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Virgil Emil CĂZĂNESCU and Costel UNGUREANU

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Virgil Emil CĂZĂNESCU*) and Costel UNGUREANU**)

November 1982

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Again on Advice on structuring compilers and proving them correct.

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ABSTRACT. We introduce an algebraic structure for which the set of Σ-flowcharts is the free structure generated by Σ. Then we show that the ADJ style for defining semantics can be also used to define compilers and proving them correct.

Introduction

We suppose the reader is familiar with calculations in an algebraic theory. From now on we shall use "theory" instead of algebraic theory. We use the ADJ terminology and notations.

The first new concept we came at was that of <u>theory with iterate</u>. Theories with iterate are more general than rationally closed theories, because they are not ordered theories. This makes them close to C.C. Elgot's iterative theories. The main difference between our theories and Elgot's is : a)every morphism has an iterate, not only the ideal morphisms, and b)the iterate, although a solution, need not be unique. The theory with iterate, even without axiom (1') from definition 1.1 below, is sufficient to define semantics in the ADJ style.

The next step, and perhaps the most important one, was the introduction of the concept of <u>flowchart over a theory</u>. This is a generalization of the ADJ flowcharts which are obtained using the initial theory, i.e. the initial object in the category of theories. We think we should explain our concept of flowchart over a theory and especially motivate the necessity of its introduction.

Let us begin with some notation.

The set of nonnegative integers is denoted by ω. It is a monoid with respect to addition. If $n \in \omega$ then the set $\{1,2,\ldots,n\}$ is denoted by $[n]$.

The free monoid generated by the set A is denoted by A^*. If $w \in A^*$ then $|w|$ is its length. If M is a monoid and $f : A \longrightarrow M$ a function then its unique extension to a monoid morphism is denoted by $f^* : A^* \longrightarrow M$.

The initial theory is denoted by N. The set N(n,m) of morphisms from n to m in this category, is the set of all functions from [n] to [m] and the composition of morphisms is the composition of functions.

We begin the explanation by recalling the ADJ definition of flowcharts.

Let Σ be a set (of statements) and let $r : \Sigma \longrightarrow \omega$ be the ranking function of Σ.

Let $n, p \in \omega$. A (normalized) Σ-flowchart from n to p of weight $s \in \omega$ consists of a triple (b, τ, e) where :

$b : [n] \longrightarrow [s+p]$ is the begin function,

$\tau : [s] \longrightarrow [s+p]^*$ is the underlying graph,

$e : [s] \longrightarrow \Sigma$ is the labeling function,

such that $|\tau(i)| = r(e(i))$ for $i \in [s]$. n is the number of inputs, p is the number of exits and s is the number of internal vertices.

We begin to use the theory N, e.g. $b \in N(n, s+p)$ instead of $b : [n] \longrightarrow [s+p]$ is a function. Notice that for every $i \in [s]$ it follows that $\tau(i) \in N(|\tau(i)|, s+p)$, and to avoid the supplementary condition $|\tau(i)| = r(e(i))$ we may write $\tau(i) \in N(r(e(i)), s+p)$. But the morphisms $\tau(i)$, $i \in [s]$, having the same target, may be replaced by their tupling :

$$\langle \tau(1), \tau(2), \ldots, \tau(s) \rangle \in N(r(e(1)) + r(e(2)) + \ldots + r(e(s)), s+p) .$$

The labeling function will be thought as an element of Σ^*; thus

$$s = |e|$$

and

$$r(e(1)) + r(e(2)) + \ldots + r(e(s)) = r^*(e) .$$

Finally we note that the triple (b, τ, e) fulfils the conditions

$$e \in \Sigma^* ,$$

$$b \in N(n, |e| + p)$$

$$\tau \in N(r^*(e), |e| + p) .$$

To obtain our generalization we replace N by a theory T. A Σ-flowchart over T with n inputs and p exits consists of a triple (b, τ, e) where

$$e \in \Sigma^* ,$$

$$b \in T(n, |e|+p) ,$$

$$\tau \in T(r^*(e), |e|+p) .$$

This formal game is very far from the real reason that led us to the above definition.

The starting point was the need of another definition of the iterate of a Σ-flowchart. We tried several definitions in order to obtain good properties fot the iterate. An iterate without new internal vertices was the only one corresponding to our exigence. To obtain it we have been obliged to extend the concept of Σ-flowchart. This extension is based on a very simple and known idea which is used when passing from finite trees to partial finite trees, namely the possible omission of some successors of certain vertices. Thus we shall allow in the definition of a Σ-flowchart that some successors of certain vertices be omitted and we shall interpret every omission as an input into a loop without exit.

That is why we have modified the initial definition of the Σ-flowchart by allowing that

$$b : [n] \longrightarrow\!\!\!\circ\!\!\!\longrightarrow [|e| + p]$$

be a partial function i.e. some inputs may be connected to loops without exit; we also allow that

$$\tau : [r^*(e)] \longrightarrow\!\!\!\circ\!\!\!\longrightarrow [|e| + p]$$

be a partial function i.e. every successor of an internal vertex is another (internal or exit) vertex or a loop without exit. Notice that this slight generalization of the initial definition is a particular case of our general definition, if T is replaced by the theory Sum_A where A is a singleton. Moreover, it is the fact that Sum_A is a theory with iterate which enabled us to obtain a new definition for the iterate of a Σ-flowchart. We think that our iterate is convenient both from the practical and theoretical points of view.

We have further modified the definition of Σ-flowcharts using two more ideas. The first one is very known: using several sorts instead of only one. The other idea is to allow, for every $\sigma \in \Sigma$, several inputs instead of only one, i.e. another ranking function of Σ is given,

$$r_1 : \Sigma \longrightarrow \omega ,$$

indicating for every $\sigma \in \Sigma$ the number $r_1(\sigma)$ of its inputs. In this case the functions b and τ will take values in the set

$$[r_1^*(s) + p]$$

i.e. every input of the Σ-flowchart and every successor of an internal vertex is either an input of another internal vertex or an exit vertex.

The category of Σ-flowcharts over a theory with iterate T is denoted by $Fl_{\Sigma,T}$. The study of this category has enabled us to introduce the concept of T-module with iterate and to prove the main theorem : the T-module with iterate $Fl_{\Sigma,T}$ is freely generated by Σ. Beside its own interest, this property of universality has many applications.

The first application is the definition of semantics. It suffices to interpret Σ into a T-module with iterate Q and to apply the property of universality in order to obtain the semantics morphism

$$Fl_{\Sigma,T} \longrightarrow Q .$$

This definition generalizes the ADJ definition of semantics because every rationally closed theory T is a theory with iterate, therefore a T-module with iterate.

The next application is to compilation. If we interpret Σ into the T-module with iterate $Fl_{\Sigma',T}$ then the semantics morphism is the compiler from Σ-flowcharts over T into Σ'-flowcharts over T. The same property of universality shows that this compiler is correct if and only if the compilation of every $\sigma \in \Sigma$ is made correctly.

Notice that we use the same method for defining semantics and compilers.

1. Algebraic theories with iterate

Let S be a set. The letters s, t will denote elements of S and the letters a, b, c, d will denote elements of S^*.

Let T be an S-sorted theory. Let 0_a be the unique morphism from the empty word λ to a.

Let $S_a^b = \langle 0_b + 1_a, 1_b + 0_a \rangle : ab \longrightarrow ba$. Notice that $S_a^b S_b^a = 1_{ab}$.

1.1 <u>Definition</u>. A <u>theory</u> T is said to be <u>with iterate</u> if for every a, b is given a mapping

$$\dagger : T(a, ab) \longrightarrow T(a, b)$$

called iterate and satisfying the following axioms :

(1) $\langle f, g \rangle^\dagger = \langle f^\dagger \langle (g \langle f^\dagger, 1_{bc} \rangle)^\dagger, 1_c \rangle, (g \langle f^\dagger, 1_{bc} \rangle)^\dagger \rangle$

for every $f \in T(a, abc)$ and $g \in T(b, abc)$,

(1') $\langle f, g \rangle^\dagger = S_a^b \langle g(S_a^b + 1_c), f(S_a^b + 1_c) \rangle^\dagger$

for every $f \in T(a, abc)$ and $g \in T(b, abc)$,

(2) $(f(1_a + g))^\dagger = f^\dagger g$

for every $f \in T(a, ab)$ and $g \in T(b, c)$,

(3) $f \langle f^\dagger, 1_b \rangle = f^\dagger$

for every $f \in T(a, ab)$ □

1.2 <u>Proposition</u>. Let T be a theory with iterate. Then :

(4) $(f(0_a + 1_b))^\dagger = f$

for every $f \in T(a, b)$,

(5) $\langle f(1_a + 0_b + 1_c), g(0_a + 1_{bc}) \rangle^\dagger = \langle f^\dagger, g^\dagger \rangle$

for every $f \in T(a, ac)$ and $g \in T(b, bc)$,

(6) $f^\dagger \langle (g \langle f^\dagger, 1_{bc} \rangle)^\dagger, 1_c \rangle = (f(S_a^b + 1_c) \langle (g(S_a^b + 1_c))^\dagger, 1_{ac} \rangle)^\dagger$

for every $f \in T(a, abc)$ and $g \in T(b, abc)$,

(7) $(g \langle f^\dagger, 1_{bc} \rangle)^\dagger = (g(S_a^b + 1_c))^\dagger \langle (f(S_a^b + 1_c) \langle (g(S_a^b + 1_c))^\dagger, 1_{ac} \rangle)^\dagger, 1_c \rangle$

for every $f \in T(a, abc)$ and $g \in T(b, abc)$,

(8) $(g \langle f^\dagger, 1_{bc} \rangle)^\dagger = (g(S_a^b + 1_c))^\dagger \langle f^\dagger \langle (g \langle f^\dagger, 1_{bc} \rangle)^\dagger, 1_c \rangle, 1_c \rangle$

for every $f \in T(a, abc)$ and $g \in T(b, abc)$.

<u>Proof</u>. Let $f \in T(a, b)$. It follows from (3) that

$(f(0_a + 1_b))^\dagger = f(0_a + 1_b) \langle (f(0_a + 1_b))^\dagger, 1_b \rangle = f$.

Let $f \in T(a,ac)$ and $g \in T(b,bc)$. It follows from (2) that

$$(f(1_a+0_b+1_c))^{\dagger} = f^{\dagger}(0_b+1_c) .$$

As

$$g(0_a+1_{bc}) \langle (f(1_a+0_b+1_c))^{\dagger}, 1_{bc} \rangle = g$$

it follows from (1) that

$$\langle f(1_a+0_b+1_c), g(0_a+1_{bc}) \rangle^{\dagger} = \langle f^{\dagger}(0_b+1_c)\langle g^{\dagger},1_c\rangle , g^{\dagger} \rangle = \langle f^{\dagger},g^{\dagger} \rangle .$$

Let $f \in T(a,abc)$ and $g \in T(b,abc)$. Using (1) in the right-hand side of (1') we deduce

$$\langle f,g \rangle^{\dagger} = \langle (f(S_a^b+1_c)\langle (g(S_a^b+1_c))^{\dagger},1_{ac} \rangle)^{\dagger},$$
$$(g(S_a^b+1_c))^{\dagger}\langle (f(S_a^b+1_c)\langle (g(S_a^b+1_c))^{\dagger},1_{ac} \rangle)^{\dagger},1_c \rangle \rangle .$$

Using (1) in the left-hand side and equalizing the two components of the tupling we obtain (6) and (7). (8) is an easy consequence of (6) and (7) □

1.3 Corollary. The condition (1')* in definition 1.1 may be replaced by (6) or by (7) .

Proof. Remark first that (6) and (7) are equivalent. Thus e.g. if we put in (6) $g(S_a^b+1_c)$ instead of f and $f(S_a^b+1_c)$ instead of g we obtain (7) .

Then, if we use (6) and (7) in (1) it follows that

$$\langle f,g \rangle^{\dagger} = S_a^b \langle (g(S_a^b+1_c))^{\dagger} \langle (f(S_a^b+1_c) \langle (g(S_a^b+1_c))^{\dagger},1_{ac} \rangle)^{\dagger},1_c \rangle ,$$
$$(f(S_a^b+1_c)\langle (g(S_a^b+1_c))^{\dagger}, 1_{ac} \rangle)^{\dagger} \rangle$$

hence applying (1) in the right-hand side of the last equality we obtain (1') □

1.4 Corollary. Let T be a theory with iterate. If $a=s_1s_2\ldots s_n$ then

$$1_a^{+} = \langle 1_{s_1}^{+},1_{s_2}^{+},\ldots,1_{s_n}^{+} \rangle .$$

Proof. By induction.

As $1_{as} = \langle 1_a+0_s,0_a+1_s \rangle$ it follows from (5) that $1_{as}^{+} = \langle 1_a^{+},1_s^{+} \rangle$ □

In a theory with iterate if we define

$$\perp_{a,b} = 1_a^{+}0_b$$

then for every $f \in T(b,c)$

$$\perp_{a,b} f = \perp_{a,c} .$$

1.5 <u>Proposition</u>. Every rationally closed theory is a theory with iterate.

<u>Proof</u>. Let T be a rationally closed theory. A proof of (1) was given in 7 . It is easy to see that

$$(if(i^{-1}+1_b))^{\dagger} = if^{\dagger}$$

for every $f \in T(a,ab)$ and for every isomorphism $i \in T(c,a)$. It follows from this remark and (1) that T fulfils (1') □

It is easy to see that the theories with iterate form a variety, hence from a well known theorem about varieties of heterogeneous algebras we deduce the existence of free theories with iterate. We conjecture that the free theory with iterate generated by Σ is RT_{Σ} i.e. the subtheory of CT_{Σ} determined by tuples of rational trees. We are interested only in the initial theory with iterate PStr because in the homogeneous case it is isomorphic to Sum_A where A is a singleton.

1.6 <u>Definition</u>. A theory morphism is said to be a morphism of theories with iterate if it preserves the iterate □

Notice that every morphism of theories with iterate preserves the morphism $\perp_{a,b}$.

We regard every $a \in S^*$ as a function $a : [|a|] \longrightarrow S$. If $a \in S^*$ and $i \in [|a|]$ we write a_i instead of $a(i)$. Therefore $a = a_1 a_2 \ldots a_{|a|}$.

Let A and B be sets. Let $[A,B]$ be the set of partial functions from A to B. We order $[A,B]$ by inclusion and obtain an ω-complete partially ordered set having the empty function as bottom element.

Let us define the theory PStr. PStr(a,b) consists of all partial functions $f : [|a|] \multimap [|b|]$ such that for each $i \in [|a|]$ if $f(i)$ is defined then $b_{f(i)} = a_i$. The composition of morphisms is naturally the composition of partial functions. PStr(a,b) is an ω-complete partially ordered set because it is a subset of $[[|a|],[|b|]]$ which is closed under least upper bounds of increasing sequences. The composition of morphisms is ω-continuous and strict.

The distinguished morphisms $x_i^a \in T(a_i, a)$ are defined by

$$x_i^a(1) = i \quad \text{for } i \in [|a|].$$

If $f \in PStr(a,c)$ and $g \in PStr(b,c)$ then

$$\langle f,g \rangle (i) = \begin{cases} f(i) & \text{if } i \in [|a|] \\ g(i-|a|) & \text{if } |a| < i \leq |ab| . \end{cases}$$

The tupling operation is monotonic.

Therefore PStr is a theory with iterate because it is an ω-continuous theory.

Notice that every $f \in PStr(a,b)$ may be written

$$f = \langle x^b_{f(1)}, x^b_{f(2)}, \ldots, x^b_{f(|a|)} \rangle$$

where by convention $x^b_{f(i)} = \perp_{a_i,b}$ if $f(i)$ is not defined.

1.7 Proposition. PStr is the initial theory with iterate.

Proof. Let T be a theory with iterate.

If $F : PStr \longrightarrow T$ is a morphism of theories with iterate then for every $f \in PStr(a,b)$

$$F(f) = \langle F(x^b_{f(1)}), F(x^b_{f(2)}), \ldots, F(x^b_{f(|a|)}) \rangle$$

therefore with the same convention as above

$$F(f) = \langle x^b_{f(1)}, x^b_{f(2)}, \ldots, x^b_{f(|a|)} \rangle .$$

This proves the uniqueness of F.

In order to prove the existence of F we define it by the latter equality.

If $f \in PStr(a,b)$, $g \in PStr(b,c)$ then

$F(f)F(g) =$

$= \langle x^b_{f(1)} \langle x^c_{g(1)}, \ldots, x^c_{g(|b|)} \rangle, \ldots, x^b_{f(|a|)} \langle x^c_{g(1)}, \ldots, x^c_{g(|b|)} \rangle \rangle =$

$= \langle x^c_{g(f(1))}, \ldots, x^c_{g(f(|a|))} \rangle =$

$= F(fg)$.

If $i \in [|a|]$ then $F(x^a_i) = x^a_i$. Therefore F is a theory morphism.

We shall prove by induction on $|a|$ that

$$F(f^+) = F(f)^+$$

for every $f \in PStr(a,ab)$.

For $|a| = 0$ the equality is obvious. For $|a| = 1$ we study three cases:

i) $f = \perp_{a,ab}$. Obviously $f^\dagger = \perp_{a,b}$ hence $F(f) = \perp_{a,ab}$ and $F(f^\dagger) = \perp_{a,b}$. Therefore

$$F(f)^\dagger = \perp_{a,ab}{}^\dagger = \perp_{a,ab} < \perp_{a,ab}{}^\dagger, 1_b > = \perp_{a,b} = F(f^\dagger) .$$

ii) $f = x_1^{ab}$. As $f^\dagger = \perp_{a,b}$ it follows that

$$F(f^\dagger) = \perp_{a,b} = 1_a^\dagger 0_b = (1_a(1_a+0_b))^\dagger = F(f)^\dagger .$$

iii) $f = x_i^{ab}$ where $2 \leq i \leq 1+|b|$. It follows that

$$F(f^\dagger) = F(x_{i-1}^b) = x_{i-1}^b = (x_{i-1}^b(0_a+1_b))^\dagger = (x_i^{ab})^\dagger = F(f)^\dagger .$$

If $|a| \geq 2$ then $a = cd$ where $|c| < |a|$ and $|d| < |a|$. It follows from (1) that

$$f^\dagger = < h^\dagger < (g < h^\dagger, 1_{db} >)^\dagger, 1_b > \ , (g < h^\dagger, 1_{db} >)^\dagger >$$

where $h = (1_c + 0_d)f$ and $g = (0_c + 1_d)f$. It follows from the inductive hypothesis that

$$F(f^\dagger) = < F(h)^\dagger < (F(g) < F(h)^\dagger, 1_{db} >)^\dagger, 1_b > \ , (F(g) < F(h)^\dagger, 1_{db} >)^\dagger$$

then by (1)

$$F(f^\dagger) = < F(h), F(g) >^\dagger = F(f)^\dagger \ \square$$

The next theorem, which answers a question of our colleague Gh. Ştefănescu, will not be used henceforth.

1.8 <u>Theorem</u>. Let T be a theory in which the iterate is defined only for morphisms with sources of length 1. Assume the following axioms:

A1 $\quad f^\dagger < (g < f^\dagger, 1_{ta} >)^\dagger, 1_a > \ = (f(S_s^t + 1_a) < (g(S_s^t + 1_a))^\dagger, 1_{sa} >)^\dagger$

for every $f \in T(s,sta)$ and $g \in T(t,sta)$,

A2 $\quad (f(1_s + g))^\dagger = f^\dagger g$

for every $f \in T(s,sa)$ and $g \in T(a,b)$,

A3 $\quad f < f^\dagger, 1_a > \ = f^\dagger$

for every $f \in T(s,sa)$.

Extend the iterate by

A4 $\quad 0_a^\dagger = 0_a$

and by induction

A5 $\quad < f, g >^\dagger = < f^\dagger < (g < f^\dagger, 1_{sb} >)^\dagger, 1_b > \ , (g < f^\dagger, 1_{sb} >)^\dagger >$

for every $f \in T(a,asb)$ and $g \in T(s,asb)$.

Then T becomes a theory with iterate.

Proof. We prove (2) by induction on $|a|$. If $|a| = 0$ we use A4 and if $|a| = 1$ then we use A2. For the inductive step we suppose $f = \langle f_1, f_2 \rangle$ where $f_1 \in T(a,asb)$ and $f_2 \in T(s,asb)$. The conclusion follows from the following calculation :

$$(f(1_{as}+g))^{\dagger} =$$

$$= \langle f_1(1_{as}+g) \ , \ f_2(1_{as}+g) \rangle^{\dagger} = \qquad \text{A5}$$

$$= \langle (f_1(1_{as}+g))^{\dagger} \langle (f_2(1_{as}+g) \langle (f_1(1_{as}+g))^{\dagger}, 1_{sc} \rangle)^{\dagger}, 1_c \rangle ,$$
$$, (f_2(1_{as}+g)) \langle (f_1(1_{as}+g))^{\dagger}, 1_{sc} \rangle)^{\dagger} \rangle =$$

by the inductive hypothesis

$$= \langle f_1^{\dagger}(1_s+g) \langle (f_2 \langle f_1^{\dagger}(1_s+g), 1_s+g \rangle)^{\dagger}, 1_c \rangle , (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle (1_s+g))^{\dagger} \rangle =$$

$$\text{A2} \quad = \langle f_1^{\dagger} \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger} g, g \rangle , (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger} g \rangle =$$

$$= \langle f_1^{\dagger} \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle , (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger} \rangle g = \qquad \text{A5}$$

$$= \langle f_1, f_2 \rangle^{\dagger} g = f^{\dagger} g .$$

We prove (3) by induction on $|a|$. If $|a| = 0$ we use A4 and if $|a| = 1$ we use A3. For the inductive step we suppose $f = \langle f_1, f_2 \rangle$ where $f_1 \in T(a,asb)$ and $f_2 \in T(s,asb)$. The conclusion follows from the following calculation :

$$f \langle f^{\dagger}, 1_b \rangle = \qquad \text{A5}$$

$$= \langle f_1, f_2 \rangle \langle f_1^{\dagger} \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle , (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle =$$

$$= \langle f_1, f_2 \rangle \langle f_1^{\dagger}, 1_{sb} \rangle \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle =$$

by the inductive hypothesis

$$= \langle f_1^{\dagger}, f_2 \langle f_1^{\dagger}, 1_{sb} \rangle \rangle \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle = \qquad \text{A3}$$

$$= \langle f_1^{\dagger} \langle (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger}, 1_b \rangle , (f_2 \langle f_1^{\dagger}, 1_{sb} \rangle)^{\dagger} \rangle = \qquad \text{A5}$$

$$= \langle f_1, f_2 \rangle^{\dagger} = f^{\dagger} .$$

We prove (1) by induction on $|b|$. If $|b| = 0$ we use A4 and if $|b| = 1$ we use A5. For the inductive step we suppose $f \in T(a,absc)$ and $g = \langle g_1, g_2 \rangle$ where $g_1 \in T(b,absc)$ and $g_2 \in T(s,absc)$. It follows from the inductive hypothesis that

$$\langle f, g_1 \rangle^{\dagger} = \langle f^{\dagger} \langle (g_1 \langle f^{\dagger}, 1_{bsc} \rangle)^{\dagger}, 1_{sc} \rangle , (g_1 \langle f^{\dagger}, 1_{bsc} \rangle)^{\dagger} \rangle$$

therefore

$$\langle\langle f,g_1\rangle^\dagger, 1_{sc}\rangle = \langle f^\dagger, 1_{bsc}\rangle \langle (g_1\langle f^\dagger, 1_{bsc}\rangle)^\dagger, 1_{sc}\rangle .$$

Let $A = (g_2\langle\langle f,g_1\rangle^\dagger, 1_{sc}\rangle)^\dagger$. It follows that

$$(g\langle f^\dagger, 1_{bsc}\rangle)^\dagger =$$

$$= \langle g_1\langle f^\dagger,1_{bsc}\rangle, g_2\langle f^\dagger,1_{bsc}\rangle\rangle^\dagger = \qquad \text{A5}$$

$$= \langle (g_1\langle f^\dagger,1_{bsc}\rangle)^\dagger\langle A,1_c\rangle, A\rangle .$$

Therefore

$$\langle f^\dagger\langle (g\langle f^\dagger,1_{bsc}\rangle)^\dagger,1_c\rangle, (g\langle f^\dagger,1_{bsc}\rangle)^\dagger\rangle =$$

$$= \langle f^\dagger\langle (g_1\langle f^\dagger,1_{bsc}\rangle)^\dagger,1_{sc}\rangle\langle A,1_c\rangle, (g_1\langle f^\dagger,1_{bsc}\rangle)^\dagger\langle A,1_c\rangle, A\rangle =$$

$$= \langle\langle f^\dagger\langle (g_1\langle f^\dagger,1_{bsc}\rangle)^\dagger,1_{sc}\rangle, (g_1\langle f^\dagger,1_{bsc}\rangle)^\dagger\rangle\langle A,1_c\rangle, A\rangle =$$

$$= \langle\langle f,g_1\rangle^\dagger\langle A,1_c\rangle, A\rangle = \qquad \text{A5}$$

$$= \langle\langle f,g_1\rangle, g_2\rangle^\dagger = \langle f,g\rangle^\dagger .$$

Thus the proof of (1) is over and we still have to prove (1').

It is easy to see that if $|a| = 0$ or $|b| = 0$ then (1') is true.

We prove (1') in the case $|a| = |b| = 1$. Let $f\in T(s,sta)$ and $g\in T(t,sta)$. It follows from A1 that

$$(g(S^t_s+1_a))^\dagger\langle (f(S^t_s+1_a)\langle (g(S^t_s+1_a))^\dagger,1_{sa}\rangle)^\dagger,1_a\rangle = (g\langle f^\dagger,1_{ta}\rangle)^\dagger$$

therefore by A1 and tupling

$$\langle f^\dagger\langle (g\langle f^\dagger,1_{ta}\rangle)^\dagger,1_a\rangle, (g\langle f^\dagger,1_{ta}\rangle)^\dagger\rangle =$$

$$= \langle (f(S^t_s+1_a)\langle (g(S^t_s+1_a))^\dagger,1_{sa}\rangle)^\dagger,$$

$$(g(S^t_s+1_a))^\dagger\langle (f(S^t_s+1_a)\langle (g(S^t_s+1_a))^\dagger,1_{sa}\rangle)^\dagger,1_a\rangle$$

then by A5

$$\langle f,g\rangle^\dagger = S^t_s\langle g(S^t_s+1_a), f(S^t_s+1_a)\rangle^\dagger .$$

We prove (1') in the case $|a| = 1$ by induction on $|b|$. Let $f\in T(t,tasc)$ and $g = \langle g_1,g_2\rangle$ where $g_1\in T(a,tasc)$ and $g_2\in T(s,tasc)$.

It follows from A5 that

$$\langle f,g\rangle^\dagger =$$

$$= \langle\langle f,g_1\rangle^\dagger\langle (g_2\langle\langle f,g_1\rangle^\dagger,1_{sc}\rangle)^\dagger,1_c\rangle, (g_2\langle\langle f,g_1\rangle^\dagger,1_{sc}\rangle)^\dagger\rangle$$

We put $\bar h = h(S^a_t+1_{sc})$ for every morphism h of target $tasc$.

By the inductive hypothesis

$$\langle f,g_1\rangle^\dagger = S^a_t\langle \bar g_1,\bar f\rangle^\dagger$$

then

$$\langle \langle f, g_1 \rangle^{\dagger}, 1_{sc} \rangle = (S_t^{s}+1_{sc}) \langle \langle \overline{g}_1, \overline{f} \rangle^{\dagger}, 1_{sc} \rangle$$

therefore using A5 and (1)

$$\langle f, g \rangle^{\dagger} =$$

$$= \langle S_t^{s} \langle \overline{g}_1, \overline{f} \rangle^{\dagger} \langle (\overline{g}_2 \langle \langle \overline{g}_1, \overline{f} \rangle^{\dagger}, 1_{sc} \rangle)^{\dagger}, 1_c \rangle, (\overline{g}_2 \langle \langle \overline{g}_1, \overline{f} \rangle^{\dagger}, 1_{sc} \rangle)^{\dagger} \rangle =$$

$$= (S_t^{a}+1_s) \langle \langle \overline{g}_1, \overline{f} \rangle, \overline{g}_2 \rangle^{\dagger} = (S_t^{a}+1_s) \langle \overline{g}_1, \langle \overline{f}, \overline{g}_2 \rangle \rangle^{\dagger} =$$

$$= (S_t^{s}+1_s) \langle \overline{g}_1^{\dagger} \langle (\langle \overline{f}, \overline{g}_2 \rangle \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle)^{\dagger}, 1_c \rangle, (\langle \overline{f}, \overline{g}_2 \rangle \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle)^{\dagger} \rangle$$

We put $\underline{h} = h(S_t^{as}+1_c)$ for every morphism h of target $tasc$.

It follows from

$$(S_t^{a}+1_{sc})(1_a+S_t^{s}+1_c) = S_t^{as}+1_c$$

that $\overline{h}(1_a+S_t^{s}+1_c) = \underline{h}$ for every morphism h of target $tasc$.

Further

$$(\langle \overline{f}, \overline{g}_2 \rangle \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle)^{\dagger} =$$

$$= \langle \overline{f} \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle, \overline{g}_2 \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle \rangle^{\dagger} = \qquad \text{(1') case } |a| = |b| = 1$$

$$= S_t^{s} \langle \overline{g}_2 \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle (S_t^{s}+1_c), \overline{f} \langle \overline{g}_1^{\dagger}, 1_{tsc} \rangle (S_t^{s}+1_c) \rangle^{\dagger} = \qquad (2)$$

$$= S_t^{s} (\langle \overline{g}_2, \overline{f} \rangle (1_a+S_t^{s}+1_c)(1_a+S_s^{t}+1_c) \langle (\overline{g}_1(1_a+S_t^{s}+1_c))^{\dagger}, S_t^{s}+1_c \rangle)^{\dagger} =$$

$$= S_t^{s} (\langle \underline{g_2}, \underline{f} \rangle \langle \underline{g_1}^{\dagger}, 1_{stc} \rangle)^{\dagger}$$

therefore

$$\langle f, g \rangle^{\dagger} =$$

$$= (S_t^{a}+1_s) \langle \overline{g}_1^{\dagger} \langle S_t^{s} (\langle \underline{g_2}, \underline{f} \rangle \langle \underline{g_1}^{\dagger}, 1_{stc} \rangle)^{\dagger}, 1_c \rangle, S_t^{s} (\langle \underline{g_2}, \underline{f} \rangle \langle \underline{g_1}^{\dagger}, 1_{stc} \rangle)^{\dagger} \rangle =$$

$$= (S_t^{a}+1_s)(1_a+S_t^{s}) \langle \underline{g_1}^{\dagger} \langle (\langle \underline{g_2}, \underline{f} \rangle \langle \underline{g_1}^{\dagger}, 1_{stc} \rangle)^{\dagger}, 1_c \rangle, (\langle \underline{g_2}, \underline{f} \rangle \langle \underline{g_1}^{\dagger}, 1_{stc} \rangle)^{\dagger} \rangle =$$

$$(1) \quad = S_t^{as} \langle \underline{g_1}, \langle \underline{g_2}, \underline{f} \rangle \rangle^{\dagger} =$$

$$= S_t^{as} \langle g(S_t^{as}+1_c), f(S_t^{as}+1_c) \rangle^{\dagger}.$$

We finish the proof of (1') by induction on $|a|$. Let $f = \langle f_1, f_2 \rangle$ where $f_1 \in T(a, asbc)$ and $f_2 \in T(s, asbc)$. Let $g \in T(b, asbc)$. It follows by the inductive hypothesis that

$$\langle f, g \rangle^{\dagger} = S_a^{sb} \langle \langle f_2, g \rangle (S_a^{sb}+1_c), f_1(S_a^{sb}+1_c) \rangle^{\dagger} =$$

$$= S_a^{sb} \langle f_2(S_a^{sb}+1_c), \langle g, f_1 \rangle (S_a^{sb}+1_c) \rangle^{\dagger}.$$

We deduce from (1') case $|a| = 1$ that

$$\langle f, g\rangle^{\dagger} = S_a^{ab}S_a^{ba}\langle\langle g,f_1\rangle(S_a^{ab}+1_c)(S_a^{ba}+1_c)\ ,\ f_2(S_a^{ab}S_a^{ba}+1_c)\rangle =$$

$$= S_{aa}^{b}\langle g(S_{aa}^{b}+1_c)\ ,\ f(S_{aa}^{b}+1_c)\rangle^{\dagger}\ \square$$

2. T-modules

Let T be an S-sorted theory.

2.1 Definition. A T-module is a category Q together with a functor $H : T \longrightarrow Q$ such that the following conditions are fulfilled :

a) The class of objects of Q is S^*,

b) For every $a,b,c \in S^*$, an operation

$$\langle\ ,\ \rangle : Q(a,c) \times Q(b,c) \longrightarrow Q(ab,c)$$

is given, called (source) tupling,

c) The functor H preserves the objects and the tupling operation,

d) The following axioms are fulfilled :

C1 $\qquad \langle\alpha,\langle\beta,\gamma\rangle\rangle = \langle\langle\alpha,\beta\rangle,\gamma\rangle$

for every $\alpha\in Q(a,d)$, $\beta\in Q(b,d)$ and $\gamma\in Q(c,d)$,

C2 $\qquad \langle\alpha,H(0_b)\rangle = \langle H(0_b),\alpha\rangle = \alpha$

for every $\alpha\in Q(a,b)$,

C3 $\qquad \langle\alpha,\beta\rangle H(f) = \langle\alpha H(f), \beta H(f)\rangle$

for every $\alpha\in Q(a,c)$, $\beta\in Q(b,c)$ and $f\in T(c,d)$,

C4 $\qquad H(f+g)\langle\alpha,\beta\rangle = \langle H(f)\alpha , H(g)\beta\rangle$

for every $f\in T(a,b)$, $g\in T(c,d)$, $\alpha\in Q(b,e)$ and $\beta\in Q(d,e)$,

C5 $\qquad \langle\alpha H(0_b+1_c),1_{bc}\rangle\langle\beta,1_c\rangle = \langle\alpha,\beta,1_c\rangle$

for every $\alpha\in Q(a,c)$ and $\beta\in Q(b,c)$ $\square$

In view of exuoms C1 and C2, the tupling operation can be extended to an arbitrary number of morphisms with the same target. The tupling of no morphism of target a is by definition $H(0_a)$. The tupling of one morphism α is by definition α.

2.2 Lemma. Let $H : T \longrightarrow Q$ be a T-module. If $\alpha,\beta\in Q(a,b)$ then

$$\langle\alpha,1_b\rangle = \langle\beta,1_b\rangle \text{ implies } \alpha=\beta .$$

Proof. $\alpha = \langle\alpha,H(0_b)\rangle = H(1_a+0_b)\langle\alpha,1_b\rangle = H(1_a+0_b)\langle\beta,1_b\rangle =$
$= \langle\beta,H(0_b)\rangle = \beta$ $\square$

We introduce in every T-module $H : T \longrightarrow Q$, for each $a,b,c,d \in S^*$, an operation

$$+ : Q(a,b) \times Q(c,d) \longrightarrow Q(ac,bd)$$

which we call sum and is defined for each $\alpha \in Q(a,b)$ and $\beta \in Q(c,d)$ by

$$\alpha + \beta = \langle \alpha H(1_b + 0_d) , \beta H(0_b + 1_d) \rangle .$$

It follows from axioms C1, C2 and C3 that the sum is associative and has the neutral element $H(0_\lambda) = 1_\lambda$.

Notise that H preserves sum.

Notice that

$$(\alpha + \beta) H(\langle f,g \rangle) = \langle \alpha H(f) , \beta H(g) \rangle$$

for every $\alpha \in Q(a,b)$, $\beta \in Q(c,d)$, $f \in T(b,e)$ and $g \in T(d,e)$.

Let T be a theory with iterate.

2.3 <u>Definition</u>. A <u>T-module</u> $H : T \longrightarrow Q$ is said to be <u>with iterate</u> if

i) for each $a,b \in S^*$ an iterate

$$^\dagger : Q(a,ab) \longrightarrow Q(a,b)$$

is defined,

ii) H preserves the iterate and

iii) the following axioms are fulfilled :

I1 $\quad (\alpha(1_a + \beta))^\dagger = \alpha^\dagger \beta$

for every $\alpha \in Q(a,ab)$ and $\beta \in Q(b,c)$,

I2 $\quad (\alpha H(0_a + 1_b))^\dagger = \alpha$

for every $\alpha \in Q(a,b)$,

I3 $\quad \langle\langle \alpha , \beta H(0_a + 1_{bc}) \rangle^\dagger , 1_c \rangle = \langle \alpha^\dagger , 1_{bc} \rangle \langle \beta^\dagger , 1_c \rangle$

for every $\alpha \in Q(a,abc)$ and $\beta \in Q(b,bc)$,

I4 $\quad (H(f) \langle \alpha^\dagger , 1_{bc} \rangle)^\dagger =$

$$= H((f(0_a^b + 1_c))^\dagger) \langle (\alpha H((0_a^b + 1_c) \langle (f(0_a^b + 1_c))^\dagger , 1_{ac} \rangle))^\dagger , 1_c \rangle$$

for every $f \in T(b,abc)$ and $\alpha \in Q(a,abc)$. □

2.4 <u>Proposition</u>. Let $H : T \longrightarrow Q$ be a T-module with iterate. If $\alpha \in Q(a,ac)$ and $\beta \in Q(b,bc)$ then

$$\langle \alpha H(1_a+0_b+1_c), \beta H(0_a+1_{bc}) \rangle^{\dagger} = \langle \alpha^{\dagger}, \beta^{\dagger} \rangle .$$

Proof. $\langle \langle \alpha H(1_a+0_b+1_c), \beta H(0_a+1_{bc}) \rangle^{\dagger}, 1_c \rangle =$ I3

$= \langle (\alpha H(1_a+0_b+1_c))^{\dagger}, 1_{bc} \rangle \langle \beta^{\dagger}, 1_c \rangle =$

$= \langle (\alpha (1_a + H(0_b+1_c)))^{\dagger}, 1_{bc} \rangle \langle \beta^{\dagger}, 1_c \rangle =$ I1

$= \langle \alpha^{\dagger} H(0_b+1_c), 1_{bc} \rangle \langle \beta^{\dagger}, 1_c \rangle =$ C5

$= \langle \alpha^{\dagger}, \beta^{\dagger}, 1_c \rangle = \langle \langle \alpha^{\dagger}, \beta^{\dagger} \rangle, 1_c \rangle$

The proof is concluded by applying lemma 2.2 □

Remark. Let $H : T \longrightarrow Q$ be a T-module with iterate. If $\alpha \in Q(a,bc)$ and $\beta \in Q(b,c)$ then

$$\langle \alpha, H(0_b+1_c) \rangle \langle \beta, 1_c \rangle = \langle \alpha \langle \beta, 1_c \rangle, 1_c \rangle .$$

Hint. Evaluate

$$\langle H(1_a+0_{bc}) \langle \langle \alpha H(0_a+1_{bc}), \beta H(0_{ab}+1_c) \rangle^{\dagger}, 1_c \rangle, 1_c \rangle$$

in two different ways.

2.5 Proposition. If T is a theory with iterate then $1_T : T \longrightarrow T$ is a T-module with iterate.

Proof. Routine calculations. E.g. for I3 :

$\langle \langle \alpha, \beta H(0_a+1_{bc}) \rangle^{\dagger}, 1_c \rangle =$ (1)

$= \langle \langle \alpha^{\dagger} \langle \beta^{\dagger}, 1_c \rangle, \beta^{\dagger} \rangle, 1_c \rangle =$

$= \langle \alpha^{\dagger} \langle \beta^{\dagger}, 1_c \rangle, \langle \beta^{\dagger}, 1_c \rangle \rangle =$

$= \langle \alpha^{\dagger}, 1_{bc} \rangle \langle \beta^{\dagger}, 1_c \rangle .$

Notice that I2 follows from (4) and I4 from (7) □

2.6 The T-module of pairs. [If T is a theory with iterate then] we shall obtain from every T-module ~~with iterate~~ $H : T \longrightarrow Q$, a ~~other~~ T-module with iterate

$$G : T \longrightarrow R .$$

The construction may seem artificial and the verification of axioms is long, but the fruits will be gathered in the next sections.

By definition

$$R(a,b) = \{ (f,\alpha) \mid f \in T(a,pb), \alpha \in Q(q,pb) ; p,q \in S^* \} .$$

If (f,α) is a morphism in R from a to b, $(f,\alpha) \in R(a,b)$, then the words $p,q \in S^*$ such that $f \in T(a,pb)$ and $\alpha \in Q(q,pb)$ are unique. In spite of this, we prefer to write (f,α,p,q) instead of (f,α).

Let $D = (f,\alpha,p,q) \in R(a,b)$ and $D' = (f',\alpha',p',q') \in R(b,c)$. We define the composition by

$$DD' = \left(f(1_p+f') \ , \ \langle \alpha H(1_p+f'), \alpha' H(0_p+1_{p'c}) \rangle \ , \ pp' \ , qq' \right)$$

Let us prove the associativity of composition. If $D'' = (f'',\alpha'',p'',q'') \in R(c,d)$ then

$$(DD')D'' =$$

$$= \big(f(1_p+f')(1_{pp'}+f''), \langle\langle \alpha H(1_p+f'), \alpha' H(0_p+1_{p'c})\rangle H(1_{pp'}+f''), \alpha'' H(0_{pp'}+1_{p''d})\rangle, (pp')p'' , (qq')q'' \big) =$$

$$= \big(f(1_p+f'(1_{p'}+f'')) \ ,$$
$$\langle \alpha H(1_p+f'(1_{p'}+f'')), \langle \alpha' H(1_{p'}+f''), \alpha'' H(0_{p'}+1_{p''d}) \rangle H(0_p+1_{p'p''d}) \rangle, \ p(p'p'') \ , \ q(q'q'') \big) =$$

$$= D \left(f'(1_{p'}+f'') \ , \ \langle \alpha' H(1_{p'}+f''), \alpha'' H(0_{p'}+1_{p''d}) \rangle \ , \ p'p'' \ , \ q'q'' \right) =$$

$$= D(D'D'') .$$

For each $f \in T(a,b)$ we define

$$G(f) = (f, H(0_b), \lambda, \lambda) \in R(a,b) .$$

It follows from the definition of composition that :

a) $$G(g)\,(f,\alpha,p,q) = (gf,\alpha,p,q)$$

for every $g \in T(c,a)$ and $(f,\alpha,p,q) \in R(a,b)$,

b) $$(f,\alpha,p,q)\,G(g) = \left(f(1_p+g) \ , \ \alpha H(1_p+g) \ , \ p \ , \ q \right)$$

for every $(f,\alpha,p,q) \in R(a,b)$ and $g \in T(b,c)$.

It follows from the above properties that $G(1_a)$ is an identity morphism of R and G is a functor.

Let $D = (f,\alpha,p,q) \in R(a,c)$ and $D' = (f',\alpha',p',q') \in R(b,c)$. We define the tupling operation by

$$\langle D,D' \rangle =$$

$$= \left(\langle f(1_p+0_{p'}+1_c), f'(0_p+1_{p'c}) \rangle \ , \ \langle \alpha H(1_p+0_{p'}+1_c), \alpha' H(0_p+1_{p'c}) \rangle \ , pp', qq' \right)$$

It follows that :

a) $\langle G(g), (f,\alpha,p,q)\rangle = (\langle g(0_p+1_c), f\rangle, \alpha, p, q)$

for every $g \in T(a,c)$ and $(f,\alpha,p,q) \in R(b,c)$,

b) $\langle (f,\alpha,p,q), G(g)\rangle = (\langle f, g(0_p+1_c)\rangle, \alpha, p, q)$

for every $(f,\alpha,p,q) \in R(a,c)$ and $g \in T(b,c)$.

Therefore R preserves the tupling operation and C2 is fulfilled.

We prove C1. If $D'' = (f'',\alpha'',p'',q'') \in R(d,c)$ then

$$\langle\langle D,D'\rangle, D''\rangle =$$

$$= (\langle\langle f(1_p+0_{p'}+1_c), f'(0_p+1_{p'c})\rangle(1_{pp'}+0_{p''}+1_c), f''(0_{pp'}+1_{p''c})\rangle,$$

$$\langle\langle \alpha H(1_p+0_{p'}+1_c), \alpha' H(0_p+1_{p'c})\rangle H(1_{pp'}+0_{p''}+1_c), \alpha'' H(0_{pp'}+1_{p''c})\rangle,$$

$$(pp')p'', (qq')q'') =$$

$$= (\langle f(1_p+0_{p'p''}+1_c), \langle f'(1_{p'}+0_{p''}+1_c), f''(0_{p'}+1_{p''c})\rangle(0_p+1_{p'p''c})\rangle,$$

$$\langle \alpha H(1_p+0_{p'p''}+1_c), \langle \alpha' H(1_{p'}+0_{p''}+1_c), \alpha'' H(0_{p'}+1_{p''c})\rangle H(0_p+1_{p'p''c})\rangle,$$

$$p(p'p''), q(q'q'')) =$$

$$= \langle D, \langle D',D''\rangle\rangle.$$

We prove C3. If $g \in T(c,d)$ then

$$\langle DG(g), D'G(g)\rangle =$$

$$= \langle (f(1_p+g), \alpha H(1_p+g), p, q), (f'(1_{p'}+g), \alpha' H(1_{p'}+g), p', q')\rangle =$$

$$= (\langle f(1_p+g)(1_p+0_{p'}+1_d), f'(1_{p'}+g)(0_p+1_{p'd})\rangle,$$

$$\langle \alpha H(1_p+g)H(1_p+0_{p'}+1_d), \alpha' H(1_{p'}+g)H(0_p+1_{p'd})\rangle, pp', qq') =$$

$$= (\langle f(1_p+0_{p'}+1_c), f'(0_p+1_{p'c})\rangle(1_{pp'}+g),$$

$$\langle \alpha H(1_p+0_{p'}+1_c), \alpha' H(0_p+1_{p'c})\rangle H(1_{pp'}+g), pp', qq') =$$

$$= \langle D,D'\rangle G(g).$$

We prove C4. If $g \in T(d,e)$ and $h \in T(e,b)$ then

$$G(g+h)\langle D,D'\rangle =$$

$$=(\langle gf(1_p+0_{p'}+1_c), hf'(0_p+1_{p'c})\rangle, \langle \alpha H(1_p+0_{p'}+1_c), \alpha' H(0_p+1_{p'c})\rangle, pp', qq') =$$

$$= \langle G(g)D, G(g)D'\rangle.$$

We prove C5.

$$\langle DG(0_b+1_c), 1_{bc}\rangle\langle D', 1_c\rangle =$$

$= (\langle f(1_p+0_b+1_c), 0_p+1_{bc}\rangle, \alpha H(1_p+0_b+1_c), p, q)\ (\langle f', 0_{p'}+1_c\rangle, \alpha', p', q') =$

$= (\langle f(1_p+0_b+1_c), 0_p+1_{bc}\rangle(1_p+\langle f', 0_{p'}+1_c\rangle),$

$\langle \alpha H(1_p+0_b+1_c)H(1_p+\langle f', 0_{p'}+1_c\rangle), \alpha' H(0_p+1_{p'c})\rangle ; pp', qq') =$

$= (\langle f(1_p+0_{p'}+1_c), f'(0_p+1_{p'c}), 0_{pp'}+1_c\rangle,$

$\langle \alpha H(1_p+0_{p'}+1_c), \alpha' H(0_p+1_{p'c})\rangle, pp', qq') =$

$= \langle D, D', 1_c\rangle$.

We define the iterate by

$$D^{\dagger} = ((f(S_p^a+1_b))^{\dagger}, \alpha H((S_p^a+1_b)\langle (f(S_p^a+1_b))^{\dagger}, 1_{pb}\rangle), p, q)$$

for every $D = (f, \alpha, p, q) \in R(a, ab)$.

It is easy to prove that G preserves the iterate.

We prove I1. Let $D = (f, \alpha, p, q) \in R(a, ab)$ and $D' = (f', \alpha', p', q') \in R(b, c)$. As

$1_a + D' = \langle G(1_a+0_c), D'G(0_a+1_c)\rangle =$

$= (\langle 0_{p'}+1_a+0_c, f'(1_{p'}+0_a+1_c)\rangle, \alpha' H(1_{p'}+0_a+1_c), p', q')$

it follows that

$D(1_a+D') = (f(1_p+\langle 0_{p'}+1_a+0_c, f'(1_{p'}+0_a+1_c)\rangle),$

$\langle \alpha H(1_p+\langle 0_{p'}+1_a+0_c, f'(1_{p'}+0_a+1_c)\rangle), \alpha' H(0_p+1_{p'}+0_a+1_c)\rangle, pp', qq')$.

We calculate the first component of $(D(1_a+D'))^{\dagger}$:

$(f\langle 1_p+0_{p'ac}, 0_{pp'}+1_a+0_c, f'(0_p+1_{p'}+0_a+1_c)\rangle(S_{pp'}^a+1_c))^{\dagger} =$

$= (f(S_p^a+1_b)\langle 1_a+0_{pp'c}, 0_a+1_p+0_{p'c}, 0_{ap}+f'\rangle)^{\dagger} =$

$= (f(S_p^a+1_b)(1_a+(1_p+f')))^{\dagger} =$ (2)

$= (f(S_p^a+1_b))^{\dagger}(1_p+f')$.

Therefore $(D(1_a+D'))^{\dagger} =$

$= ((f(S_p^a+1_b))^{\dagger}(1_p+f'),$

$\langle \alpha H(S_p^a+1_b)H(1_{ap}+f'), \alpha' H(0_{ap}+1_{p'c})\rangle H(\langle (f(S_p^a+1_b))^{\dagger}(1_p+f'), 1_{pp'c}\rangle),$
$pp', qq') =$

$= ((f(S_p^a+1_b))^{\dagger}(1_p+f'),$

$\langle \alpha H((S_p^a+1_b)\langle (f(S_p^a+1_b))^{\dagger}, 1_{pb}\rangle)H(1_p+f'), \alpha' H(0_p+1_{p'c})\rangle, pp', qq') =$

$= D^\dagger D'$.

We prove I2. If $D = (f, \alpha, p, q) \in R(a,b)$ then $(DG(0_a+1_b))^\dagger =$

$= (f(1_p+0_a+1_b) , \alpha H(1_p+0_a+1_b) , p , q)^\dagger = (f, \alpha, p, q) = D$.

We prove I3. Let $D = (f, \alpha, p, q) \in R(a,abc)$ and

$D' = (f', \alpha', p', q') \in R(b,bc)$.

Let us put $F = (f(S^a_p+1_{bc}))^\dagger$ and $F' = (f'(S^b_{p'}+1_c))^\dagger$. As

$\langle D^\dagger, 1_{bc} \rangle = (\langle F, 0_p+1_{bc} \rangle , \alpha H((S^a_p+1_{bc}) \langle F, 1_{pbc} \rangle) , p , q)$ and

$\langle D'^\dagger, 1_c \rangle = (\langle F', 0_{p'}+1_c \rangle , \alpha' H((S^b_{p'}+1_c) \langle F', 1_{p'c} \rangle) , p' , q')$

therefore

$$\langle D^\dagger, 1_{bc} \rangle \langle D'^\dagger, 1_c \rangle =$$

$$= (\langle F, 0_p+1_{bc} \rangle (1_p+ \langle F', 0_{p'}+1_c \rangle) ,$$

$$\langle \alpha H((S^a_p+1_{bc}) \langle F, 1_{pbc} \rangle (1_p+ \langle F', 0_{p'}+1_c \rangle)), \alpha' H((S^b_{p'}+1_c) \langle F', 1_{p'c} \rangle (0_p+1_{p'c})) \rangle ,$$

$$pp' , qq') =$$

$$= (\langle F(1_p+ \langle F', 0_{p'}+1_c \rangle) , 0_p+F' , 0_{pp'}+1_c \rangle ,$$

$$\langle \alpha H((S^a_p+1_{bc}) \langle F(1_p+ \langle F', 0_{p'}+1_c \rangle), 1_p+0_{p'c}, 0_p+F', 0_{pp'}+1_c \rangle) ,$$

$$\alpha' H((S^b_{p'}+1_c) \langle 0_p+F', 0_p+1_{p'c} \rangle) \rangle , pp' , qq') .$$

We pass to the calculation of left-hand side of I3.

$D'G(0_a+1_{bc}) = (f'(1_{p'}+0_a+1_{bc}) , \alpha' H(1_{p'}+0_a+1_{bc}) , p' , q')$.

$\langle D , D'G(0_a+1_{bc}) \rangle = (\langle f(1_p+0_{p'}+1_{abc}) , f'(0_p+1_{p'}+0_a+1_{bc}) \rangle ,$

$\langle \alpha H(1_p+0_{p'}+1_{abc}), \alpha' H(0_p+1_{p'}+0_a+1_{bc}) \rangle , pp', qq')$.

We calculate the first component of $\langle D , D'G(0_a+1_{bc}) \rangle^\dagger$:

$F'' = (\langle f(1_p+0_{p'}+1_{abc}), f'(0_p+1_{p'}+0_a+1_{bc}) \rangle (S^{ab}_{pp'}+1_c))^\dagger =$

$= \langle f(S^a_p+1_{bc})(1_a+S^b_p+0_{p'}+1_c), f'(S^b_{p'}+1_c)(0_a+1_b+0_p+1_{p'c}) \rangle^\dagger =$ (1),(2)

$= \langle F(S^b_p+0_{p'}+1_c) \langle F'(0_p+1_{p'c}), 1_{pp'c} \rangle , F'(0_p+1_{p'c}) \rangle =$

$= \langle F(1_p+ \langle F', 0_{p'}+1_c \rangle) , 0_p+F' \rangle$.

It follows that $\langle D , D'G(0_a+1_{bc}) \rangle^\dagger =$

$= (F'' , \langle \alpha H((S^a_p+1_{bc})(1_a+S^b_p+0_{p'}+1_c) \langle F'', 1_{pp'c} \rangle) ,$

$$\alpha' H((S^b_{p'}+1_c)(0_a+1_b+0_p+1_{p'c}) \langle F'', 1_{pp'c} \rangle) \rangle , pp' , qq') =$$

$$= (\langle F(1_p + \langle F', 0_{p'}+1_c \rangle) , 0_p+F' \rangle ,$$

$$\langle \alpha H((S^a_p+1_{bc}) \langle F(1_p+\langle F', 0_{p'}+1_c \rangle), 1_p+0_{p'c}, 0_p+F', 0_{pp'}+1_c \rangle) ,$$

$$\alpha' H(S^b_{p'}+1_c) \langle 0_p+F', 0_p+1_{p'c} \rangle) \quad pp' , qq') .$$

It follows easily from this that

$$\langle \langle D , D'G(0_a+1_{bc}) \rangle^\dagger , 1_c \rangle = \langle D^\dagger , 1_{bc} \rangle \langle D'^\dagger , 1_c \rangle .$$

We still have to prove I4. Let $D = (f, \alpha, p, q) \in R(a, abc)$ and $g \in T(b, abc)$.

Let us put

$$i = f(S^{ab}_p+1_c) \in T(a, abpc)$$

and $$j = g(1_{ab}+0_p+1_c) \in T(b, abpc) .$$

It follows from (2) that

$$(f(S^a_p+1_{bc}))^\dagger (S^b_p+1_c) = (f(S^a_p+1_{bc})(1_a+S^b_p+1_c))^\dagger = i^\dagger .$$

We prepare the calculation of the left-hand side of I4.

$$G(g) \langle D^\dagger , 1_{bc} \rangle =$$

$$= (g \langle (f(S^a_p+1_{bc}))^\dagger , 0_p+1_{bc} \rangle , \alpha H((S^a_p+1_{bc}) \langle (f(S^a_p+1_{bc}))^\dagger , 1_{pbc} \rangle), p, q)$$

The first component of $(G(g) \langle D^\dagger , 1_{bc} \rangle)^\dagger$ is

$$(g \langle (f(S^a_p+1_{bc}))^\dagger , 0_p+1_{bc} \rangle (S^b_p+1_c))^\dagger = (g \langle i^\dagger , 1_b+0_p+1_c \rangle)^\dagger = (j \langle i^\dagger , 1_{bpc} \rangle)^\dagger$$

therefore

$$(G(g) \langle D^\dagger , 1_{bc} \rangle)^\dagger = ((j \langle i^\dagger , 1_{bpc} \rangle)^\dagger ,$$

$$\alpha H((S^a_p+1_{bc}) \langle (f(S^a_p+1_{bc}))^\dagger , 1_{pbc} \rangle (S^b_p+1_c) \langle (j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , 1_{pc} \rangle), p, q) =$$

$$= ((j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , \alpha H((S^a_p+1_{bc}) \langle i^\dagger , S^b_p+1_c \rangle \langle (j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , 1_{pc} \rangle), p, q) =$$

$$= ((j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , \alpha H((S^{ab}_p+1_c) \langle i^\dagger , 1_{bpc} \rangle \langle (j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , 1_{pc} \rangle), p, q) =$$

$$= ((j \langle i^\dagger , 1_{bpc} \rangle)^\dagger , \alpha H((S^{ab}_p+1_c) \langle \langle i, j \rangle^\dagger , 1_{pc} \rangle) , p , q) .$$

The last equality follows from (1).

Now (2) implies that

$$(g(S^b_a+1_c))^\dagger (1_a+0_p+1_c) = (g(S^b_a+1_c)(1_{ba}+0_p+1_c))^\dagger =$$

$$= (g(1_{ab}+0_p+1_c)(S_a^b+1_{pc}))^\dagger = (j(S_a^b+1_{pc}))^\dagger .$$

We prepare the calculation of the right-hand side of I4.

$$D\ G((S_a^b+1_c) < (g(S_a^b+1_c))^\dagger, 1_{ac} >) =$$

$$= (f(1_p+(S_a^b+1_c) < (g(S_a^b+1_c))^\dagger, 1_{ac} >) ,$$

$$\alpha H(1_p+(S_a^b+1_c) < (g(S_a^b+1_c))^\dagger, 1_{ac} >) , p , q) .$$

Further, since

$$(1_p+(S_a^b+1_c) < (g(S_a^b+1_c))^\dagger, 1_{ac} >)(S_p^a+1_c) =$$

$$= < 0_a+1_p+0_c , (S_a^b+1_c) < (g(S_a^b+1_c))^\dagger, 1_{ac} > (1_a+0_p+1_c) > =$$

$$= < 0_a+1_p+0_c , (S_a^b+1_c) < (j(S_a^b+1_{pc}))^\dagger, 1_a+0_{pc}, 0_{ap}+1_c > > =$$

$$= < 0_a+1_p+0_c , 1_a+0_{pc} , (j(S_a^b+1_{pc}))^\dagger , 0_{ap}+1_c > =$$

$$= (S_p^{ab}+1_c)(S_a^b+1_{pc}) < (j(S_a^b+1_{pc}))^\dagger, 1_{apc} >$$

it follows that if we put

$$A = (i(S_a^b+1_{pc}) < (j(S_a^b+1_{pc}))^\dagger , 1_{apc} >)^\dagger$$

then

$$(D\ G((S_a^b+1_c) < (g(S_a^b+1_c))^\dagger , 1_{ac} >))^\dagger =$$

$$= (A , \alpha H((S_p^{ab}+1_c)(S_a^b+1_{pc}) < (j(S_a^b+1_{pc}))^\dagger, 1_{apc} > < A, 1_{pc} >) , p , q)$$

Using in turn (1) and (1'), we see that the right-hand side of I4 is

$$((g(S_a^b+1_c))^\dagger < A , 0_p+1_c > ,$$

$$\alpha H((S_p^{ab}+1_c) < S_a^b < (j(S_a^b+1_{pc}))^\dagger < A, 1_{pc} >, A >, 1_{pc} >) , p, , q) =$$

$$= ((g(S_a^b+1_c))^\dagger (1_a+0_p+1_c) < A, 1_{pc} > ,$$

$$\alpha H((S_p^{ab}+1_c) < S_a^b < j(S_a^b+1_{pc}), i(S_a^b+1_{pc}) >^\dagger , 1_{pc} >) , p , q) =$$

$$= ((j(S_a^b+1_{pc}))^\dagger < A, 1_{pc} > , \alpha H((S_p^{ab}+1_c) < < i, j >^\dagger, 1_{pc} >) , p , q) .$$

We deduce from (7) that it is equal to the left-hand side of I4.

We have proved that $G : T \longrightarrow R$ is a T-module with iterate.

It will be called the <u>T-module of pairs</u> over $H : T \longrightarrow Q$.

-

2.7 <u>Definition</u>. Let $H : T \longrightarrow Q$ be a T-module with iterate. A subcategory of Q having the same objects as Q will be called <u>a T-submodule with iterate of $H : T \longrightarrow Q$</u> if it is closed under tupling operation and iterate and if it contain $H(f)$ for every $f \in T(a,b)$ □

2.8 <u>The T-module of normal descriptions</u>. Let T be a theory with iterate, let $H : T \longrightarrow Q$ be a T-module ~~with iterate~~ and let $G : T \longrightarrow R$ be its T-module of pairs. A morphism (f,α,p,q) from R is called a <u>normal description</u> over $H : T \longrightarrow Q$ if $p = q$. All the normal descriptions over $H : T \longrightarrow Q$ form a T-submodule with iterate of $G : T \longrightarrow R$, denoted by ND_Q. We shall prefer to write (f,α,p) instead of (f,α,p,p) for every normal description.

2.9 <u>Definition</u>. Let $H : T \longrightarrow Q$ and $H' : T \longrightarrow Q'$ be two T-modules with iterate. A functor $F : Q \longrightarrow Q'$ is called a <u>morphism of T-modules with iterate</u> if F preserves the objects, the tupling operation and the iterate; moreover, $HF = H'$ □

Notice that every morphism of T-modules with iterate preserves the sum.

Of course the composition of two morphisms of T-modules with iterate is a morphism of T-modules with iterate.

2.10 <u>Proposition</u>. Let $H : T \longrightarrow Q$ be a T-module with iterate. If for each $(f,\alpha,p) \in ND_Q(a,b)$ we define

$$|(f,\alpha,p)|_Q = H(f)\langle \alpha^{\dagger}, 1_b \rangle$$

then

$$|\ |_Q : ND_Q \longrightarrow Q$$

is a morphism of T-modules with iterate.

<u>Proof</u>. If $f \in T(a,b)$ then

$|G(f)|_Q = H(f)\langle H(0_b)^{\dagger}, 1_b \rangle = H(f)\langle H(0_b^{\dagger}), 1_b \rangle = H(f)$.

It follows from this that $|1_a|_Q = 1_a$.

If $(f,\alpha,p) \in ND_Q(a,b)$ and $(g,\beta,q) \in ND_Q(b,c)$ then

$|(f,\alpha,p)(g,\beta,q)|_Q =$

$= H(f(1_p+g)) < < \alpha H(1_p+g), \beta H(0_p+1_{qc}) >^{\dagger} , 1_c > =$ I3

$= H(f)\ H(1_p+g) < (\alpha(1_p+H(g)))^{\dagger} , 1_{qc} > < \beta^{\dagger} , 1_c > =$ I1, C4

$= H(f) < \alpha^{\dagger} H(g) , H(g) > < \beta^{\dagger} , 1_c > =$ C3

$= H(f) < \alpha^{\dagger} , 1_b > H(g) < \beta^{\dagger} , 1_c > =$

$= |(f, \alpha, p)|_{\alpha}\ |(g, \beta, q)|_{\alpha}$.

If $(f,\alpha,p) \in MD_{\alpha}(a,c)$ and $(g,\beta,q) \in MD_{\alpha}(b,c)$ then using proposition 2.4 in the second equality we obtain

$|<(f,\alpha,p) , (g,\beta,q)>|_{\alpha} =$

$= H(<f(1_p+0_q+1_c), g(0_p+1_{qc})>) < < \alpha H(1_p+0_q+1_c), \beta H(0_p+1_{qc}) >^{\dagger} , 1_c > =$

$= H(f+g)\ H(1_p + <0_q+1_c, 1_{qc}>) < \alpha^{\dagger}, \beta^{\dagger}, 1_c > =$ C4

$= H(f+g) < \alpha^{\dagger} , < H(0_q+1_c), 1_{qc} > < \beta^{\dagger}, 1_c > > =$ C5

$= H(f+g) < \alpha^{\dagger} , 1_c , \beta^{\dagger} , 1_c > =$ C4

$= < H(f) < \alpha^{\dagger}, 1_c > , H(g) < \beta^{\dagger}, 1_c > > =$

$= < |(f,\alpha,p)|_{\alpha} , |(g,\beta,q)|_{\alpha} >$.

If $(f,\alpha,p) \in MD_{\alpha}(a,ab)$ then I4 implies

$|(f,\alpha,p)^{\dagger}|_{\alpha} =$

$= H((f(S^a_p+1_b))^{\dagger}) < (\alpha H((S^a_p+1_b) < (f(S^a_p+1_b))^{\dagger}, 1_{pb} >))^{\dagger} , 1_b > =$

$= (H(f) < \alpha^{\dagger}, 1_{ab} >)^{\dagger} = |(f,\alpha,p)|_{\alpha}^{\dagger}$ □

3. The T-module of flowcharts

Let T be an S-sorted theory with iterate.

3.1 We made every ~~left-cancellable~~ monoid M into a T-module with iterate $H : T \longrightarrow Q_M$ by

$Q_M(a,b) = \{ (a,m,b) \mid m \in M \}$,

composition $(a,m,b)(b,m',c) = (a,mm',c)$,

tupling $<(a,m,c) , (b,m',c)> = (ab,mm',c)$,

iterate $(a,m,ab)^{\dagger} = (a,m,b)$,

$H(f) = (a,\lambda,b)$ if $f \in T(a,b)$.

~~The left-cancellation of M is necessary for defining the iterate~~

3.2 Proposition. The category of T-modules with iterate has products.

Proof. Let $H' : T \longrightarrow Q'$ and $H'' : T \longrightarrow Q''$ be two T-modules with iterate. Their product $H : T \longrightarrow Q$ is defined by

$Q(a,b) = Q'(a,b) \times Q''(a,b)$,

composition $(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', \beta\beta')$,

tupling $\langle(\alpha, \beta),(\alpha', \beta')\rangle = (\langle\alpha,\alpha'\rangle, \langle\beta, \beta'\rangle)$,

iterate $(\alpha, \beta)^{\dagger} = (\alpha^{\dagger}, \beta^{\dagger})$,

$H(f) = (H'(f), H''(f))$ for every $f \in T(a,b)$.

It is easy to prove that $H : T \longrightarrow Q$ is a T-module with iterate. The functors $F' : Q \longrightarrow Q'$ and $F'' : Q \longrightarrow Q''$ defined by

$$F'(\alpha, \beta) = \alpha \quad \text{and} \quad F''(\alpha, \beta) = \beta$$

are morphisms of T-modules with iterate.

Let $H_1 : T \longrightarrow Q_1$ be a T-module with iterate. Let further $G' : Q_1 \longrightarrow Q'$ and $G'' : Q_1 \longrightarrow Q''$ be two morphisms of T-modules with iterate. The unique morphism of T-modules with iterate $G : Q_1 \longrightarrow Q$ such that $GF' = G'$ and $GF'' = G''$ is defined by $G(\alpha) = (G'(\alpha), G''(\alpha))$ for every $\alpha \in Q_1(a,b)$ □

Let Σ be a set and

$$r_1 : \Sigma \longrightarrow S^* \quad \text{and} \quad r_2 : \Sigma \longrightarrow S^*$$

be two functions. For each $\sigma \in \Sigma$, $r_1(\sigma)$ indicates the number and the sorts of the inputs of σ. For each $\sigma \in \Sigma$, $r_2(\sigma)$ indicates the number and the sorts of the exits of σ.

3.3 Definition. Let $a, b \in S^*$. A Σ-flowchart over T with input a and exit b consists of a triple (i, τ, e) where $e \in \Sigma^*$, $i \in T(a, r_1^*(e)b)$ and $\tau \in T(r_2^*(e), r_1^*(e)b)$ □

Let $Fl_{\Sigma,T}(a,b)$ be the set of all Σ-flowcharts over T with input a and exit b.

Let $K : T \longrightarrow M$ be the T-module of pairs over $1_T : T \longrightarrow T$ and let $H : T \longrightarrow Q_{\Sigma^*}$ be the T-module with iterate obtained from the monoid Σ^* as in 3.1 . We shall identify each $(i, \tau, e) \in Fl_{\Sigma,T}(a,b)$

with $((i, \zeta, r_1^*(e), r_2^*(e)), (a, e, b)) \in H(a,b) \times Q_{\Sigma^*}(a,b)$.

3.4 Proposition. $Fl_{\Sigma,T}$ is a T-module with iterate.

Proof. It is easy to see that by the above identification $Fl_{\Sigma,T}$ becomes a T-submodule with iterate of the product of $H : T \longrightarrow$ and $H : T \longrightarrow Q_{\Sigma^*}$ □

Let $\hat{} : T \longrightarrow Fl_{\Sigma,T}$ be the structural functor of $Fl_{\Sigma,T}$. Obviously $\hat{f} = (f, 0_b, \lambda)$ for every $f \in T(a,b)$.

3.5 Definition. An interpretation I of Σ into a T-module with iterate $H : T \longrightarrow Q$ consists of a Σ-family of morphisms of Q such that for each $\sigma \in \Sigma$

$$I(\sigma) \in Q(r_1(\sigma), r_2(\sigma)) \ \square$$

If $e \in \Sigma^*$ then we shall write

$$I^*(e) = I(e_1)+I(e_2)+\ldots+I(e_{|e|}) .$$

Obviously $I^*(\lambda) = 1_\lambda$.

3.6 Definition. The standard interpretation of Σ is the interpretation I_Σ of Σ in $Fl_{\Sigma,T}$ defined for each $\sigma \in \Sigma$ by

$$I_\Sigma(\sigma) = (1_{r_1(\sigma)}+0_{r_2(\sigma)} , 0_{r_1(\sigma)}+1_{r_2(\sigma)} , \sigma) \ \square$$

3.7 Proposition. If $(i, \zeta, e) \in Fl_{\Sigma,T}(a,b)$ then

$$(i, \zeta, e) = \hat{i} < (I_\Sigma^*(e)\hat{\zeta})^\dagger , 1_b > .$$

Proof. An easy calculation shows that

$$I_\Sigma^*(e) = (1_{r_1^*(e)}+0_{r_2^*(e)} , 0_{r_1^*(e)}+1_{r_2^*(e)} , e) .$$

Therefore $(I_\Sigma^*(e)\hat{\zeta})^\dagger = (1_{r_1^*(e)}+0_{r_1^*(e)b} , 0_{r_1^*(e)}+\zeta , e)^\dagger =$

$= (1_{r_1^*(e)}+0_b , \zeta , e)$.

Hence $\hat{i} < (I_\Sigma^*(e))\hat{\zeta})^\dagger, 1_b> = \hat{i}(1_{r_1^*(e)b} , \zeta , e) = (i, \zeta, e)$ □

3.8 Proposition. Let I be an interpretation of Σ in the T-module ~~with iterate~~ $H : T \longrightarrow Q$. For each $(i, \zeta, e) \in Fl_{\Sigma,T}(a,b)$, define

$$\bar{I}(i, \zeta, e) = (i, I^*(e)H(\zeta), r_1^*(e)) \in ND_Q(a,b) .$$

Then

$$\bar{I} : Fl_{\Sigma,T} \longrightarrow ND_Q$$

is a morphism of T-modules with iterate.

<u>Proof</u>. If $f \in T(a,b)$ then

$\overline{I}(\hat{f}) = \overline{I}(f, 0_b, \lambda) = (f, H(0_b), \lambda) = \sigma(f)$.

It follows from this that $\overline{I}(1_a) = 1_a$.

If $(i, \tau, e) \in Fl_{\Sigma,T}(a,b)$ and $(i', \tau', e') \in Fl_{\Sigma,T}(b,c)$ then

$\overline{I}(i, \tau, e)\, \overline{I}(i', \tau', e') =$

$= (i, I^*(e)H(\tau), r_1^*(e))\,(i', I^*(e')H(\tau'), r_1^*(e')) =$

$= (i(1_{r_1^*(e)}+i')$,

$\langle I^*(e)H(\tau)H(1_{r_1^*(e)}+i'), I^*(e')H(\tau')H(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle, r_1^*(e)r_1^*(e')) =$

$= (i(1_{r_1^*(e)}+i'), I^*(ee')H(\langle \tau(1_{r_1^*(e)}+i'), \tau'(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle), r_1^*(ee')) =$

$= \overline{I}(\ i(1_{r_1^*(e)}+i')\ , \langle \tau(1_{r_1^*(e)}+i'), \tau'(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle\ , ee'\ ) =$

$= \overline{I}((i, \tau, e)(i', \tau', e'))$.

If $(i, \tau, e) \in Fl_{\Sigma,T}(a,c)$ and $(i', \tau', e') \in Fl_{\Sigma,T}(b,c)$ then

$\langle \overline{I}(i, \tau, e)\ ,\ \overline{I}(i', \tau', e') \rangle =$

$= \langle\ (i, I^*(e)H(\tau), r_1^*(e))\ ,\ (i', I^*(e')H(\tau'), r_1^*(e'))\ \rangle =$

$= (\ \langle i(1_{r_1^*(e)}+0_{r_1^*(e')}+1_c)\ ,\ i'(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle$,

$\langle I^*(e)H(\tau)\,(1_{r_1^*(e)}+0_{r_1^*(e')}+1_c), I^*(e')H(\tau')H(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle$,

$r_1^*(e)r_1^*(e')\ ) =$

$= (\ \langle i(1_{r_1^*(e)}+0_{r_1^*(e')}+1_c)\ ,\ i'(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle$,

$I^*(ee')H(\langle \tau(1_{r_1^*(e)}+0_{r_1^*(e')}+1_c), \tau'(0_{r_1^*(e)}+1_{r_1^*(e')c})\rangle), r_1^*(ee')\ ) =$

$= \overline{I}(\langle (i, \tau, e)\ ,\ (i', \tau', e')\rangle)$.

If $(i, \tau, e) \in Fl_{\Sigma,T}(a, ab)$ then

$\overline{I}(\ (i, \tau, e)^{\dagger}\ ) =$

$= \overline{I}(\ (i(S^a_{r_1^*(e)}+1_b))^{\dagger}, \tau(S^a_{r_1^*(e)}+1_b)\langle (i(S^a_{r_1^*(e)}+1_b))^{\dagger}, 1_{r_1(e)b}\rangle\ ,\ e\ ) =$

$= (\ (i(S^a_{r_1^*(e)}+1_b))^{\dagger}$,

$I^*(e)H(\tau)H((S^a_{r_1^*(e)}+1_b)\langle (i(S^a_{r_1^*(e)}+1_b))^{\dagger}, 1_{r_1(e)b}\rangle)\ ,\ r_1^*(e)\ ) =$

$= (\ i\ ,\ I^*(e)H(\tau)\ ,\ r_1^*(e)\ )^{\dagger} = (\ \overline{I}(i, \tau, e)\ )^{\dagger}$ ∎

3.9 <u>Main Theorem</u>. $Fl_{\Sigma,T}$ is the T-module with iterate freely generated by Σ.

<u>Proof</u>. Let $H : T \longrightarrow Q$ be a T-module with iterate. Let I be an interpretation of Σ in Q. We have to prove that there is a unique morphism of T-modules with iterate

$$F : Fl_{\Sigma,T} \longrightarrow Q$$

such that $I_{\Sigma}F = I$, i.e. for each $\sigma \in \Sigma$

$$F(I_{\Sigma}(\sigma)) = I(\sigma) .$$

It follows from propositions 2.10 and 3.8 that

$$\overline{I} \,|\; |_Q : Fl_{\Sigma,T} \longrightarrow Q$$

is a morphism of T-modules with iterate. Moreover, for each $\sigma \in \Sigma$

$$|\overline{I}(\,I_{\Sigma}(\sigma)\,)|_Q =$$

$$= |\,(\,1_{r_1(\sigma)}+0_{r_2(\sigma)}\,,\; I(\sigma)H(0_{r_1(\sigma)}+1_{r_2(\sigma)})\,,\; r_1(\sigma)\,)\,|_Q =$$

$$= H(1_{r_1(\sigma)}+0_{r_2(\sigma)})\langle I(\sigma), 1_{r_2(\sigma)}\rangle = \langle I(\sigma), H(0_{r_2(\sigma)})\rangle = I(\sigma) .$$

Let $F : Fl_{\Sigma,T} \longrightarrow Q$ be a morphism of T-modules with iterate such that $I_{\Sigma}F = I$. If $(i, \tau, e) \in Fl_{\Sigma,T}(a,b)$ then it follows from proposition 3.7 that

$$F(\,i,\tau,e\,) = F(\,\hat{i}\langle(I_{\Sigma}^{*}(e)\hat{\tau})^{\dagger}\,,\; 1_b\rangle) =$$

$$= H(i)\langle(F(I_{\Sigma}^{*}(e))H(\tau))^{\dagger}\,,\; 1_b\rangle = H(i)\langle(I^{*}(e)H(\tau))^{\dagger}\,,\; 1_b\rangle =$$

$$= |\,(i, I^{*}(e)H(\tau)\,,\; r_1^{*}(e))\,|_Q = |\,\overline{I}(\,i,\tau,e\,)\,|_Q \quad \square$$

4. Proving compilers correct

Let $G : T \longrightarrow T'$ be a morphism of theories with iterate.

4.1 <u>Lemma</u>. If $H' : T' \longrightarrow Q$ is a T'-module with iterate then $GH' : T \longrightarrow Q$ is a T-module with iterate $\square$

4.2 <u>Lemma</u>. Every morphism of the T'-modules with iterate $H' : T' \longrightarrow Q'$ and $H'' : T' \longrightarrow Q''$ is a morphism of the T-modules with iterate $GH' : T \longrightarrow Q'$ and $GH'' : T \longrightarrow Q''$ $\square$

The triple (Σ, r_1, r_2) together with an interpretation I of Σ in the T-module with iterate $H : T \longrightarrow Q$ define a language whose syntax and semantics were defined in the previous section. Let

$$F : Fl_{\Sigma,T} \longrightarrow \mathcal{Q}$$

be the semantics morphism given by theorem 3.9. Obviously $I_{\Sigma}F = I$.

Let (Σ', r'_1, r'_2) and I' be another language where $r'_1 : \Sigma' \longrightarrow S^*$, $r'_2 : \Sigma' \longrightarrow S^*$ are functions and I' is an interpretation of Σ' in the T-module with iterate $H' : T \longrightarrow \mathcal{Q}'$. Let $F' : Fl_{\Sigma',T'} \longrightarrow \mathcal{Q}'$ be the corresponding semantics morphism. Obviously $I_{\Sigma'}F' = I'$.

4.3 Definition. A morphism

$$C : Fl_{\Sigma,T} \longrightarrow Fl_{\Sigma',T'}$$

of the T-modules with iterate $\Lambda : T \longrightarrow Fl_{\Sigma,T}$ and $G\Lambda' : T \longrightarrow Fl_{\Sigma',T'}$ is called compiler □

Let $E : \mathcal{Q} \longrightarrow \mathcal{Q}'$ be a morphism of the T-modules with iterate $H : T \longrightarrow \mathcal{Q}$ and $GH' : T \longrightarrow \mathcal{Q}'$. This morphism has the following intuitive meaning : if $\alpha \in \mathcal{Q}(a,b)$ is the behaviour of $P \in Fl_{\Sigma,T}(a,b)$ then $E(\alpha)$ must be the behaviour of $C(P)$.

4.4 Definition. The compiler C is correct if $CF' = FE$ □

4.5 Proposition. The compiler C is correct if and only if

$$I_{\Sigma}CF' = IE .$$

Proof. It follows from lemma 4.2 that F' is a morphism of the T-modules with iterate $G\Lambda' : T \longrightarrow Fl_{\Sigma',T'}$ and $GH' : T \longrightarrow \mathcal{Q}'$. Then theorem 3.9 implies that the equality $CF' = FE$ of morphisms of T-modules with iterate holds if and only if $I_{\Sigma}CF' = I_{\Sigma}FE$, i.e. $I_{\Sigma}CF' = IE$ □

4.6 Proposition. There exists a correct compiler C if and only if there exists an interpretation J of Σ in $Fl_{\Sigma',T'}$ such that

$$JF' = IE .$$

Proof. If C is correct then we take $J = I_{\Sigma}C$ and apply proposition 4.5.

Let J be an interpretation of Σ in $Fl_{\Sigma',T'}$ such that $JF' = IE$. It follows from theorem 3.9 that there exists a unique compiler C such that $I_\Sigma C = J$. As $I_\Sigma CF' = JF' = IE$ the compiler C is correct □

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