THE BETTI NUMBERS OF STANLEY-REISNER IDEALS OF SIMPLICIAL TREES

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Abstract

We provide a simple method to compute the Betti numbers of the Stanley-Reisner ideal of a simplicial tree and its Alexander dual.

Keywords: resolution, monomial ideal, simplicial tree, Stanley-Reisner ideal

Simplicial trees [F1] are a class of flag complexes initially studied for the properties of their facet ideals. In this short note we give a short and straightforward method to compute the Betti numbers of their Stanley-Reisner ideals.

The *Betti numbers* of a homogeneous ideal I in a polynomial ring R over a field are the ranks of the free modules appearing in a minimal free resolution

$$0 \to \oplus_d R(-d)^{\beta_{p,d}} \to \cdots \to \oplus_d R(-d)^{\beta_{0,d}} \to I \to 0$$

of I. Here $R(-d)$ denotes the graded free module obtained by shifting the degrees of elements in R by d. The numbers $\beta_{i,d}$, which we shall refer to as the ith $\mathbb{N}$-*graded Betti numbers* of degree d of I, are independent of the choice of the graded minimal finite free resolution.

Definition 1 (simplicial complex)**.** A *simplicial complex* Δ over a set of vertices $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. A *subcollection* of Δ is a simplicial complex whose facets are also facets of Δ; in other words a simplicial complex generated by a subset of the set of facets of Δ. $A \subseteq V$, the *induced subcomplex of* Δ *on* A, denoted by Δ_A, is defined as $\Delta_A = \{F \in \Delta \mid F \subseteq A\}$.

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Definition 2. Let Δ be a simplicial complex with vertex set $x_1, \dots, x_n$ and $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. The *Stanley-Reisner ideal* of Δ is defined as $I_\Delta = (\prod_{x_i \in F} x_i \mid F \notin \Delta)$.

Definition 3 ([F1] leaf, joint)**.** A facet F of a simplicial complex is called a *leaf* if either F is the only facet of Δ or for some facet $G \neq F$ of Δ we have $F \cap H \subseteq G$ for all other facets H of Δ. Such a facet G is called a *joint* of F.

Definition 4 ([F1] tree, forest)**.** A connected simplicial complex Δ is a *tree* if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a *forest*.

Theorem 5 ([F2] Theorem 2.5)**.** *An induced subcomplex of a simplicial tree is a simplicial forest.*

Definition 6 (link)**.** Let Δ be a simplicial complex over a vertex set V and let F be a face of Δ. The *link* of F is defined as $\operatorname{lk}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset \ \&\ F \cup G \in \Delta\}$.

Lemma 7 (A link in a tree is a forest)**.** *If Δ is a tree and F is a face of Δ, then $\operatorname{lk}_\Delta(F)$ is a forest.*

Proof. Suppose $\operatorname{lk}_\Delta(F) = \langle G_1, \dots, G_s \rangle$ where G_i is a subset of a facet $F_i = F \cup G$ of Δ. Now suppose $\Gamma = \langle G_{a_1}, \dots, G_{a_r} \rangle$ is a subcollection of $\operatorname{lk}_\Delta(F)$. We need to show that Γ has a leaf. Let $\langle F_{a_1}, \dots, F_{a_r} \rangle$ be the corresponding subcollection of Δ, which must have a leaf, say F_{a_1} and a joint, say F_{a_2}. Then we have $F_{a_i} \cap F_{a_1} \subseteq F_{a_2}$ for $i = 3, \dots, r$. But since $F_{a_i} = F \cup G_{a_i}$ and $F \cap G_{a_i} = \emptyset$ for all i, we must have $G_{a_i} \cap G_{a_1} \subseteq G_{a_2}$ for $i = 3, \dots, r$ which means that G_{a_1} is a leaf of Γ. $\square$

We will combine the above two facts with Hochster's formula for Betti numbers of the ideal and its dual [BCP].

Theorem 8 ([BCP])**.** *Let k be a field and Δ a simplicial complex over vertex set V. Then*

$$\beta_{i,j}(I_\Delta) = \sum_{A \subseteq V,\ |A|=j} \dim_k \widetilde{H}_{j-i-2}(\Delta_A; k) \tag{1}$$

$$\beta_{i,j}(I_\Delta^\vee) = \sum_{A \subseteq V,\ |A|=j} \dim_k \widetilde{H}_{i-1}(\operatorname{lk}_\Delta(V \setminus A; k)). \tag{2}$$

If Δ is a tree, the following theorem shows how to find Betti numbers of I_Δ, and along the way also gives a proof of the fact that I_Δ has a linear resolution. This last statement is not unknown, it follows also from Fröberg's characterizations of edge ideals with linear resolutions [Fr] along with observations in [HHZ], and is also proved in [CF].

Theorem 9. *Let Δ be a simplicial tree with vertex set V. Then Δ is a flag complex, I_Δ has a linear resolution, and the Betti numbers of I_Δ can be computed by*

$$\beta_{i,j}(I_\Delta) = \begin{cases} \displaystyle\sum_{A \subseteq V,\ |A|=j} (\textit{number of connected components of } \Delta_A - 1) & j = i+2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By (1) we know that we are looking at the reduced homology modules of Δ_A for various $A \subseteq V$. For a given A, we know that Δ_A is a forest, and every connected component is a tree and therefore acyclic ([F2] Theorem 2.9). Therefore, for each such A the only possible nonzero reduced homology is the 0th one, that is when $|A| - i - 2 = 0$ or $|A| = i + 2$. The formula now just follows.

In particular, β_0 is only positive in degree 2, which implies that Δ is a flag complex, and the fact that the resolution is linear is evident from the way the Betti numbers grow. □

Theorem 10. *Let Δ be a simplicial tree with vertex set V of cardinality n. Then the $I_\Delta^\vee$ has projective dimension 1, and its Betti numbers are*

$$\beta_{i,j}(I_\Delta^\vee) = \begin{cases} \textit{number of facets of } \Delta \textit{ of cardinality } n - j & i = 0 \\ \displaystyle\sum_{A \subseteq V,\ |A|=j} (\textit{number of connected components of } \mathrm{lk}_\Delta(V \setminus A) - 1) & i = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from (2). Note that in this case we are looking at the homology modules of $\mathrm{lk}_\Delta(V \setminus A)$ for $A \subseteq V$. By Lemma 7 $\mathrm{lk}_\Delta(V \setminus A)$ is a forest, and so since all the connected components are acyclic, we only have possible homology in degrees -1 (if the link is empty) and 0.

The case $i = 1$ is the 0th homology, and we are counting the numbers of connected components minus 1, which is straightforward.

In the case $i = 0$, we are counting only those $A \subset V$ where $\mathrm{lk}_\Delta(V \setminus A) = \{\emptyset\}$, or equivalently $V \setminus A$ is a facet of Δ. So the formula for the case $i = 0$ follows. □

References

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