 1, \quad \left\{\left[\begin{array}{c} 7 \\ -2 \\ 2 \end{array}\right]\right\} \text{ for eigenvalue } 3.$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

8.4.3 The eigenvalues are -1 and 1. The eigenvectors corresponding to the eigenvalues are:

$$\left\{ \begin{bmatrix} 10 \\ -2 \\ 3 \end{bmatrix} \right\} \text{ for eigenvalue } -1, \quad \left\{ \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} \right\} \text{ for eigenvalue } 1.$$

Since there are only 2 linearly independent eigenvectors, this matrix is not diagonalizable.

8.4.4 The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ for eigenvalue } -3, \quad \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ for eigenvalue } 3, \quad \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for eigenvalue } -2.$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 0 & -2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

8.5.1 First we write $A = PDP^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore $A^{10} = PD^{10}P^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^{10}+1 & 3^{10}-1 \\ 3^{10}-1 & 3^{10}+1 \end{bmatrix}.$$

8.5.4 $B = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}.$

8.5.5 $A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$.

8.8.3 The solution is

$$e^{At}C = \begin{bmatrix} 8e^{2t} - 6e^{3t} \\ 18e^{3t} - 16e^{2t} \end{bmatrix}.$$

8.9.1 (a) $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1. Since the sum of the geometric multiplicities of all eigenvalues is 1, the matrix is not diagonalizable.

(b) $\lambda = 1$ has algebraic and geometric multiplicity 1; $\lambda = -3$ has algebraic and geometric multiplicity 1. Since the sum of the geometric multiplicities is 2, the matrix is diagonalizable.

(c) $\lambda = 2$ has algebraic and geometric multiplicity 2, $\lambda = 3$ has algebraic and geometric multiplicity 1. Since the sum of the geometric multiplicities is 3, the matrix is diagonalizable.

(d) $\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2. Since the sum of the geometric multiplicities of all eigenvalues is 2, the matrix is not diagonalizable.

(e) $\lambda = -1$ has algebraic multiplicity 3 and geometric multiplicity 1. Since the sum of the geometric multiplicities of all eigenvalues is 1, the matrix is not diagonalizable.

8.9.2 (a) Not diagonalizable.

(b) Diagonalizable.

(c) Not diagonalizable.

8.10.3 (a) The characteristic polynomial of A is a quadratic polynomial, and therefore it is of the form $p(\lambda) = \lambda^2 + b\lambda + c$, for some $r, s \in \mathbb{R}$. By the Cayley-Hamilton theorem, $p(A) = 0$, therefore $A^2 = -bA - cI$. This proves that A^2 is a linear combination of A and I.

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

8.11.1 (a) The characteristic polynomial is $\lambda^2 - 4\lambda + 5$, the eigenvalues are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$, and the corresponding basic eigenvectors are $\mathbf{v}_1 = [1, i]^T$ and $\mathbf{v}_2 = [1, -i]^T$, respectively. The matrix A is diagonalizable as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

(b) The characteristic polynomial is $\lambda^2 + 1$, the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, and the corresponding basic eigenvectors are $\mathbf{v}_1 = [1 - i, -1]^T$ and $\mathbf{v}_2 = [1 + i, -1]^T$, respectively. Therefore $B = PDP^{-1}$, where

$$P = \begin{bmatrix} 1-i & 1+i \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

(c) The characteristic polynomial is $x^2 - 2x + 2 - (x^3 - 2x^2 + 2x) - \lambda^3 - \lambda^2 + 2 = (1 - \lambda)(\lambda^2 - 2\lambda + 2)$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1 + i$, and $\lambda_3 = 1 - i$. The corresponding basic eigenvectors are $\mathbf{v}_1 = [1, 1, 0]^T$, $\mathbf{v}_2 = [-1 - 2i, -i, 1]^T$, and $\mathbf{v}_2 = [1 + 2i, i, 1]^T$, respectively. Therefore $C = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & -1-2i & 1+2i \\ 1 & -i & i \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}.$$

(d) The characteristic polynomial is $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)(\lambda - 1)(\lambda - 2)$. Therefore the eigenvalues are $\lambda = 1$ with algebraic multiplicity 2 and $\lambda = 2$ with algebraic multiplicity 1. The eigenspace for $\lambda = 1$ is 1-dimensional; it is spanned by $[1, 1, -1]^T$. Since the geometric multiplicity of the eigenvalue 1 is less than its algebraic multiplicity, the matrix is not diagonalizable, not even over the complex numbers.

8.11.2 (a) By Proposition 8.51, the other eigenvalue is $1 - 2i$ with corresponding eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(b) From the eigenvalues and eigenvectors, we know that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}.$$

(c) $A = PDP^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

9.1.13 Let $f(i)$ be the i^{th} component of a vector $\mathbf{x} \in \mathbb{R}^n$. Thus a typical element in $\mathbb{R}^n$ is $(f(1), \ldots, f(n))$.

9.2.1 $\mathbf{0} = 0\mathbf{u}_1 + \ldots + 0\mathbf{u}_k$.

9.2.5 Let $S = \{e^0, e^1, e^2, \ldots\}$. To prove the left-to-right implication, assume $a \in \operatorname{span} S$. Then there exists some finite subset $\{e^{k_1}, e^{k_2}, \ldots, e^{k_n}\}$ of S and scalars $b_1, \ldots, b_n$ such that $a = b_1 e^{k_1} + \ldots + b_n e^{k_n}$. Let $N \in \mathbb{N}$ be a number that is greater than $k_1, \ldots, k_n$. Since each of the sequences $e^{k_1}, \ldots, e^{k_n}$ is zero after the first N elements, the same is true for their linear combination a. Thus, a is finitely supported.

To prove the right-to-left direction, assume a is finitely supported. Let $N \in \mathbb{N}$ such that $a_k = 0$ for all $k \geq N$. Then $a = (a_0, a_1, \ldots, a_{N-1}, 0, 0, 0, \ldots)$, with infinitely many zeros following. Therefore, $a = a_0 e^0 + a_1 e^1 + \ldots + a_{N-1} e^{N-1}$, and it follows that $a \in \operatorname{span} S$.

9.2.8 Let $p_i(x)$ denote the i^{th} of these polynomials. Suppose $C_1 p_1(x) + \ldots + C_4 p_4(x) = 0$. Then collecting terms according to the exponent of x, we have

$$\begin{aligned} C_1 a_1 + C_2 a_2 + C_3 a_3 + C_4 a_4 &= 0, \\ C_1 b_1 + C_2 b_2 + C_3 b_3 + C_4 b_4 &= 0, \\ C_1 c_1 + C_2 c_2 + C_3 c_3 + C_4 c_4 &= 0, \\ C_1 d_1 + C_2 d_2 + C_3 d_3 + C_4 d_4 &= 0. \end{aligned}$$

The matrix of coefficients is just the transpose of the above matrix. There exists a non-trivial solution if and only if the determinant of this matrix equals 0.

9.2.9 To determine whether R is linearly independent, we must solve the equation

$$a(2\mathbf{u}-\mathbf{w})+b(\mathbf{w}+\mathbf{v})+c(3\mathbf{v}+\tfrac{1}{2}\mathbf{u}) \;=\; \mathbf{0}.$$

If the only solution is the trivial solution, the set is linearly independent. We rewrite the equation as follows.

$$(2a+\tfrac{1}{2}c)\mathbf{u}+(b+3c)\mathbf{v}+(-a+b)\mathbf{w} \;=\; \mathbf{0}.$$

Since $\mathbf{u},\mathbf{v},\mathbf{w}$ are linearly independent, the coefficients in the last equation must all equal 0. In other words:

$$\begin{aligned} 2a+\tfrac{1}{2}c &= 0, \\ b+3c &= 0, \\ -a+b &= 0. \end{aligned}$$

We solve and find that the unique solution is $a=b=c=0$. Therefore, the set R is linearly independent.

9.3.2 No. It is not closed under scalar multiplication.

9.3.3 No. It is not closed under addition.

9.3.8 This is not a subspace. $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in it, but $5\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not.

9.4.1 (a) Yes. Suppose

$$c_1(x^3+1)+c_2(x^2+x)+c_3(2x^3+x^2)+c_4(2x^3-x^2-3x+1)=0.$$

Then collect equal powers of x:

$$(c_1+2c_3+2c_4)x^3+(c_2+c_3-c_4)x^2+(c_2-3c_4)x+(c_1+c_4)=0.$$

Does the system

$$\begin{gathered} c_1+2c_3+2c_4=0 \\ c_2+c_3-c_4=0 \\ c_2-3c_4=0 \\ c_1+c_4=0 \end{gathered}$$

have a non-trivial solution? The only solution is

$$c_1=0, \quad c_2=0, \quad c_3=0, \quad c_4=0,$$

and therefore, the polynomials are linearly independent. Since there are 4 linearly independent polynomials in a 4-dimensional space, they form a basis.

(b) Yes.

9.4.6 Yes, because the set of 5 linearly independent vectors can be extended to a basis B of V. But since V is 5-dimensional, B has only 5 elements, which must be the original 5 vectors.

9.4.7 No. Since V has a spanning set of size 5, the 6 vectors cannot be linearly independent by the Exchange Lemma.

9.4.11 (a) In case $k = 3$, the following sequences form a basis for W_3:

$$(1,0,0,1,0,0,1,0,0,1,\ldots),$$
$$(0,1,0,0,1,0,0,1,0,0,\ldots),$$
$$(0,0,1,0,0,1,0,0,1,0,\ldots).$$

Therefore, W_3 is a 3-dimensional space. For general k, the situation is analogous and the dimension of W_k is k.

(b) Let U be the set of all periodic sequences of all periods. We can find an (infinite) spanning set for U by taking all the basis vectors for all of the spaces W_k:

$$(1,1,1,1,1,1,1,1,1,1,\ldots) \quad \text{(period 1)},$$

$$(1,0,1,0,1,0,1,0,1,0,\ldots) \quad \text{(period 2)},$$
$$(0,1,0,1,0,1,0,1,0,1,\ldots) \quad \text{(period 2)},$$

$$(1,0,0,1,0,0,1,0,0,1,\ldots) \quad \text{(period 3)},$$
$$(0,1,0,0,1,0,0,1,0,0,\ldots) \quad \text{(period 3)},$$
$$(0,0,1,0,0,1,0,0,1,0,\ldots) \quad \text{(period 3)},$$

$$(1,0,0,0,1,0,0,0,1,0,\ldots) \quad \text{(period 4)},$$
$$(0,1,0,0,0,1,0,0,0,1,\ldots) \quad \text{(period 4)},$$
$$(0,0,1,0,0,0,1,0,0,0,\ldots) \quad \text{(period 4)},$$
$$(0,0,0,1,0,0,0,1,0,0,\ldots) \quad \text{(period 4)},$$

and so on. However, these sequences are not linearly independent. For example, we can obtain the sequence $(1,0,1,0,1,0,1,0,\ldots)$ of period 2 as a linear combination of two sequences of period 4, namely $(1,0,0,0,1,0,0,0,\ldots)$ and $(0,0,1,0,0,0,1,0,\ldots)$. By Proposition 9.47, we know that it is possible to shrink the above spanning set to a basis by removing certain sequences. How exactly to do this is an interesting question. One way to construct a basis is to keep exactly those sequences of period k that start with ℓ zeros, where $\gcd(\ell,k) = 1$. Proving that this really works is an interesting project.

9.4.12 When we add two number of the form $a+b\sqrt{2}$, we get another number of the same form. When we multiply a number of the form $a+b\sqrt{2}$ by a (rational) scalar, we get another number of the same form. Also, $0 = 0+0\sqrt{2}$ is of the required form. The 8 axioms of a vector space are satisfied because all of them are laws of the arithmetic of real numbers. A basis is $\left\{1,\sqrt{2}\right\}$. By definition, the span of these gives the collection of vectors. To prove that they are linearly independent, assume $a+b\sqrt{2} = 0$, where a,b are rational numbers. If $b \neq 0$, then $\sqrt{2} = -\frac{a}{b}$, which cannot happen because $\sqrt{2}$ is irrational. If $a \neq 0$, then $\frac{1}{\sqrt{2}} = -\frac{b}{a}$, which again cannot happen since $\frac{1}{\sqrt{2}}$ is irrational. Hence both $a,b = 0$. Therefore, 1 and $\sqrt{2}$ are linearly independent over the rational numbers, and form a basis. The dimension of the space is 2.

9.5.6 The block length is $n = r-1 = 15$ and the message length is $k = n-r = 11$. The check matrix and generator matrix are not unique, because there are different ways of ordering the columns of the check matrix. Here is one possible answer:

$$H = \left[\begin{array}{ccccccccccc|cccc} 1&1&0&1&1&0&1&0&1&0&1&1&0&0&0\\ 1&0&1&1&0&1&1&0&0&1&1&0&1&0&0\\ 0&1&1&1&0&0&0&1&1&1&1&0&0&1&0\\ 0&0&0&0&1&1&1&1&1&1&1&0&0&0&1 \end{array}\right], \quad G = \left[\begin{array}{ccccccccccc} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ \hline 1&1&0&1&1&0&1&0&1&0&1\\ 1&0&1&1&0&1&1&0&0&1&1\\ 0&1&1&1&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1 \end{array}\right].$$

With these matrices, the encoding of 00110011000 10100000001 is 001100110000110 101101110110101. The error syndrome for 001000010010000 is 1010, which corresponds to an error in bit 2. We find that the corrected code block is 011000010010000, and the decoding is 01100001001. The error syndrome for 010000100000001 is 0110, which corresponds to an error in bit 3. The corrected code block is 011000100000001 and the decoding is 01100010000. However, if you have used a different check matrix and/or generator matrix, your answers might differ.

10.1.1 (a) We have $T\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$, so T does not preserve the zero vector.

(b) We have $T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$ and $T\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}$, so T does not preserve scalar multiplication.

(c) We have $T\begin{bmatrix}\pi/2\\0\\0\end{bmatrix} = \begin{bmatrix}1\\\pi/2\end{bmatrix}$ and $T\begin{bmatrix}\pi\\0\\0\end{bmatrix} = \begin{bmatrix}0\\\pi\end{bmatrix}$, so T does not preserve scalar multiplication.

10.1.3 Yes, yes, no, yes.

10.1.6 (a) Let $a = (a_0, a_1, a_2, \ldots)$. Then the equation $\text{shift}(a) = a$ is equivalent to

$$(a_1, a_2, a_3, \ldots) = (a_0, a_1, a_2, \ldots),$$

which translates to the recurrence $a_{n+1} = a_n$. The only free variable is a_0, and the only solutions are the constant sequences. They form a 1-dimensional space with basis $\{(1, 1, 1, \ldots)\}$.

(b) The equation $\text{shift}(a) = -a$ is equivalent to

$$(a_1, a_2, a_3, \ldots) = (-a_0, -a_1, -a_2, \ldots),$$

which translates to the recurrence $a_{n+1} = -a_n$. The only free variable is a_0, and the solutions form a 1-dimensional space with basis $\{(1, -1, 1, -1, \ldots)\}$.

(c) The equation shift(shift(a)) = shift(a) + $2a$ is equivalent to

$$(a_2, a_3, a_4, \ldots) = (a_1 + 2a_0, a_2 + 2a_1, a_3 + 2a_2, \ldots),$$

which translates to the recurrence $a_{n+2} = a_{n+1} + 2a_n$. The only free variables are a_0 and a_1. The most obvious basis vectors are obtained by letting $(a_0, a_1) = (1, 0)$ and $(a_0, a_1) = (0, 1)$, giving the sequences $(1, 0, 2, 2, 6, 10, 22, \ldots)$ and $(0, 1, 1, 3, 5, 11, 21, \ldots)$. Thus,

$$\{(1, 0, 2, 2, 6, 10, 22, \ldots), (0, 1, 1, 3, 5, 11, 21, \ldots)\}$$

is a basis for the solutions. Another basis, which is slightly more convenient, is

$$\{(1, -1, 1, -1, 1, -1, \ldots), (1, 2, 4, 8, 16, 32, \ldots)\}.$$

10.2.1 Yes, no, yes, no.

10.3.1 By linearity we have $T(x^2) = 1$, $T(x) = T(x^2 + x - x^2) = T(x^2 + x) - T(x^2) = 5 - 1 = 4$, and $T(1) = T((x^2 + x + 1) - (x^2 + x)) = T(x^2 + x + 1) - T(x^2 + x) = -1 - 5 = -6$. Thus $T(ax^2 + bx + c) = aT(x^2) + bT(x) + cT(1) = a + 4b - 6c$.

10.3.2 The matrix A is invertible if and only if its rank is n, which means that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and therefore a basis of $\mathbb{R}^n$. The existence of T then follows from Theorem 10.20.

10.4.2 $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$

10.4.3 (a) The i^{th} column of $[T]_{E,E}$ is $[T(\mathbf{e}_i)]_E = T(\mathbf{e}_i)$, so

$$[T]_{E,E} = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 2 & 3 \\ 3 & 3 & -1 \end{bmatrix}.$$

(b) The i^{th} column of $[T]_{B,B}$ is $[T(\mathbf{v}_i)]_B$. This requires a calculation:

$$T(\mathbf{v}_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{therefore } [T(\mathbf{v}_1)]_B = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}.$$

$$T(\mathbf{v}_2) = T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = -7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{therefore } [T(\mathbf{v}_2)]_B = \begin{bmatrix} -7 \\ -1 \\ 6 \end{bmatrix}.$$

$$T(\mathbf{v}_3) = T\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix} = -11\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{therefore } [T(\mathbf{v}_3)]_B = \begin{bmatrix} -11 \\ 3 \\ 8 \end{bmatrix}.$$

We therefore have

$$[T]_{B,B} = \begin{bmatrix} -3 & -7 & -11 \\ 0 & -1 & 3 \\ 3 & 6 & 8 \end{bmatrix}.$$

10.4.4 $M_{B_2B_1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

10.4.7 Recall that $\text{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v}$. The desired matrix has i^{th} column equal to $\text{proj}_{\mathbf{v}}(\mathbf{e}_i)$. Therefore, the desired matrix is

$$\frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}.$$

10.4.8 We have $T(\mathbf{v}_1) = \mathbf{v}_1$, $T(\mathbf{v}_2) = \mathbf{0}$, and $T(\mathbf{v}_3) = \mathbf{0}$. Therefore

$$[T]_{B,B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

10.4.9 Since M is invertible, its columns $[\mathbf{v}_1]_B, \ldots, [\mathbf{v}_n]_B$ are linearly independent and span $\mathbb{R}^n$; it follows that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V. To show that $[T]_{C,B} = NM^{-1}$, it is sufficient to check that $NM^{-1}[\mathbf{v}_i]_B = [\mathbf{w}_i]_C$, for all $i = 1, \ldots, n$. But by assumption, $[\mathbf{v}_i]_B$ is the i^{th} column of M, so that $[\mathbf{v}_i]_B = M\mathbf{e}_i$, where $\mathbf{e}_i$ is the i^{th} basis vector. Therefore $M^{-1}[\mathbf{v}_i]_B = \mathbf{e}_i$. On the other hand, $N\mathbf{e}_i$ is the i^{th} column of N, i.e., $[\mathbf{w}_i]_C$. We therefore have

$$NM^{-1}[\mathbf{v}_i]_B = N\mathbf{e}_i = [\mathbf{w}_i]_C,$$

as desired.

11.1.1 (a) Yes, it is an inner product. Since $A = A^T$, symmetry and linearity follow as in Example 11.3. For the positive definite property, note that $\langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + 2u_2^2 \geq 0$, and equality holds if and only if $u_1, u_2 = 0$.

(b) Yes, it is an inner product. Since $A = A^T$, symmetry and linearity follow as in Example 11.3. For the positive definite property, we have $\langle \mathbf{u}, \mathbf{u} \rangle = 3u_1^2 + 2u_1u^2 + 3u_2^2 = (u_1 + u_2)^2 + 2u_1^2 + 2u_2^2 \geq 0$. Equality holds if and only if $u_1 + u_2$, u_1, and u_2 are all zero, which is the case if and only if $\mathbf{u} = 0$.

(c) No, it is not symmetric. For example, $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = 1$ but $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle = 0$.

(d) No, it is not positive definite. For example, $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = -1$.

(e) No, it is not positive definite. For example, $\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rangle = 0$ although $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \mathbf{0}$.

11.1.2 (a) $\langle 1, x \rangle = \int_0^1 1 \cdot x\,dx = \int_0^1 x\,dx = \frac{1}{2}$.

(b) $\langle x, x^2 \rangle = \int_0^1 x \cdot x^2\,dx = \int_0^1 x^3\,dx = \frac{1}{4}$.

(c) $\langle 1 + x, 2 + x^2 \rangle = \int_0^1 (1+x)(2+x^2)\,dx = \int_0^1 2 + x^2 + 2x + x^3\,dx = \frac{43}{12}$.

11.1.3 (a) $\|1\|^2 = \langle 1, 1 \rangle = \int_0^1 1\,dx = 1$, therefore $\|1\| = 1$.

(b) $\|x\|^2 = \langle x,x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}$, therefore $\|x\| = \frac{1}{\sqrt{3}}$.

(c) $\left\|x^2+1\right\|^2 = \langle x^2+1, x^2+1\rangle = \int_0^1 x^4+2x^2+1\,dx = \frac{1}{5}+\frac{2}{3}+1 = \frac{28}{15}$, therefore $\left\|x^2+1\right\| = \sqrt{\frac{28}{15}}$.

11.1.4 The operation $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}*\mathbf{v}$ is an inner product on $\mathbb{R}^3$. Symmetry and linearity are straightforward to check, as is the positive definite property. Therefore, the claimed inequality holds by Proposition 2.27.

11.1.5 (a) We have $\langle x,x\rangle = \int_{-1}^1 x^2\,dx = \frac{2}{3}$, $\langle x^2,x^2\rangle = \int_{-1}^1 x^4\,dx = \frac{2}{5}$, and $\langle x,x^2\rangle = \int_{-1}^1 x^3\,dx = 0$. Therefore

$$\cos\theta = \frac{\langle x,x^2\rangle}{\|x\|\,\|x^2\|} = 0.$$

Therefore, the angle θ is $\pi/2$ radians, or 90 degrees. In other words, x and x^2 are orthogonal in $C[-1,1]$.

(b) We have $\langle x,x\rangle = \int_{-1}^1 x^2\,dx = \frac{2}{3}$, $\langle x^3,x^3\rangle = \int_{-1}^1 x^6\,dx = \frac{2}{7}$, and $\langle x,x^3\rangle = \int_{-1}^1 x^4\,dx = \frac{2}{5}$. Therefore

$$\cos\theta = \frac{\langle x,x^3\rangle}{\|x\|\,\|x^3\|} = \frac{\frac{2}{5}}{\sqrt{\frac{2}{3}}\sqrt{\frac{2}{7}}} = \frac{\sqrt{21}}{5}.$$

The angle θ is $\cos^{-1}(\frac{\sqrt{21}}{5})$, which is approximately 0.4115 radians or 23.58 degrees.

11.1.6 (a) By assumption, a and b are square summable. Let $N = a_0^2+a_1^2+\ldots$ and $M = b_0^2+b_1^2+\ldots$. By the Cauchy-Schwarz inequality, for all n, we have

$$|a_0|\,|b_0|+\ldots+|a_n|\,|b_n| \leq \sqrt{|a_0|^2+\ldots+|a_n|^2}\sqrt{|b_0|^2+\ldots+|b_n|^2} \leq \sqrt{N}\sqrt{M}.$$

Therefore the series $|a_0b_0|+|a_1b_1|+\ldots$ is bounded. By the absolute convergence test from calculus, it follows that the series $a_0b_0+a_1b_1+\ldots$ converges.

(b) It is clear that the zero sequence is square summable, and also that a scalar multiple of a square summable sequence is square summable. Hence $\mathbf{Hilb}_{\mathbb{R}}$ contains the zero vector and is closed under scalar multiplication. To show that it is closed under addition, assume $a,b \in \mathbf{Hilb}_{\mathbb{R}}$, and let $c = a+b$. We must show that c is square summable. But

$$c_0^2+c_1^2+\ldots = (a_0+b_0)^2+(a_1+b_1)^2+\ldots = (a_0^2+a_1^2+\ldots)+(b_0^2+b_1^2+\ldots)+(2a_0b_0+2a_1b_1+\ldots).$$

The series $a_0^2+a_1^2+\ldots$ and $b_0^2+b_1^2+\ldots$ converge by assumption, and the series $2a_0b_0+2a_1b_1+\ldots$ converges by part (a). It follows that $\mathbf{Hilb}_{\mathbb{R}}$ is closed under addition.

(c) Symmetry and linearity follow straightforwardly from properties of convergent series. For example,

$$\begin{aligned}
\langle a, kb+\ell c\rangle &= a_0(kb_0+\ell c_0)+a_1(kb_1+\ell c_1)+\ldots \\
&= k(a_0b_0+a_1b_1+\ldots)+\ell(a_0c_0+a_1c_1+\ldots) \\
&= k\langle a,b\rangle+\ell\langle a,c\rangle.
\end{aligned}$$

As for the positive definite property, note that

$$\langle a,a\rangle = a_0^2+a_1^2+\ldots \geq 0.$$

Moreover, since all terms in the series are ≥ 0, it follows that $\langle a,a\rangle = 0$ if and only if $a_i = 0$ for all i.

11.2.1 We have $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \mathbf{u}_1^T A \mathbf{u}_2 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \mathbf{u}_1^T A \mathbf{u}_3 = -5$, $\langle \mathbf{u}_1, \mathbf{u}_4 \rangle = \mathbf{u}_1^T A \mathbf{u}_4 = 0$, $\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \mathbf{u}_2^T A \mathbf{u}_3 = 0$, $\langle \mathbf{u}_2, \mathbf{u}_4 \rangle = \mathbf{u}_2^T A \mathbf{u}_4 = 32$, and $\langle \mathbf{u}_3, \mathbf{u}_4 \rangle = \mathbf{u}_3^T A \mathbf{u}_4 = 54$. Therefore, $\mathbf{u}_1 \perp \mathbf{u}_2$, $\mathbf{u}_1 \perp \mathbf{u}_4$, and $\mathbf{u}_2 \perp \mathbf{u}_3$. None of the other pairs of vectors are orthogonal.

11.2.2 $f_1 \perp f_2$, $f_1 \perp f_4$, $f_2 \perp f_3$, and $f_3 \perp f_4$.

11.2.3 (a) Let $p(x) = ax^3 + bx^2 + cx + d$. Then $\langle p(x), x^2 \rangle = \frac{2}{5}b + \frac{2}{3}d$ and $\langle p(x), x \rangle = \frac{2}{5}a + \frac{2}{3}c$. Therefore, $p(x)$ is in the orthogonal complement of $\{x^2, x\}$ if and only if $\frac{2}{5}b + \frac{2}{3}d = 0$ and $\frac{2}{5}a + \frac{2}{3}c = 0$. It follows that $b = -\frac{5}{3}d$ and $a = -\frac{5}{3}c$. The general solution is $p(x) = -\frac{5}{3}cx^3 - \frac{5}{3}dx^2 + cx + d$. A basis for the orthogonal complement is $\left\{-\frac{5}{3}x^3 + x, -\frac{5}{3}x^2 + 1\right\}$.

(b) A basis for the orthogonal complement of $\{x+1\}$ is $\left\{5x^3 - 1, 3x^2 - 1, 3x - 1\right\}$.

11.2.4 (a) Orthonormal (therefore also orthogonal). (b) Neither orthogonal nor orthonormal. (c) Orthogonal (not orthonormal). (d) Orthonormal (therefore also orthogonal).

11.2.5 $\mathbf{v} = \frac{1}{4}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 - \frac{4}{5}\mathbf{u}_3$.

11.2.6 We have $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ where $a_1 = \frac{\langle \mathbf{u}_1, \mathbf{v} \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} = \frac{1}{2}$. So the first coordinate is $\frac{1}{2}$.

11.3.1 $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$.

11.3.2 $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

11.3.3 $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

11.3.4 $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

11.3.5 $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

11.3.6 $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x - 1$, $x^2 - 2x + \frac{2}{3}$.

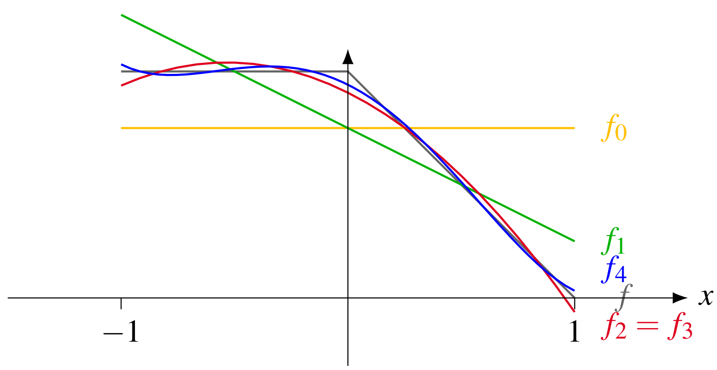
11.3.7 The orthonormal basis is $\left\{\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}\right\} = \left\{\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}, \frac{1}{\sqrt{26}}\begin{bmatrix}1\\4\\-3\end{bmatrix}\right\}$.

11.4.1 The best approximation is $\mathbf{v}' = 3\mathbf{u}_1 - \mathbf{u}_2 = \begin{bmatrix}-2\\5\\2\end{bmatrix}$.

11.4.2 The best approximation is $\mathbf{v}' = \mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 = \begin{bmatrix}1\\3\\-2\\4\end{bmatrix}$.

11.4.3 Let $p_0, p_1, \ldots$ be the Legendre polynomials from Section 11.3. The approximating polynomials are:

$$\begin{aligned}
f_0(x) &= \frac{3}{4}p_0,\\
f_1(x) &= \frac{3}{4}p_0 - \frac{1}{2}p_1,\\
f_2(x) &= \frac{3}{4}p_0 - \frac{1}{2}p_1 - \frac{15}{32}p_2,\\
f_3(x) &= \frac{3}{4}p_0 - \frac{1}{2}p_1 - \frac{15}{32}p_2 + 0p_3,\\
f_4(x) &= \frac{3}{4}p_0 - \frac{1}{2}p_1 - \frac{15}{32}p_2 + 0p_3 + \frac{105}{256}p_4.
\end{aligned}$$



11.4.4 $\frac{\pi^2}{3} - \frac{4}{1}\cos x + \frac{4}{4}\cos 2x - \frac{4}{9}\cos 3x + \frac{4}{16}\cos 4x - \frac{4}{25}\cos 5x \pm \ldots$.

11.5.1 $(x,y,z) = (-1,-1,2)$.

11.5.2 $(x,y,z) = (2,2,-1)$.

11.5.3 $y = 2 + 2x$.

11.5.4 $y = 1 - x + x^2$.

11.6.1 (a) Yes. (b) Yes. (c) No. (d) Yes.

11.6.2 A and C are orthogonal (and therefore isometries). D is an isometry but not orthogonal. B is neither orthogonal nor an isometry.

11.6.3 A and D are orthogonal, but B and C are not.

11.7.1 (a) $A = PDP^{-1}$ where $D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$, $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

(b) $A = PDP^{-1}$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(c) $A = PDP^{-1}$ where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$.

(d) $A = PDP^{-1}$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$.

11.7.2 Suppose A is orthogonally diagonalizable. Then $A = PDP^{-1}$, where D is diagonal and P is orthogonal. Since P is orthogonal, we have $P^{-1} = P^T$, and therefore $A = PDP^T$. Moreover, since D is diagonal, we have $D = D^T$. It follows that $A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$, so A is symmetric.

11.8.1 (a) Positive semidefinite. (b) Positive definite. (c) Not symmetric (therefore neither). (d) Positive semidefinite. (e) Neither.

11.8.2 (a) Eigenvalues: $\{1,6\}$. Positive definite. (b) Eigenvalues: $\{0,13\}$. Positive semidefinite. (c) Eigenvalues: $\{0,2,3\}$. Positive semidefinite. (d) Eigenvalues: $\{-1,4,8\}$. Neither.

11.8.3 (a) No (neither positive definite nor semidefinite). (b) Yes. (c) No (positive semidefinite but not definite). (d) Yes.

11.8.4 The characteristic polynomials are:

$$\begin{aligned} \det(A-\lambda I) &= \lambda^2 - 3\lambda - 2, \\ \det(B-\lambda I) &= -\lambda^3 + 4\lambda^2 - 4\lambda + 1, \\ \det(C-\lambda I) &= \lambda^4 - 6\lambda^3 + 9\lambda^2 - 3\lambda + 0. \end{aligned}$$

For A, the coefficients are not weakly alternating, so A is not positive semidefinite. For B, the coefficients are strongly alternating, so B is positive definite. For C, the coefficients are weakly, but not strongly alternating, so C is positive semidefinite, but not positive definite.

11.9.1 All of them except (e) and (g).

11.9.3 $f(x,y,z) = \mathbf{v}^T A\mathbf{v}$, where

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1.5 \\ -2 & 1.5 & 2 \end{bmatrix}.$$

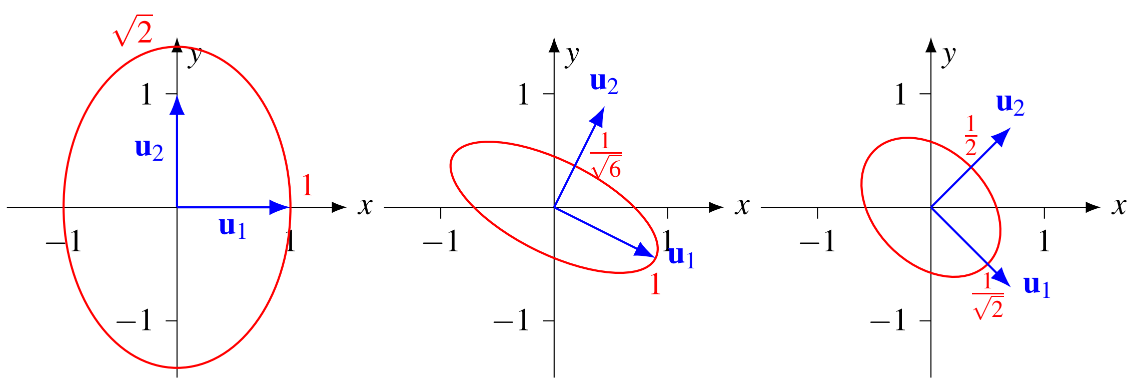
11.9.4 $3u^2 + v^2 + vw + w^2$.

11.9.5 $x = u + v$ and $y = u - v$ gives $f = 4u^2 + 8v^2$.

11.9.7 (a) Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$. Principal axes: $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\mathbf{u}_1$-intercept: 1, $\mathbf{u}_2$-intercept: $\sqrt{2}$.

(b) Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 6$. Principal axes: $\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\mathbf{u}_1$-intercept: 1, $\mathbf{u}_2$-intercept: $\frac{1}{\sqrt{6}}$.

(c) Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 4$. Principal axes: $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\mathbf{u}_1$-intercept: $\frac{1}{\sqrt{2}}$, $\mathbf{u}_2$-intercept: $\frac{1}{2}$.



11.9.8 The principal axes are $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

11.10.1 (a) $1 - i$, (b) $1 + 5i$, (c) $1 - 5i$, (d) 0.

11.10.2 (a) $\sqrt{2}$, (b) $\sqrt{6}$, (c) $\sqrt{13}$.

11.10.3 (a) $-6i$, (b) $5i$.

11.10.7 $\mathbf{u}_1 \perp \mathbf{u}_2$, $\mathbf{u}_1 \perp \mathbf{u}_4$, and $\mathbf{u}_2 \perp \mathbf{u}_3$.

11.10.9 Orthogonal: $[0,i,2]^T$, $[1,2,i]^T$. Orthonormal: $\frac{1}{\sqrt{5}}[0,i,2]^T$, $\frac{1}{\sqrt{6}}[1,2,i]^T$.

11.11.1 (a) Yes. (b) Yes. (c) No. (d) Yes.

11.11.2 (a) Yes. (b) No. (c) Yes.

11.11.3 (a) No. (b) Yes. (c) Yes.

11.11.4 All except (b) and (e) are unitary.

11.11.5 Only (f) is unitary.

11.11.8 All except (a), (c), and (d) are hermitian.

11.11.9 Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$. Normalized eigenvectors: $\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

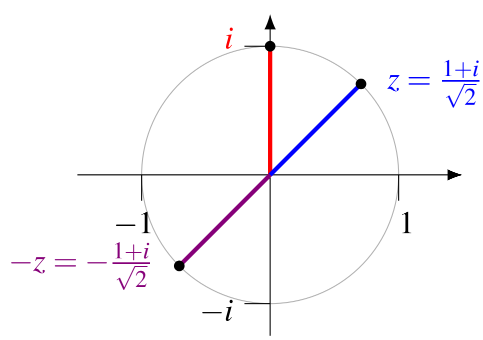
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad P = \frac{1}{\sqrt{3}}\begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}.$$

A.1.1 $z+w = 5-i$, $z-2w = -4+23i$, $zw = 62+5i$, and $\dfrac{w}{z} = -\dfrac{50}{53} - \dfrac{37}{53}i$.

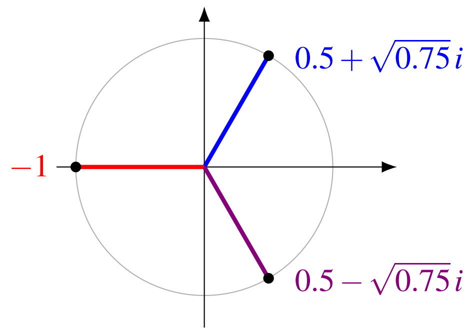
A.1.4 If $z = 0$, let $w = 1$. If $z \neq 0$, let $w = \dfrac{|z|}{z}$. Note that $wz = \dfrac{|z|}{z}z = |z|$ and $|w| = \dfrac{|z|}{|z|} = 1$.

A.1.5 The problem is that there is no single $\sqrt{-1}$. In the complex numbers, -1 has two square roots, namely i and $-i$. Since each complex number has two square roots, and generally neither of them is positive or even real, the notation $\sqrt{z}$ does not have a fixed meaning in the complex numbers. Therefore, the equation $\sqrt{z}\sqrt{w} = \sqrt{zw}$ cannot be used. At best, we could maybe say $\sqrt{z}\sqrt{w} = \pm\sqrt{zw}$.

A.2.2 Since i has magnitude 1 and argument $\pi/2$ (or $90°$), the number z must have magnitude 1 and argument $\pi/4$. It therefore lies at $45°$ on the unit circle. The solution is $z = \frac{1+i}{\sqrt{2}}$. A second solution is $-z = -\frac{1+i}{\sqrt{2}}$, whose argument is $-3\pi/4$ or $-135°$. Note that if we double this angle, we get $-270°$, which is the same as $+90°$.



A.2.3 The three solutions can be found on the unit circle at 60°, 180°, and 300°. If we triple any of these angles, we get 180° (up to multiples of 360°). Thus, the three cube roots of -1 are $z = -1$ and $z = 0.5 \pm \sqrt{0.75}\,i$.



A.3.1 By the quadratic formula, $z = \dfrac{6 \pm \sqrt{16}}{2} = 3 \pm 2i$.

A.3.2 $p(z)$ factors as $(z-1)(z^2+6z+10)$. The roots are $z = 1$, $z = -3+i$, and $z = -3-i$.

A.3.3 By trial and error, we find the roots $z = 1$ and $z = -2$. Moreover, $z = 1$ is a double root. We have $z^3 - 3z + 2 = (z-1)(z-1)(z+2)$.

A.3.4 If $p(z) = 0$ and $a_0, \ldots, a_k$ are real, we have

$$\begin{aligned}
p(\bar{z}) &= a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \ldots + a_1 \bar{z} + a_0 \\
&= \overline{a_n}\, \overline{z}^n + \overline{a_{n-1}}\, \overline{z}^{n-1} + \ldots + \overline{a_1}\, \overline{z} + \overline{a_0} \\
&= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_1 z} + \overline{a_0} \\
&= \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} \\
&= \overline{p(z)} \\
&= 0.
\end{aligned}$$

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