Primary Decomposition in a Sequentially Cohen-Macaulay Module

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March 17, 2005

The notion of a sequentially Cohen-Macaulay module was introduced by Stanley [?], following the introduction of a nonpure shellable simplicial complex by Björner and Wachs [BW]. It was known that the Stanley-Reisner ideal of a shellable simplicial complex is Cohen-Macaulay (see [BH]). A shellable simplicial complex is by definition pure (all facets have the same dimension), which is equivalent to its Stanley-Reisner ideal being unmixed. A nonpure shellable simplicial complex, on the other hand, may not be pure, so its Stanley-Reisner ideal may not be unmixed, and hence not Cohen-Macaulay. As it turns out, however, the Stanley-Reisner ideal of a nonpure simplicial complex is "sequentially Cohen-Macaulay" (Definition 1 below).

If the Stanley-Reisner ideal of a simplicial complex is sequentially Cohen-Macaulay, the complex has Cohen-Macaulay pure subcomplexes (see Duval [D] Theorem 3.3, or Stanley [?] Chapter III, Proposition 2.10). In the language of commutative algebra, this is equivalent to all equidimensional components appearing in the primary decomposition of a square-free monomial ideal being Cohen-Macaulay (see [F] for more details).

The purpose of this note is to establish that, more generally, this is what being sequentially Cohen-Macaulay means for any module. Below we use basic facts about primary decomposition of modules to study the structure of the submodules appearing in the (unique) filtration of a sequentially Cohen-Macaulay module. The main result (Theorem 5) states that each submodule appearing in the filtration of a sequentially Cohen-Macaulay module M is the intersection of all primary submodules whose associated primes have a certain height and appear in an irredundant primary decomposition of the 0-submodule of M. Similar results, stated in a different language, appear in [Sc]; the author thanks Jürgen Herzog for pointing this out.

Definition 1 ([St] Chapter III, Definition 2.9). Let M be a finitely generated $\mathbb{Z}$-graded module over a finitely generated $\mathbb{N}$-graded k-algebra, with $R_0 = k$. We say that M is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of M by graded submodules M_i satisfying the following two conditions.

(a) Each quotient M_i/M_{i-1} is Cohen-Macaulay;

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This research was supported by a NSERC Postdoctoral Fellowship.

(b) dim $(M_1/M_0) <$ dim $(M_2/M_1) < \ldots <$ dim (M_r/M_{r-1}), where "dim " denotes Krull dimension.

Before we begin our study of sequentially Cohen-Macaulay modules, we record two basic lemmas that we shall use later. Throughout the discussions below, we assume that R is a finitely generated algebra over a field, and M is a finite module over R.

Lemma 2. *Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_t, \mathcal{P}$ all be primary submodules of an R-module M, such that $\operatorname{Ass}(M/\mathcal{Q}_i) = \{q_i\}$ and $\operatorname{Ass}(M/\mathcal{P}) = \{\wp\}$. If $\mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_t \subseteq \mathcal{P}$ and $\mathcal{Q}_i \not\subseteq \mathcal{P}$ for some i, then there is a $j \neq i$ such that $q_j \subseteq \wp$.*

Proof. Let $x \in \mathcal{Q}_i \setminus \mathcal{P}$. For each $j \neq i$, pick the positive integer m_j such that

$$q_j^{m_j} x \subseteq \mathcal{Q}_j.$$

So we have that

$$q_1^{m_1} \ldots q_{i-1}^{m_{i-1}} q_{i+1}^{m_{i+1}} \ldots q_t^{m_t} x \subseteq \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_t \subseteq \mathcal{P}$$

which implies that, since $x \notin \mathcal{P}$,

$$q_1^{m_1} \ldots q_{i-1}^{m_{i-1}} q_{i+1}^{m_{i+1}} \ldots q_t^{m_t} \subseteq \wp$$

and hence for some $j \neq i$, $q_j \subseteq \wp$. □

Lemma 3. *Let M be an R-module and N be a submodule of M. Then for every $\wp \in \operatorname{Ass}(M/N)$, if $\wp \not\supseteq \operatorname{Ann}(N)$, then $\wp \in \operatorname{Ass}(M)$.*

Proof. Since $\wp \in \operatorname{Ass}(M/N)$, there exists $x \in M \setminus N$ such that $\wp = \operatorname{Ann}(x)$; in other words

$$\wp x \subseteq N.$$

Suppose $\operatorname{Ann}(N) \not\subseteq \wp$, and let $y \in \operatorname{Ann}(N) \setminus \wp$. Now $y\wp x = 0$, and so $\wp \subseteq \operatorname{Ann}(yx)$ in M.

On the other hand, if $z \in \operatorname{Ann}(yx)$, then $zyx = 0 \subseteq N$ and so $zy \in \wp$. But $y \notin \wp$, so $z \in \wp$. Therefore $\wp \in \operatorname{Ass}(M)$.

□

Suppose M is a sequentially Cohen-Macaulay module with filtration as in Definition 1. We adopt the following notation. For a given integer j, we let

$$\operatorname{Ass}(M)_j = \{\wp \in \operatorname{Ass}(M) \mid \text{height } \wp = j\}.$$

Suppose that all the j where $\operatorname{Ass}(M)_j \neq \emptyset$ form the sequence of integers

$$0 \leq h_1 < \ldots < h_c \leq \dim R$$

so that

$$\operatorname{Ass}(M) = \bigcup_{1 \leq j \leq c} \operatorname{Ass}(M)_{h_j}.$$

We can now make the following observations.

Proposition 4. *For all $i = 0, \ldots, r-1$, we have*

1. $\operatorname{Ass}(M_{i+1}/M_i) \cap \operatorname{Ass}(M) \neq \emptyset$;

2. $\mathrm{Ass}(M)_{h_{r-i}} \subseteq \mathrm{Ass}(M_{i+1}/M_i)$ *and* $c = r$*;*

3. *If* $\wp \in \mathrm{Ass}(M_{i+1})$*, then* height $\wp \geq h_{r-i}$*;*

4. *If* $\wp \in \mathrm{Ass}(M_{i+1}/M_i)$*, then* $\mathrm{Ann}(M_i) \not\subseteq \wp$*;*

5. $\mathrm{Ass}(M_{i+1}/M_i) \subseteq \mathrm{Ass}(M)$*;*

6. $\mathrm{Ass}(M_{i+1}/M_i) = \mathrm{Ass}(M)_{h_{r-i}}$*;*

7. $\mathrm{Ass}(M/M_i) = \mathrm{Ass}(M)_{\leq h_{r-i}}$*;*

8. $\mathrm{Ass}(M_{i+1}) = \mathrm{Ass}(M)_{\geq h_{r-i}}$.

Proof. 1. We use induction on the length r of the filtration of M. The case $r = 1$ is clear, as we have a filtration $0 \subset M$, and the assertion follows. Now suppose the statement holds for sequentially Cohen-Macaulay modules with filtrations of length less than r. Notice that M_{r-1} that appears in the filtration of M in Definition 1 is also sequentially Cohen-Macaulay, and so by the induction hypothesis, we have

$$\mathrm{Ass}(M_{i+1}/M_i) \cap \mathrm{Ass}(M_{r-1}) \neq \emptyset \text{ for } i = 0, \ldots, r-2$$

and since $\mathrm{Ass}(M_{r-1}) \subseteq \mathrm{Ass}(M)$ it follows that

$$\mathrm{Ass}(M_{i+1}/M_i) \cap \mathrm{Ass}(M) \neq \emptyset \text{ for } i = 0, \ldots, r-2.$$

It remains to show that $\mathrm{Ass}(M/M_{r-1}) \cap \mathrm{Ass}(M) \neq \emptyset$.

For each i, $M_{i-1} \subset M_i$, so we have ([B] Chapter IV)

$$\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \tag{1}$$

The inclusion $M_2 \subseteq M_3$ along with the inclusions in (1) imply that

$$\mathrm{Ass}(M_2) \subseteq \mathrm{Ass}(M_3) \subseteq \mathrm{Ass}(M_2) \cup \mathrm{Ass}(M_3/M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \mathrm{Ass}(M_3/M_2).$$

If we continue this process inductively, at the i-th stage we have

$$\begin{aligned} \mathrm{Ass}(M_i) &\subseteq \mathrm{Ass}(M_{i-1}) \cup \mathrm{Ass}(M_i/M_{i-1}) \\ &\subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \mathrm{Ass}(M_3/M_2) \cup \ldots \cup \mathrm{Ass}(M_i/M_{i-1}) \end{aligned}$$

and finally, when $i = r$ it gives

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \mathrm{Ass}(M_3/M_2) \cup \ldots \cup \mathrm{Ass}(M/M_{r-1}). \tag{2}$$

Because of Condition (b) in Definition 1, and the fact that each M_{i+1}/M_i is Cohen-Macaulay (and hence all its associated primes have the same height; see [BH] Chapter 2), if for every i we pick $\wp_i \in \mathrm{Ass}(M_{i+1}/M_i)$, then

$$h_c \geq \text{height } \wp_0 > \text{height } \wp_1 > \ldots > \text{height } \wp_{r-1}.$$

where the left-hand-side inequality comes from the fact that $\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M)$. By our induction hypothesis, $\mathrm{Ass}(M)$ intersects $\mathrm{Ass}(M_{i+1}/M_i)$ for all $i \leq r-2$, and so because of (2) we conclude that

$$\text{height } \wp_i = h_{c-i}, \text{ and } \mathrm{Ass}(M)_{h_{c-i}} \subseteq \mathrm{Ass}(M_{i+1}/M_i) \text{ for } 0 \leq i \leq r-2.$$

And now $\mathrm{Ass}(M)_{h_0}$ has no choice but to be included in $\mathrm{Ass}(M/M_{r-1})$, which settles our claim. It also follows that $c = r$.

2. See the proof for part 1.

3. We use induction. The case $i = 0$ is clear, since for every $\wp \in \mathrm{Ass}(M_1) = \mathrm{Ass}(M_1/M_0)$ we know from part 2 that height $\wp = h_r$. Suppose the statement holds for all indices up to $i-1$. Consider the inclusion

$$\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_{i+1}/M_i).$$

From part 2 and the induction hypothesis it follows that if $\wp \in \mathrm{Ass}(M_{i+1})$ then height $\wp \geq h_{r-i}$.

4. Suppose $\mathrm{Ann}(M_i) \subseteq \wp$. Since $\sqrt{\mathrm{Ann}(M_i)} = \bigcap_{\wp' \in \mathrm{Ass}(M_i)} \wp'$, we have

$$\bigcap_{\wp' \in \mathrm{Ass}(M_i)} \wp' \subseteq \wp$$

so there is a $\wp' \in \mathrm{Ass}(M_i)$ such that $\wp' \subseteq \wp$. But by part 2 and part 3 above

$$\text{height } \wp' \geq h_{r-i+1} \text{ and height } \wp = h_{r-i}$$

which is a contradiction.

5. From part 4 and Lemma 3, it follows that

$$\mathrm{Ass}(M_{i+1}/M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M).$$

6. This follows from parts 2 and 5, and the fact that M_{i+1}/M_i is Cohen-Macaulay, and hence all associated primes have the same height.

7. We show this by induction on $e = r - i$. The case $e = 1$ (or $i = r-1$) is clear, because by part 6

$$\mathrm{Ass}(M/M_{r-1}) = \mathrm{Ass}(M)_{h_1} = \mathrm{Ass}(M)_{\leq h_1}.$$

Now suppose the equation holds for all integers up to $e-1$ (namely $i = r-e+1$), and we would like to prove the statement for e (or $i = r - e$). Since $M_{i+1}/M_i \subseteq M/M_i$, we have

$$\mathrm{Ass}(M_{i+1}/M_i) \subseteq \mathrm{Ass}(M/M_i) \subseteq \mathrm{Ass}(M_{i+1}/M_i) \cup \mathrm{Ass}(M/M_{i+1}) \tag{3}$$

By the induction hypothesis and part 6 we know that

$$\mathrm{Ass}(M/M_{i+1}) = \mathrm{Ass}(M)_{\leq h_{r-i-1}} \text{ and } \mathrm{Ass}(M_{i+1}/M_i) = \mathrm{Ass}(M)_{h_{r-i}},$$

which put together with (3) implies that

$$\mathrm{Ass}(M)_{h_{r-i}} \subseteq \mathrm{Ass}(M/M_i) \subseteq \mathrm{Ass}(M)_{\leq h_{r-i}}$$

We still have to show that $\mathrm{Ass}(M/M_i) \supseteq \mathrm{Ass}(M)_{\leq h_{r-i-1}}$.

Let

$$\wp \in \mathrm{Ass}(M)_{\leq h_{r-i-1}} = \mathrm{Ass}(M/M_{i+1}) = \mathrm{Ass}((M/M_i)/(M_{i+1}/M_i)).$$

If $\wp \supseteq \mathrm{Ann}(M_{i+1}/M_i)$, then (by part 6)

$$\wp \supseteq \bigcap_{q \in \mathrm{Ass}(M)_{h_{r-i}}} q \implies \wp \supseteq q \text{ for some } q \in \mathrm{Ass}(M)_{h_{r-i}}$$

which is a contradiction, as height $\wp \leq h_{r-i-1} <$ height q.

It follows from Lemma 3 that $\wp \in \mathrm{Ass}(M/M_i)$.

8. The argument is based on induction, and exactly the same as the one in part 4, using more information; from

$$\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_{i+1}/M_i),$$

the induction hypothesis, and part 6 we deduce that

$$\mathrm{Ass}(M)_{\geq h_{r-i+1}} \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M)_{\geq h_{r-i+1}} \cup \mathrm{Ass}(M)_{h_{r-i}},$$

which put together with part 4, along with Lemma 3 produces the equality.

□

Now suppose that as a submodule of M, $M_0 = 0$ has an irredundant primary decomposition of the form:

$$M_0 = 0 = \bigcap_{1 \leq j \leq r} \mathcal{Q}_1^{h_j} \cap \ldots \cap \mathcal{Q}_{s_j}^{h_j} \tag{4}$$

where for a fixed $j \leq r$ and $e \leq s_j$, $\mathcal{Q}_e^{h_j}$ is a primary submodule of M with

$$\mathrm{Ass}(M/\mathcal{Q}_e^{h_j}) = \{\wp_e^{h_j}\} \text{ and } \mathrm{Ass}(M)_{h_j} = \{\wp_1^{h_j}, \ldots, \wp_{s_j}^{h_j}\}.$$

Theorem 5. *Let M be a sequentially Cohen-Macaulay module with filtration as in Definition 1, and suppose that $M_0 = 0$ has a primary decomposition as in (4). Then for each $i = 0, \ldots, r-1$, M_i has the following primary decomposition*

$$M_i = \bigcap_{1 \leq j \leq r-i} \mathcal{Q}_1^{h_j} \cap \ldots \cap \mathcal{Q}_{s_j}^{h_j}. \tag{5}$$

Proof. We prove this by induction on r (length of the filtration). The case $r = 1$ is clear, as the filtration is of the form $0 = M_0 \subset M$. Now consider M with filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M.$$

Since M_{r-1} is a sequentially Cohen-Macaulay module of length $r-1$, it satisfies the statement of the theorem. We first show that M_{r-1} has a primary decomposition as described in (5). From part 7 of Proposition 4 it follows that

$$\mathrm{Ass}(M/M_{r-1}) = \mathrm{Ass}(M)_{h_1}$$

and so for some $\wp_e^{h_1}$-primary submodules $\mathcal{P}_e^{h_1}$ of M $(1 \leq e \leq s_j)$, we have

$$M_{r-1} = \mathcal{P}_1^{h_1} \cap \ldots \cap \mathcal{P}_{s_1}^{h_1}. \tag{6}$$

We would like to show that $\mathcal{Q}_e^{h_1} = \mathcal{P}_e^{h_1}$ for $e = 1, \ldots, s_1$.

Fix $e = 1$ and assume $\mathcal{Q}_1^{h_1} \not\subset \mathcal{P}_1^{h_1}$. From the inclusion $M_0 \subset \mathcal{P}_1^{h_1}$ and Lemma 2 it follows that for some e and j (with $e \neq 1$ if $j = 1$), we have $\wp_e^{h_j} \subseteq \wp_1^{h_1}$. Because of the difference in heights of these ideals the only conclusion is $\wp_e^{h_j} = \wp_1^{h_1}$, which is not possible. With a similar argument we deduce that $\mathcal{Q}_e^{h_1} \subset \mathcal{P}_e^{h_1}$, for $e = 1, \ldots, s_1$.

Now fix $j \in \{1, \ldots, r\}$ and $e \in \{1, \ldots, s_j\}$. If $M_{r-1} = \mathcal{Q}_e^{h_j}$ we are done. Otherwise, note that for every j and $\wp_e^{h_j}$-primary submodule $\mathcal{Q}_e^{h_j}$ of M,

$$\mathcal{Q}_e^{h_j} \cap M_{r-1}$$

is a $\wp_e^{h_j}$-primary submodule of M_{r-1} (as $\emptyset \neq \mathrm{Ass}(M_{r-1}/(\mathcal{Q}_e^{h_j} \cap M_{r-1})) = \mathrm{Ass}((M_{r-1} + \mathcal{Q}_e^{h_j})/\mathcal{Q}_e^{h_j}) \subseteq \mathrm{Ass}(M/\mathcal{Q}_e^{h_j}) = \{\wp_e^{h_j}\}$). So $M_0 = 0$ as a submodule of M_{r-1} has a primary decomposition

$$M_0 \cap M_{r-1} = 0 = \bigcap_{1 \leq j \leq r} (\mathcal{Q}_1^{h_j} \cap M_{r-1}) \cap \ldots \cap (\mathcal{Q}_{s_j}^{h_j} \cap M_{r-1}).$$

From Proposition 4 part 8 it follows that

$$\mathrm{Ass}(M_{r-1}) = \mathrm{Ass}(M)_{\geq h_2}$$

so the components $\mathcal{Q}_t^{h_1} \cap M_{r-1}$ are redundant for $t = 1, \ldots, s_1$, so for each such t we have

$$\bigcap_{\mathcal{Q}_e^{h_j} \neq \mathcal{Q}_t^{h_1}} (\mathcal{Q}_1^{h_j} \cap M_{r-1}) \subseteq \mathcal{Q}_t^{h_1} \cap M_{r-1}.$$

If $\mathcal{Q}_e^{h_j} \cap M_{r-1} \not\subseteq \mathcal{Q}_t^{h_1} \cap M_{r-1}$ for some e and j (with $\mathcal{Q}_e^{h_j} \neq \mathcal{Q}_t^{h_1}$), then by Lemma 2 for some such e and j we have $\wp_e^{h_j} \subseteq \wp_t^{h_1}$, which is a contradiction (because of the difference of heights).

Therefore, for each t ($1 \leq t \leq s_1$), there exists indices e and j (with $\mathcal{Q}_e^{h_j} \neq \mathcal{Q}_t^{h_1}$) such that

$$\mathcal{Q}_e^{h_j} \cap M_{r-1} \subseteq \mathcal{Q}_t^{h_1} \cap M_{r-1}.$$

It follows now, from the primary decomposition of M_{r-1} in (6) that for a fixed t

$$\mathcal{P}_1^{h_1} \cap \ldots \cap \mathcal{P}_{s_1}^{h_1} \cap \mathcal{Q}_e^{h_j} \subseteq \mathcal{Q}_t^{h_1}.$$

Assume $\mathcal{P}_t^{h_1} \not\subseteq \mathcal{Q}_t^{h_1}$. Applying Lemma 2 again, we deduce that

$$\wp_e^{h_j} \subseteq \wp_t^{h_1}, \text{ or there is } t' \neq t \text{ such that } \wp_{t'}^{h_1} \subseteq \wp_t^{h_1}.$$

Neither of these is possible, so $\mathcal{P}_t^{h_1} \subseteq \mathcal{Q}_t^{h_1}$ for all t.

We have therefore proved that

$$M_{r-1} = \mathcal{Q}_1^{h_1} \cap \ldots \cap \mathcal{Q}_{s_1}^{h_1}.$$

By induction hypothesis, for each $i \leq r-2$, M_i has the following primary decomposition

$$M_i = \bigcap_{2 \leq j \leq r-i} (\mathcal{Q}_1^{h_j} \cap M_{r-1}) \cap \ldots \cap (\mathcal{Q}_{s_j}^{h_j} \cap M_{r-1}) = \bigcap_{1 \leq j \leq r-i} \mathcal{Q}_1^{h_j} \cap \ldots \cap \mathcal{Q}_{s_j}^{h_j}$$

which proves the theorem.

□

References

[B] Bourbaki, N. *Commutative algebra, Chapters 1–7,* Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989.

[BH] Bruns, W., Herzog, J. *Cohen-Macaulay rings*, vol. 39, Cambridge studies in advanced mathematics, revised edition, 1998.

[BW] Björner, A., Wachs, M.L. *Shellable nonpure complexes and posets*, I. Trans. Amer. Math. Soc. 348 (1996), no. 4, 1299–1327.

[D] Duval, A.M. *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes*, Electron. J. Combin. 3 (1996), no. 1, Research Paper 21.

[F] Faridi, S. *Simplicial trees are sequentially Cohen-Macaulay*, J. Pure and Applied Algebra, Volume 190, Issues 1-3, Pages 121-136 (June 2004).

[Sc] P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, *Commutative algebra and algebraic geometry (Ferrara)*, 245–264, Lecture Notes in Pure and Appl. Math., **206**, Dekker, New York, 1999.

[St] Stanley, R.P. *Combinatorics and commutative algebra*, Second edition, Progress in Mathematics 41, Birkhuser Boston, Inc., Boston, MA, 1996.