Local Cohomology and the Cohen-Macaulay property

Abstract

These lectures represent an extended version of the contents of a one hour introductory talk prepared by Florian Enescu and Sara Faridi for the Minnowbrook workshop to assist the lectures of one of the main speakers, Paul Roberts.

Part of this talk was given earlier at the 2004 Utah mini-course on *Classical problems in commutative algebra* by the first author. The notes for that talk have been been typed and prepared by Bahman Engheta and we used part of them quite extensively in preparing this version. We would like to thank Bahman for his work.

The references listed at the end were used in depth in preparing these notes and the authors make no claim of originality. Moreover, the reader is encouraged to consult these references for more details and many more results that were omitted due to time constraints.

1 Injective modules and essential extensions

Throughout, let R be a commutative Noetherian ring.

Definition 1 (injective module). An R-module E is called *injective* if one of the following equivalent conditions hold.

1. Given any R-module monomorphism $f : N \to M$, every homomorphism $u : N \to E$ can be extended to a homomorphism $g : M \to E$, i.e. $g\,f = u$.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

2. For each ideal J of R, any homomorphism $u : J \to E$ can be extended to a homomorphism $R \to E$.

3. The functor $\operatorname{Hom}(\_, E)$ is exact.

Example 2. From part 2 of Definition 1 it follows that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

Definition 3 (divisible module). An R-module M is called *divisible* if for every $m \in M$ and M-regular element $r \in R$, there is an $m' \in M$ such that $m = rm'$.

Exercise 4. An injective R-module is divisible. The converse holds if R is a principal ideal domain; see the example above.

A note on existence: If $R \to S$ is a ring homomorphism and E is an injective R-module, then $\operatorname{Hom}_R(S, E)$ is an injective S-module. [The S-module structure is given by $s \cdot \varphi(\_) := \varphi(s\_)$.] In particular, $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q})$ is an injective R-module and any R-module can be embedded in an injective R-module.

We now show how to embed a given module into a minimal injective module.

Definition 5 (essential extension). Let $M \subseteq N$ be R-modules. We say that N is an *essential extension* of M if one of the following equivalent conditions holds:

a) $M \cap N' \neq 0 \quad \forall\, 0 \neq N' \subseteq N$.

b) $\forall\, 0 \neq n \in N \quad \exists\, r \in R \quad$ such that $\quad 0 \neq rn \in M$.

c) $\forall\, N \xrightarrow{\varphi} Q$, if $\varphi|_M$ is injective, then φ is injective.

If additionally $M \neq N$, then we say that N is a *proper* essential extension of M.

More generally, an injective map $M \xrightarrow{h} N$ is called an *essential extension* if in the conditions above we replace M by $h(M)$.

Example 6. If R is a domain and $Q(R)$ its fraction field, then $R \subseteq Q(R)$ is an essential extension.

Example 7. Let $(R, \mathfrak{m}, k)$ be a local ring and N an R-module such that every element of N is annihilated by some power of the maximal ideal $\mathfrak{m}$. Let $\mathrm{Soc}(N) := \mathrm{Ann}_N(\mathfrak{m})$ denote the socle of N (which is a k-vector space). Then $\mathrm{Soc}(N) \subseteq N$ is an essential extension.

Exercise 8. An R-module N is Artinian if and only if $\mathrm{Soc}(N)$ is finite dimensional (as a k-vector space) and $\mathrm{Soc}(N) \subseteq N$ is essential.

Exercise 9. Let M be a submodule of N. Use Zorn's lemma to show that there is a maximal submodule $N' \subseteq N$ containing M such that $M \subset N'$ is an essential extension.

Definition 10 (maximal essential extension). If $M \subseteq N$ is an essential extension such that N has no proper essential extension, then $M \subseteq N$ is called a *maximal essential extension*.

The following proposition characterizes injective modules in terms of essential extensions.

Proposition 11. *a) An R-module E is injective if and only if it has no proper essential extension.*

b) Let M be an R-module and E an injective R-module containing M. Then any maximal essential extension N of M with $N \subseteq E$ is a maximal essential extension. In particular, N is injective and thus a direct summand of E.

c) If $M \subseteq E$ and $M \subseteq E'$ are two maximal essential extensions, then there is an isomorphism $E \cong E'$ that fixes M.

Definition 12 (injective hull). A maximal essential extension of an R-module M is called an *injective hull* of M, denoted $E_R(M)$.

Proposition 11 states that every module M has a unique injective hull up to isomorphism. Moreover, if $M \subseteq E$ where E is an injective module, then M has a maximal essential extension $E_R(M)$ that is contained in E, and $E_R(M)$ is a direct summand of E.

Definition 13 (injective resolution). Let M be an R-module. Set $E^{-1} := M$ and $E^0 := E_R(M)$. Inductively define $E^n := E_R({}^{E^{n-1}}\!/_{\mathrm{im}(E^{n-2})})$. Then the acyclic complex

$$\mathbb{I}: \quad 0 \to E^0 \to E^1 \to \cdots \to E^n \to \cdots$$

is called an *injective resolution* of M, where the maps are given by the composition

$$E^{n-1} \;\to\; E^{n-1}/\mathrm{im}(E^{n-2}) \;\hookrightarrow\; E_R(E^{n-1}/\mathrm{im}(E^{n-2})).$$

Conversely, an acyclic complex $\mathbb{I}$ of injective R-modules is a minimal injective resolution of M if

- $M = \ker(E^0 \to E^1)$,
- $E^0 = E_R(M)$,
- $E^n = E_R(\mathrm{im}(E^{n-1} \to E^n))$.

For a Noetherian ring R one can give a specific description of each E^i appearing in the injective resolution of M as a direct sum of $E_R(R/p)$ for $p \in \mathrm{Spec}R$. We refer interested reader to the sources mentioned in the references for more on this.

2 Local cohomology

Definition 14 (I-torsion). Given an ideal $I \subseteq R$ and an R-module M, set $\Gamma_I(M) := \bigcup_n (0 :_M I^n)$. The covariant functor $\Gamma_I(\_)$ over the category of R-modules is called the *I-torsion functor,* and for a homomorphism $f : M \to N$, $\Gamma_I(f)$ is given by the restriction $f\big|_{\Gamma_I(M)}$.

Proposition 15. *$\Gamma_I(\_)$ is a left exact functor.*

Proof. Let

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be a short exact sequence. We want to show that

$$0 \to \Gamma_I(L) \xrightarrow{\Gamma_I(f)} \Gamma_I(M) \xrightarrow{\Gamma_I(g)} \Gamma_I(N)$$

is an exact sequence.

Exactness at $\Gamma_I(L)$: $\Gamma_I(f)$ is injective as it is the restriction of the injective map f. Exactness at $\Gamma_I(M)$: It is clear that $\operatorname{im}(\Gamma_I(f)) \subseteq \ker(\Gamma_I(g))$. Conversely, let $m \in \ker(\Gamma_I(g))$. Then $m \in \ker(g)$ and therefore $m = f(l)$ for some $l \in L$. It remains to show that $l \in \Gamma_I(L)$. As $m \in \Gamma_I(M)$, we have $I^k m = 0$ for some integer k. Then $f(I^k l) = I^k f(l) = I^k m = 0$. As f is injective, $I^k l = 0$ and $l \in \Gamma_I(L)$. $\square$

Exercise 16. $\Gamma_I = \Gamma_J$ if and only if $\sqrt{I} = \sqrt{J}$.

Definition 17 (local cohomology). The *i-th local cohomology functor* $H^i_I(\_)$ is defined as the i-th right derived functor of $\Gamma_I(\_)$. More precisely, given an R-module M, let $\mathbb{I}$ be an injective resolution of M:

$$\mathbb{I}: \quad 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \to E^n \xrightarrow{d^n} \cdots$$

Apply $\Gamma_I(\_)$ to $\mathbb{I}$ and obtain the complex:

$$\Gamma_I(\mathbb{I}): \quad 0 \to \Gamma_I(E^0) \to \Gamma_I(E^1) \to \cdots \to \Gamma_I(E^n) \to \cdots$$

Then set $H^0_I(M) := \Gamma_I(M)$ and $H^i_I(M) := \ker(\Gamma_I(d^i))/\operatorname{im}(\Gamma_I(d^{i-1}))$ for $i > 0$. Note that $H^i_I(\_)$ is a covariant functor.

Clearly, if E is an injective R-module, then $H^i_I(E) = 0$ for $i > 0$.

Proposition 18. *1) If $0 \to L \to M \to N \to 0$ is a short exact sequence, then we have an induced long exact sequence*

$$\begin{aligned} 0 \to H^0_I(L) \to H^0_I(M) \to H^0_I(N) \to \\ H^1_I(L) \to H^1_I(M) \to H^1_I(N) \to \cdots \end{aligned}$$

2) Given a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

then we have the following commutative diagram with exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & H^i_I(M) \arrow[r] \arrow[d] & H^i_I(N) \arrow[r] \arrow[d] & H^{i+1}_I(L) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^i_I(M') \arrow[r] & H^i_I(N') \arrow[r] & H^{i+1}_I(L') \arrow[r] & \cdots \end{tikzcd}$$

3) $\sqrt{I} = \sqrt{J}$ if and only if $H^i_I(\_) = H^i_J(\_)$.

4) Localization: Let $S \subseteq R$ be a multiplicatively closed set. Then $S^{-1}\,\Gamma_I(M) = \Gamma_{S^{-1}I}(S^{-1}M)$ and the same holds for the higher local cohomology modules.

2.1 Alternate construction of local cohomology

An alternate way of constructing the local cohomology modules: Consider the module $\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n)$. Now, if $n \geqslant m$, then one has a natural map $R/I^n \to R/I^m$, forming an inverse system. Applying $\operatorname{Hom}_R(\_, M)$, we get a direct system of maps:

$$\varinjlim_n \operatorname{Hom}_R(R/I^n, M) \cong \bigcup_n (0 :_M I^n) = \Gamma_I(M).$$

As one might guess (or hope), it is also the case that

$$\varinjlim_n \operatorname{Ext}^i_R(R/I^n, M) \cong H^i_I(M).$$

This follows from the theory of negative strongly connected functors – see [R].

Definition 19 (strongly connected functors). Let R, R' be commutative rings. A sequence of covariant functors $\{T^i\}_{i\geqslant 0}$ from the category of R-modules to the category of R'-modules is said to be *negative (strongly) connected* if

(i) Any short exact sequence $0 \to L \to M \to N \to 0$ induces a long exact sequence

$$\begin{aligned} 0 \to T^0(L) \to T^0(M) \to T^0(N) \to \\ T^1(L) \to T^1(M) \to T^1(N) \to \cdots \end{aligned}$$

(ii) For any commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

there is a chain map between the long exact sequences given in (i).

Definition 20 (natural equivalence of functors). Let T and U be two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* ψ from T to U associates to every object X in $\mathcal{C}$ a morphism $\psi_X : T(X) \to U(X)$ in $\mathcal{D}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$ we have $\psi_Y o T(f) = U(f) o \psi_X$. If, for every object X in $\mathcal{C}$, the morphism ψ_X is an isomorphism in $\mathcal{D}$, then ψ is said to be a *natural equivalence*.

Theorem 21. *Let $\psi^0 : T^0 \to U^0$ be a natural equivalence, where $\{T^i\}_{i\geqslant 0}, \{U^i\}_{i\geqslant 0}$ are strongly connected. If $T^i(E) = U^i(E) = 0$ for all $i > 0$ and injective modules E, then there is a natural equivalence of functors $\psi = \{\psi^i\}_{i\geqslant 0} : \{T^i\}_i \to \{U^i\}_i$.*

The above theorem implies that $\varinjlim_n \operatorname{Ext}^i_R(R/I^n, M) \cong H^i_I(M)$.

Remark 22. i) One can replace the sequence of $\{I^n\}$ by any decreasing sequence of ideals $\{J_t\}$ which are cofinal with $\{I^n\}$, i.e. $\forall\, t\, \exists\, n$ such that $I^n \subseteq J_t$ and $\forall\, n\, \exists\, t$ such that $J_t \subseteq I^n$.

ii) Every element of $H^i_I(M)$ is killed by some power of I, as every $m \in H^i_I(M)$ is the image of some $\operatorname{Ext}^i_R(R/I^n, M)$ which is killed by I^n.

iii) For any $x \in R$, the homomorphism $M \xrightarrow{\cdot x} M$ induces a homomorphism $H^i_I(M) \xrightarrow{\cdot x} H^i_I(M)$.

Proposition 23. *Let M be a finitely generated R-module and $I \subseteq R$ an ideal. Then $IM = M \iff H^i_I(M) = 0\, \forall\, i$. If $IM \neq M$, then* $\min\{i \mid H^i_I(M) \neq 0\} = \operatorname{depth}_I(M)$.

Proof. We have $IM = M \iff I^tM = M\,\forall\, t$. So $I^t + \operatorname{Ann}(M) = R$, as otherwise $I^t + \operatorname{Ann}(M) \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \text{max-Spec}(R)$. Since $I^t + \operatorname{Ann}(M)$ annihilates $\operatorname{Ext}^i_R(R/I^t, M)$, we have $\operatorname{Ext}^i_R(R/I^t, M) = 0$ and therefore $H^i_I(M) = 0$.

It suffices to assume now that $IM \neq M$. Set $d := \operatorname{depth}_I(M)$ and let $x_1, \dots, x_d$ be a maximal M-regular sequence in I. We show by induction on d that $H^i_I(M) = 0$ for $i < d$ and $H^d_I(M) \neq 0$.

If $d = 0$, that is, if $\operatorname{depth}_I(M) = 0$, then there is an $0 \neq m \in M$ killed by I. So $m \in H^0_I(M) \neq 0$. Now let $d \geqslant 1$ and set $x := x_1$. Consider the short exact sequence

$$0 \to M \stackrel{\cdot x}{\to} M \to M/xM \to 0$$

and the induced long exact sequence

$$\cdots \to H^{i-1}_I(M/xM) \to H^i_I(M) \stackrel{\cdot x}{\to} H^i_I(M) \to \cdots$$

If $i < d$, then $H^{i-1}_I(M/xM) = 0$ by induction hypothesis and x is a nonzerodivisor on $H^i_I(M)$. As all elements of $H^i_I(M)$ are killed by some power of I, we conclude that $H^i_I(M) = 0$.

It remains to show that $H^d_I(M) \neq 0$. This follows from the induction hypothesis and the long exact sequence

$$\cdots \stackrel{\cdot x}{\to} H^{d-1}_I(M) \to H^{d-1}_I(M/xM) \to H^d_I(M) \stackrel{\cdot x}{\to} \cdots$$

which yield $0 \neq H^{d-1}_I(M/xM) \hookrightarrow H^d_I(M)$. $\square$

2.2 The Koszul interpretation

Let $\mathbb{K}, \mathbb{L}$ be two complexes of R-modules with differentials d', d'', respectively. One can define the complex $\mathbb{M} := \mathbb{K} \otimes \mathbb{L}$ via $\mathbb{M}_k := \bigoplus_{i+j=k} \mathbb{K}_i \otimes \mathbb{L}_j$ with differential $d(a_i \otimes b_j) := d'(a_i) \otimes b_j + (-1)^i a_i \otimes d''(b_j)$ where $a_i \in K_i,\ b_j \in L_j$. Similarly, if $\mathbb{K}^{(1)}, \dots, \mathbb{K}^{(n)}$ are n complexes, then $\mathbb{K}^{(1)} \otimes \cdots \otimes \mathbb{K}^{(n)}$ is defined inductively as $(\mathbb{K}^{(1)} \otimes \cdots \otimes \mathbb{K}^{(n-1)}) \otimes \mathbb{K}^{(n)}$.

To any $x \in R$ one can associate complexes

$$K_\bullet(x; R):\ 0 \to R \stackrel{\cdot x}{\to} R \to 0 \quad \text{or} \quad K^\bullet(x; R):\ 0 \to R \stackrel{\cdot x}{\to} R \to 0$$

where the degrees of the components R, from left to right, are 1, 0 for $K_\bullet(x; R)$ and 0,1 for $K^\bullet(x; R)$.

Given a sequence $\underline{x} = x_1, \dots, x_n$ of elements in R, we define the (homological) Koszul complex $K_\bullet(\underline{x}; R) := \bigotimes_{i=1}^n K_\bullet(x_i; R)$, and cohomological Koszul complex $K^\bullet(\underline{x}; R) := \bigotimes_{i=1}^n K^\bullet(x_i; R)$.

For an R-module M we define $K_\bullet(\underline{x}; M) := K_\bullet(\underline{x}; R) \otimes M$ and $K^\bullet(\underline{x}; M) := K^\bullet(\underline{x}; R) \otimes M$. The following isomorphism holds: $K^\bullet(\underline{x}; M) \cong \operatorname{Hom}_R(K_\bullet(\underline{x}; R), M)$.

Let $x \in R$ and M an R-module. Consider the complex

$$M \stackrel{\cdot x}{\to} M \stackrel{\cdot x}{\to} M \to \cdots$$

and set $N := \ker(M \to M_x)$ and $M' := M/N$. Note that $N = H^0_{xR}(M)$. Then

$$\varinjlim(M \stackrel{\cdot x}{\to} M \stackrel{\cdot x}{\to} M \to \cdots) \cong$$
$$\varinjlim(M' \stackrel{\cdot x}{\to} M' \stackrel{\cdot x}{\to} M' \to \cdots) =$$
$$\varinjlim(M' \subseteq x^{-1}M' \subseteq \cdots \subseteq x^{-t}M' \subseteq \cdots) \cong M'_x \cong M_x.$$

Notation: Whenever $\underline{x} = x_1, \dots, x_n$ denotes a sequence of elements in R, we will denote by $\underline{x}^t$ the sequence of the individual powers $x_1^t, \dots, x_n^t$

For $x \in R$ we have a chain map of complexes $K^\bullet(x^t; R) \to K^\bullet(x^{t+1}; R)$ via the commutative diagram

$$\begin{tikzcd}
K^\bullet(x^t; R): & 0 \arrow[r] & R \arrow[r, "\cdot x^t"] \arrow[d, "\mathrm{id}"] & R \arrow[r] \arrow[d, "\cdot x"] & 0 \\
K^\bullet(x^{t+1}; R): & 0 \arrow[r] & R \arrow[r, "\cdot x^{t+1}"] & R \arrow[r] & 0
\end{tikzcd} \tag{1}$$

By tensoring these maps for all the x_i, and then tensoring with M, we get $K^\bullet(\underline{x}^t; M) \to K^\bullet(\underline{x}^{t+1}; M)$. Take the direct limit and denote the resulting complex by $K^\bullet(\underline{x}^\infty; M)$. We can look at $H^i(K^\bullet(\underline{x}^\infty; M)) = \varinjlim_t H^i(K^\bullet(\underline{x}^t; M))$ for which we simply write $H^i(\underline{x}^\infty; M)$.

Theorem 24. *If $I = (x_1, \ldots, x_n)$ and M is an R-module, then there is a canonical isomorphism $H^i(\underline{x}^\infty; M) \cong H^i_I(M)$.*

Consider the case where $\underline{x} = x_1, \ldots, x_n$ is an R-regular sequence. It is known that $K_\bullet(\underline{x}^t; R)$ is a projective resolution of $R/\underline{x}^t R$. Apply $\operatorname{Hom}_R(\_, M)$ and note that on the one hand $K^\bullet(\underline{x}^t; M)$ gives $H^\bullet(\underline{x}^t; M)$ while on the other hand we get $\operatorname{Ext}^\bullet_R(R/\underline{x}^t; M)$. Now take the direct limit: $H^i(\underline{x}^\infty; M) \cong H^i_I(M)$.

Discussion 25 (The Cěch complex and a detailed look at $K^\bullet(\underline{x}^\infty; M)$). Let $x \in R$. Then $K^0(x^\infty; R) = R$ and $K^1(x^\infty; R) = R_x$. (Recall diagram (1).) So $K^\bullet(\underline{x}^\infty; R) = \bigotimes_{i=1}^n (0 \to R \to R_{x_i} \to 0)$, that is,

$$K^j(\underline{x}^\infty; R) = \bigoplus_{|S|=j} R_{x(S)}$$

where $S \subseteq \{1, \ldots, n\}$ and $x(S) = \prod\limits_{i \in S} x_i$. Similarly, $K^j(\underline{x}^\infty; M) = \bigoplus_{|S|=j} M_{x(S)}$. The map $R_{x(S)} \to R_{x(T)}$, where $|T| = |S| + 1$, is the zero map unless $S \subset T$, in which case it is the localization map times $(-1)^a$ where a is the number of elements in S preceding the element in $T \setminus S$.

Exercise 26. Let $x, y, z \in R$. Write down the maps in

$$0 \to R \to R_x \oplus R_y \oplus R_z \to R_{xy} \oplus R_{yz} \oplus R_{zx} \to R_{xyz} \to 0.$$

Discussion 25 and Theorem 24 imply the following.

Corollary 27. *If $I \subseteq R$ is an ideal which can be generated by n elements up to radical, then $H^i_I(R) = 0$ for $i > n$.*

Remark 28. The modules occurring in $K^\bullet(\underline{x}^\infty; R)$ are flat.

3 Properties of local cohomology

Proposition 29. *Let R and S be Noetherian rings.*

1. *Let $R \to S$ be a ring homomorphism, I an ideal of R, and M an S-module. Then $H^i_I(M) \cong H^i_{IS}(M)$ as S-modules.*
2. *Let Λ be a directed set and $\{M_\lambda\}_{\lambda \in \Lambda}$ a direct system of R-modules. Then $\varinjlim_\lambda H^i_I(M_\lambda) \cong H^i_I(\varinjlim_\lambda M_\lambda)$.*
3. *If S is flat over R, then $H^i_I(M) \otimes_R S = H^i_{IS}(M \otimes_R S)$.*
4. *If $\mathfrak{m} \subseteq R$ is a maximal ideal, then $H^i_\mathfrak{m}(M) \cong H^i_{\mathfrak{m}R_\mathfrak{m}}(M_\mathfrak{m})$.*
5. *If $(R, \mathfrak{m})$ is local, then $H^i_\mathfrak{m}(M) \cong H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ which is isomorphic to $H^i_{\mathfrak{m}\hat{R}}(\hat{M})$ if M is finitely generated.*

Proposition 30. *Let $I \subseteq R$ be an ideal which is the radical of an ideal generated by a regular sequence of length n. Then $H^i_I(M) \cong \operatorname{Tor}_{n-i}(M, H^n_I(R))$ for $i \leqslant n$.*

Proof. If $i < \operatorname{depth}_I(R) = n$, then $H^i_I(R) = 0$. Therefore $K^\bullet(\underline{x}^\infty; R)$ gives a flat resolution of $H^n_I(R)$, numbered backwards:

$$\cdots \to K^{n-1} \to K^n \to H^n_I(R) = K^n / \operatorname{im}(d^{n-1}) \to 0.$$

On the one hand $\operatorname{Tor}^R_{n-i}(M, H^n_I(R)) = H^i(K^\bullet(\underline{x}^\infty; R) \otimes_R M)$ by definition of Tor. On the other hand, by the preceding theorem, $H^i(K^\bullet(\underline{x}^\infty; R) \otimes_R M) \cong H^i_I(M)$. $\square$

Corollary 31. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension n. Then $H^i_{\mathfrak{m}}(M) \cong \operatorname{Tor}^R_{n-i}(M, H^n_{\mathfrak{m}}(R))$.*

Proof. The maximal ideal $\mathfrak{m}$ is the radical of an ideal generated by a(ny) regular sequence of length n. □

Theorem 32 (Grothendieck's Theorems).

(Vanishing Theorem) *Let $I \subseteq R$ be an ideal and M an R-module. Then $H^i_I(M) = 0$ for $i > \dim(M)$.*

(Non-Vanishing Theorem) *Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = \dim(M)$.*

Proof of 1. We may assume that R is local with maximal ideal $\mathfrak{m}$. Further, as M is the direct limit of its finitely generated submodules, we may also assume that M is finitely generated. Set $S := R/\operatorname{Ann}(M)$ so that $n := \dim(M) = \dim(S)$. The maximal ideal of S is generated by n elements up to radical, so $H^i_{\mathfrak{m}S}(M) = 0$ for $i > n$.

We want to show that $H^i_I(M) = 0$ for $i > \dim(M)$. By induction, we assume the theorem is true for all finitely generated modules of dimension less than n. We leave the case $n = 0$ as an exercise and assume $n > 0$.

Note that if a module is I-torsion, then all its higher local cohomology modules vanish. So, as $\Gamma_I(M)$ is I-torsion, without loss of generality $\Gamma_I(M) \neq M$. Also, the long exact sequence induced by

$$0 \to \Gamma_I(M) \to M \to {}^{M}\!/_{\Gamma_I(M)} \to 0$$

yields that $H^i_I(M) \cong H^i_I(M/\Gamma_I(M))$ for all $i > 0$. Hence, by passing to $M/\Gamma_I(M)$, we may assume that $M \neq 0$ is I-torsionfree. It follows that I contains an M-regular element r. (Otherwise I is contained in the union of the associated primes of M, and by prime avoidance I is contained in one of those primes which is of the form $(0 :_R m)$ for some $0 \neq m \in M$. That is $Im = 0$ — a contradiction.)

Let $i > n$ and let t be an integer. Consider the short exact sequence

$$0 \to M \xrightarrow{\cdot r^t} M \to {}^{M}\!/_{r^t M} \to 0$$

and the induced long exact sequence

$$\cdots \to H^{i-1}_I(M/r^t M) \to H^i_I(M) \xrightarrow{\cdot r^t} H^i_I(M) \to \cdots$$

Since $\dim(M/r^t M) < \dim(M)$, $H^{i-1}_I(M/r^t M) = 0$ by induction hypothesis and r^t is a nonzerodivisor on $H^i_I(M)$. But any element of $H^i_I(M)$ is killed by a power of I. As $r \in I$, that implies $H^i_I(M) = 0$. □

Remark 33. If $\dim(R) = n$ and M is an R-module, then $H^n_I(M) \cong M \otimes_R H^n_I(R)$. This follows from the fact that $H^n_{\mathfrak{m}}(-)$ is a right exact functor, a consequence of Grothedieck's Vanishing Theorem.

4 Kunneth formula and applications

We would like to discuss the local cohomology modules of a graded ring. Let $R = \oplus_{n \geqslant 0} R_n$ be a $\mathbf{N}$-graded ring over $R_0 = K$, where K is an algebraically closed field. Assume that R is finitely generated over K. Let $\mathfrak{m}_R$ be the maximal ideal $\oplus_{n \geqslant 1} R_n$. Do the modules $H^i_{\mathfrak{m}_R}(R)$ inherit a natural grading from the ring R? The answer is yes and we plan to explain how this goes.

Let I be a graded ideal of R. We can consider the category of graded R-modules and homogeneous R-homomorphisms which is an Abelian category which has enough projective objects and injective objects so the construction used to define the local cohomology modules can be imitated in this setting.

Let M, N be two $\mathbb{Z}$-graded R-modules. Let i be an integer and denote $\underline{\operatorname{Hom}}_R(M, N)_i$ the Abelian group of R-linear maps $f : M \to N$ such that $f(M_n) = N_{n+i}$ for all n. The R-module of *homogeneous homomorphisms* between M and N is by definition $\underline{\operatorname{Hom}}_R(M, N) := \oplus_{i \geqslant 0} \underline{\operatorname{Hom}}_R(M, N)_i$. One can check that, for a fixed graded R-module N, $\underline{\operatorname{Hom}}_R(-, N)$ defines a functor on the category of graded R-modules.

For every graded R-module N, the ith right derived functor of $\underline{\operatorname{Hom}}_R(-, N)$ will be denoted $\underline{\operatorname{Ext}}^i_R(-, N)$. Then, $\underline{H}^i_I(N) = \varinjlim \underline{\operatorname{Ext}}^i_R(R/I^n, N)$.

Since R/I^n are finitely generated for all n, then $\underline{\mathrm{Ext}}^i_R(R/I^n, N) = \mathrm{Ext}^i_R(R/I^n, N)$ as R-modules (ignoring the grading on the first module). As a result, we see that $\underline{H}^i_I(N)$ has a natural **Z**-grading, and, if we ignore this grading, we recover $H^i_I(N)$. The graded local cohomology can also be defined via Cěch or Koszul complexes and all these approaches agree with each other. In the end, we obtain a naturally graded object such that if we drop its grading we recover the standard local cohomology module. Important information resides in this extra structure given by the grading on the local cohomology module and the *a-invariant* of a ring R, $a(R)$ captures some of it. By definition, $a(R) = \max\{n : (\underline{H}^d_{\mathfrak{m}_R}(R))_n \neq 0\}$, where $d = \dim(R)$. This number will become relevant in the discussion that follows.

Let $A = \oplus A_n$ and $B = \oplus B_n$ be two **N**-graded rings such that $A_0 = B_0 = K$. The *Segre product* of these two rings is the ring $A \sharp B = \oplus_n A_n \otimes_K B_n$. This ring is naturally **N**-graded and a direct summand in the ring $A \otimes_K B$. More generally, if M is a **Z**-graded A-module and N is a **Z**-graded B-module, *Segre product* of M and B is the graded $A \sharp B$-module defined by $M \sharp N = \oplus_n M_n \otimes_K N_n$.

Goto and Watanabe [GW] have proved a formula describing the local cohomology of the Segre product R with support at the maximal ideal $\mathfrak{m}_R = \mathfrak{m}_A \sharp \mathfrak{m}_B$. In fact, their result is more general:

Theorem 34 (Goto-Watanabe). *With notations as above, assume that $\underline{H}^j_{\mathfrak{m}_A}(M) = \underline{H}^j_{\mathfrak{m}_B}(N) = 0$ for $j = 0, 1$. Then for all $i \geqslant 0$, we have the following isomorphism:*

$$\underline{H}^i_{\mathfrak{m}_R}(M \sharp N) \simeq (\underline{H}^i_{\mathfrak{m}_A}(M) \sharp N) \oplus (M \sharp \underline{H}^i_{\mathfrak{m}_B}(N)) \oplus (\oplus_{r+s=i+1} \underline{H}^r_{\mathfrak{m}_A}(M) \sharp \underline{H}^s_{\mathfrak{m}_B}(N)).$$

Corollary 35. *Let A and B as above and assume that $\dim(A) = d$, $\dim(B) = e$. Then*

1. *If $d, e \geqslant 1$, then $\dim(A \sharp B) = d + e - 1$.*

2. *If $d, e \geqslant 2$ and if A, B are Cohen-Macaulay, then $A \sharp B$ is Cohen-Macaulay if and only if $\underline{H}^d_{\mathfrak{m}_A}(A) \sharp B = A \sharp \underline{H}^e_{\mathfrak{m}_B}(B) = 0$.*

3. *If $d = 1$ and $e \geqslant 2$, then if A, B are Cohen-Macaulay then $A \sharp B$ is Cohen-Macaulay if and only if $\underline{H}^1_{\mathfrak{m}_A}(A) \sharp B = 0$.*

4. *If $d = e = 1$, and if A, B are Cohen-Macaulay, then $A \sharp B$ is Cohen-Macaulay .*

This corollary is a source of examples of normal rings that are not Cohen-Macaulay: if A, B are normal and Cohen-Macaulay, then $A \sharp B$ is normal as a direct summand in a normal ring $A \otimes_K B$. But often $A \sharp B$ is not Cohen-Macaulay. If $\dim(A) = d \geqslant 2, \dim(B) = e \geqslant 2$, then $\underline{H}^d_{\mathfrak{m}_A}(A) \sharp B = A \sharp \underline{H}^e_{\mathfrak{m}_B}(B) = 0$ is a restrictive condition equivalent to $a(A) < 0$, $a(B) < 0$. For example, $A = k[x, y, z]/(x^3 + y^3 + z^3)$, then the a-invariant of A is 0, and hence, for any Cohen-Macaulay ring B, $A \sharp B$ is not Cohen-Macaulay.

Two recent papers dealing with Segre products and local cohomology are [Si, SW], where in [SW] an example of a normal, but not Cohen-Macaulay ring, obtained as above as a Segre product is studied in detail.

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