Commuting nilpotent matrices and pairs of partitions

Roberta Basili

”Algebraic Combinatorics Meets Inverse Systems”

Montréal, January 19-21, 2007

We will explain some results on commuting $n \times n$ matrices and on commuting $n \times n$ nilpotent matrices over an algebraically closed field K. Tony Iarrobino will explain some applications of this subjects. Let $\mathcal{M}(n, K)$ be the set of all the $n \times n$ matrices over K. Let $\mathcal{N}(n, K)$ be the subset of $\mathcal{M}(n, K)$ of the nilpotent matrices. For $C \in \mathcal{M}(n, K)$ let $\mathcal{Z}_C$ be the centralizer of C in $\mathcal{M}(n, K)$. We first need to talk about some elementary results on commuting matrices.

If C is a diagonal block matrix and any eigenvalue of C is eigenvalue of only one block then any $D \in \mathcal{Z}_C$ is also a diagonal block matrix with blocks of the same orders as in C. This applies in particular to the case in which any diagonal block of C has only one eigenvalue, which happens if the matrix C is in Jordan canonical form.

Example Let

$$C = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix}$$

where

$$C_1 = \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_1 & & \\ & & & \lambda_1 & 1 \\ & & & & \lambda_1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \lambda_2 & 1 & \\ & \lambda_2 & \\ & & \lambda_2 \end{pmatrix}$$

and $\lambda_1 \neq \lambda_2$ (the entries which are omitted are 0). If $D \in \mathcal{Z}_C$ then

$$D = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}$$

where $D_1 \in \mathcal{Z}_{C_1}$ and $D_2 \in \mathcal{Z}_{C_2}$.

Let I be the identity matrix in $\mathcal{M}(n, K)$. A matrix with only one eigenvalue λ is the sum of λI and a nilpotent matrix, then its centralizer is the centralizer of its nilpotent part. Hence in the study of many properties of commuting matrices one can consider only nilpotent matrices.

Let us fix $B \in \mathcal{N}(n, K)$ and let $u_1 \geq \ldots \geq u_t$ be the orders of the Jordan blocks of B. We will say that $(u_1, \ldots, u_t)$ is the partition (of n) of the matrix B and we will denote this partition by P.

Let Δ_B be a basis of K^n with respect to which B is in Jordan canonical form. For any $X \in \mathcal{M}(n, K)$ let us consider the matrix which represents X with respect to Δ_B as a block matrix (X_{hk}), $h, k = 1, \ldots, t$, where X_{hk} is an $u_h \times u_k$ matrix.

Example If

$$B = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

and $X \in \mathcal{M}(5, K)$ we write

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

where $X_{11} \in \mathcal{M}(3, K)$, $X_{22} \in \mathcal{M}(2, K)$ and X_{12}, X_{21} are respectively a 3×2 and a 2×3 matrix over K.

Lemma 1 (H.W. Turnbull and A.C. Aitken, 1931, [2]). *If $A \in \mathcal{M}(n, K)$ we have $AB = BA$ if and only if for $1 \leq k \leq h \leq t$ the matrices A_{hk} and A_{kh} have the following form:*

$$A_{hk} = \begin{pmatrix} 0 & \dots & 0 & a_{hk}^1 & a_{hk}^2 & \dots & a_{hk}^{u_h} \\ \vdots & & & 0 & a_{hk}^1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & a_{hk}^2 \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{hk}^1 \end{pmatrix},$$

$$A_{kh} = \begin{pmatrix} a_{kh}^1 & a_{kh}^2 & \dots & a_{kh}^{u_h} \\ 0 & a_{kh}^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{kh}^2 \\ \vdots & & 0 & a_{kh}^1 \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

where for $u_h = u_k$ we omit the first $u_k - u_h$ columns and the last $u_k - u_h$ rows respectively.

A matrix $R \in \mathcal{M}(n,K)$ is said regular if the minimum polynomial of R has degree n, that is if $I,\ R,\ R^2,\ \ldots, R^{n-1}$ are linearly independent. The subset of $\mathcal{M}(n,K)$ of the regular matrices is open. The regular matrices of $\mathcal{N}(n,K)$ are the matrices conjugated to the Jordan block $J_n \in \mathcal{N}(n,K)$. R is regular iff different Jordan blocks of R have different eigenvalues. By Lemma 1 we get that B is regular iff $\mathcal{Z}_B$ has minimum dimension n; this property extends to any $n \times n$ matrix. If R is regular then $\mathcal{Z}_R$ is the vector space generated by the powers of R (including also $R^0 = I$).

Let

$$\mathcal{C}(n,K) = \{(C,D) \in \mathcal{M}(n,K) \times \mathcal{M}(n,K)\ :\ CD = DC\}$$

which is said commuting variety. The open subset of $\mathcal{C}(n,K)$ of all (C,D) such that D is regular is irreducible and is dense, since in the centralizer of any matrix the subset of all the regular matrices is dense. This is the easiest proof of the irreducibility of $\mathcal{C}(n,K)$; this result was first proved by Motzkin and Taussky ([8]) and, independently, by Gerstenhaber ([5]). From this it follows that $\mathcal{C}(n,K)$ has dimension n^2+n, that is has codimension n^2-n. It is a conjecture, due to Artin and Hochster, that the n^2 equations for $\mathcal{C}(n,K)$ given by the condition $CD = DC$ generate a radical ideal.
The irreducibility of $\mathcal{C}(n,K)$ also shows that the maximum dimension of an algebra generated by two commuting $n \times n$ matrices is n, since it is the dimension of the algebra generated by two commuting matrices such that one of them is regular. This result was pointed out by Gerstenhaber ([5]). It isn't known if the maximum dimension of the algebra generated by three $n \times n$ matrices such that any two of them commute is still n. This is obviously true if one of the matrices is regular or, more generally, if two of the matrices commute with a regular matrix. Neubauer and Sethuraman ([10]) have shown that this is still true if two of the matrices commute with a 2-regular matrix, that is a matrix such that for any eigenvalue there are at most two

Jordan blocks. This is a quite interesting open problem.
The irreducibility of $\mathcal{C}(n, K)$ has been extended by Richardson ([12]) to the variety of the commuting pairs of elements of a reductive Lie algebra.

The variety $\mathcal{N}(n, K)$ is irreducible of dimension $n^2 - n$. In fact, let $\mathcal{D}$ be the subspace of $\mathcal{M}(n, K)$ of all $X = (x_{ij}),\ i, j = 1, \ldots, n$, such that $x_{ij} = 0$ unless $j - i = 1$. Since any matrix has a Jordan canonical form, there exists a surjective morphism

$$\mathrm{GL}(n, K) \times \mathcal{D} \rightarrow \mathcal{N}(n, K);$$

moreover $\mathcal{N}(n, K)$ is the closure of the orbit of J_n, which has dimension $n^2 - n$.
We now recall some properties of the following variety:

$$\mathcal{H}(n, K) = \{(C, D) \in \mathcal{N}(n, K) \times \mathcal{N}(n, K)\ :\ CD = DC\}$$

One of the main reasons of interest in this subject is that the variety $\mathcal{C}(n, K)$ is closely related to the Hilbert scheme of n points of an algebraic surface; this relation will be explained by Tony Iarrobino. $\mathcal{H}(n, K)$ is related to the local Hilbert scheme, that is to the case of lenght n schemes concentrated at a single point.
The irreducibility of $\mathcal{H}(n, K)$ is much harder to prove than that of $\mathcal{C}(n, K)$. It has been proved by Baranovsky ([1]) showing that it is equivalent to the irreducibility of the local Hilbert scheme of n points on a surface and using the quite difficult proof by Briançon ([4]) of the irreducibility of this scheme (which assumes char $K = 0$; a variant by Iarrobino ([7]) assumes char $K > n$). Then I ([3]) proved the irreduciblity of $\mathcal{H}(n, K)$ only by calculations with matrices and with the small extension to the case char $K \geq \dfrac{n}{2}$. Recently Premet ([11]) proved this result as a particular case of a theorem on the irreducible components of the variety of the pair of commuting nilpotent

elements of a reductive Lie algebra, without any hypothesis on char K. These methods give both independent proofs of Briançon's irreducibility theorem.

Let

$$\mathcal{N}_B = \mathcal{Z}_B \cap \mathcal{N}(n, K).$$

My proof of the irreducibility of $\mathcal{H}(n, K)$ uses some results on $\mathcal{N}_B$ which can be used to get other information and I am going to explain.

Let $q_i \in \{1, \ldots, t\}$, $i = 1, \ldots, \bar{t}$, be such that $q_1 = 1$, $u_{q_i} = u_{q_{i+1}-1} > u_{q_{i+1}}$ for $i = 1, \ldots, \bar{t} - 1$, $u_{q_{\bar{t}}} = u_t$.

Example If $P = (5, 5, 5, 4, 4, 3, 2, 2)$ then $q_1 = 1$, $q_2 = 4$ (that is the index of the first 4), $q_3 = 6$, $q_4 = 7$. There are four subpartitions such that the numbers of each of them are equal.

Let $A \in \mathcal{Z}_B$ and let us consider again the matrix (A_{hk}) for $h, k = 1, \ldots, t$. For $i = 1, \ldots, \bar{t}$ let

$$\bar{A}_i = (a^1_{hk}) \quad \text{where} \quad \begin{cases} q_i \le h, k \le q_{i+1} - 1 & \text{if} \quad i = 1, \ldots, \bar{t} - 1 \\ q_{\bar{t}} \le h, k \le t & \text{if} \quad i = \bar{t} \end{cases}$$

Lemma 2 *$A \in \mathcal{N}_B$ iff $\bar{A}_i$ is nilpotent for $i = 1, \ldots, \bar{t}$, hence $\mathcal{N}_B$ is irreducible.*
Moreover for any $A \in \mathcal{N}_B$ it is possible to choose Δ_B such that, besides B being in Jordan canonical form, $\bar{A}_i$ is upper triangular for $i = 1, \ldots, \bar{t}$.

Example If

$$B = \begin{pmatrix} J_3 & & & \\ & J_3 & & \\ & & J_3 & \\ & & & J_2 \end{pmatrix}$$

the generic element A of $\mathcal{N}_B$ is

$$\begin{pmatrix}
a_{11}^1 & a_{11}^2 & a_{11}^3 & a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{13}^1 & a_{13}^2 & a_{13}^3 & a_{14}^1 & a_{14}^2 \\
 & a_{11}^1 & a_{11}^2 & & a_{12}^1 & a_{12}^2 & & a_{13}^1 & a_{13}^2 & & a_{14}^1 \\
 & & a_{11}^1 & & & a_{12}^1 & & & a_{13}^1 & & \\
 & & & & & & & & & & \\
a_{21}^1 & a_{21}^2 & a_{21}^3 & a_{22}^1 & a_{22}^2 & a_{22}^3 & a_{23}^1 & a_{23}^2 & a_{23}^3 & a_{24}^1 & a_{24}^2 \\
 & a_{21}^1 & a_{21}^2 & & a_{22}^1 & a_{22}^2 & & a_{23}^1 & a_{23}^2 & & a_{24}^1 \\
 & & a_{21}^1 & & & a_{22}^1 & & & a_{23}^1 & & \\
 & & & & & & & & & & \\
a_{31}^1 & a_{31}^2 & a_{31}^3 & a_{32}^1 & a_{32}^2 & a_{32}^3 & a_{33}^1 & a_{33}^2 & a_{33}^3 & a_{34}^1 & a_{34}^2 \\
 & a_{31}^1 & a_{31}^2 & & a_{32}^1 & a_{32}^2 & & a_{33}^1 & a_{33}^2 & & a_{34}^1 \\
 & & a_{31}^1 & & & a_{32}^1 & & & a_{33}^1 & & \\
 & & & & & & & & & & \\
 & a_{41}^1 & a_{41}^2 & & a_{42}^1 & a_{42}^2 & & a_{43}^1 & a_{43}^2 & 0 & a_{44}^2 \\
 & & a_{41}^1 & & & a_{42}^1 & & & a_{43}^1 & & 0
\end{pmatrix}$$

where $\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 \\ a_{21}^1 & a_{22}^2 & a_{23}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 \end{pmatrix}$ is nilpotent.

For any $A \in \mathcal{N}_B$ it is possible to choose Δ_B such that A has the following form:

$$
\begin{pmatrix}
0 & a_{11}^2 & a_{11}^3 & a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{13}^1 & a_{13}^2 & a_{13}^3 & a_{14}^1 & a_{14}^2 \\
 & 0 & a_{11}^2 & & a_{12}^1 & a_{12}^2 & & a_{13}^1 & a_{13}^2 & & a_{14}^1 \\
 & & 0 & & & a_{12}^1 & & & a_{13}^1 & & \\
0 & a_{21}^2 & a_{21}^3 & 0 & a_{22}^2 & a_{22}^3 & a_{23}^1 & a_{23}^2 & a_{23}^3 & a_{24}^1 & a_{24}^2 \\
 & 0 & a_{21}^2 & & 0 & a_{22}^2 & & a_{23}^1 & a_{23}^2 & & a_{24}^1 \\
 & & 0 & & & 0 & & & a_{23}^1 & & \\
0 & a_{31}^2 & a_{31}^3 & 0 & a_{32}^2 & a_{32}^3 & 0 & a_{33}^2 & a_{33}^3 & a_{34}^1 & a_{34}^2 \\
 & 0 & a_{31}^2 & & 0 & a_{32}^2 & & 0 & a_{33}^2 & & a_{34}^1 \\
 & & 0 & & & 0 & & & 0 & & \\
 & a_{41}^1 & a_{41}^2 & & a_{42}^1 & a_{42}^2 & & a_{43}^1 & a_{43}^2 & 0 & a_{44}^2 \\
 & & a_{41}^1 & & & a_{42}^1 & & & a_{43}^1 & & 0
\end{pmatrix}
$$

In the study of the geometrical properties of these varieties it is helpful to describe the relation between the partitions of two nilpotent matrices with the property that the orbit of one of them is contained in the closure of the orbit of the other. For this purpose, in the set of the orbits of $\mathcal{N}(n, K)$ under the action of $\mathrm{GL}(n, K)$ the following partial order is defined. Let $\mathcal{O}_C$ and $\mathcal{O}_D$ be respectively the orbits of C and D. We have that $\mathcal{O}_C = \mathcal{O}_D$ iff $\operatorname{rank} C^m = \operatorname{rank} D^m$ for any $m \in \mathbb{N}$. We say that $\mathcal{O}_C < \mathcal{O}_D$ if $\operatorname{rank} C^m \leq \operatorname{rank} D^m$ for any $m \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ such that $\operatorname{rank} C^m < \operatorname{rank} D^m$. The following claim is the basic theorem on degenerations of orbits, due to Hesselink ([6]): $\mathcal{O}_C < \mathcal{O}_D$ iff $\mathcal{O}_C \subset \overline{\mathcal{O}_D}$.

We have that $\mathcal{O}_C < \mathcal{O}_D$ and there are no orbits between them if and only if the partition of C can be obtained from the partition of D in the following way: subtracting 1 to a number and adding 1 to another number which is smaller than the previous one of at least 2 (in particular, also to the number

0 considered as last number of the partition). We can consider this order on the orbits as an order on the partitions of n.

Examples

$$(5,3,2,1) < (6,3,1,1) < (6,4,1)$$

$$(6,4,3) < (6,5,2) < (6,6,1)$$

Let Y be any irreducible subvariety of $\mathcal{N}(n,K)$. For any $m \in \mathbb{N}$, $m < n$, the subset of Y of all A such that rank A^m is the maximum possible is open. Since Y is irreducible, the intersection of these open subsets is non-empty. Hence there is a maximum partition for the elements of Y and there exists an open subset of Y whose elements are the elements which have this partition. Since these elements form a dense subset of Y, the knowledge of the maximum partition can be useful in the study of many properties of Y. This applies in particular to the subvariety $\mathcal{N}_B$.

With Tony Iarrobino we have studied the problem of finding the possible partitions, or the maximum possible partition $Q(P)$, for the elements of $\mathcal{N}_B$. We have solved this problem only in some special cases, by using the following results.

For $s \in \mathbb{N} - \{0\}$ let q and r be the quotient and the remainder of the division of n by s. Then $(J_n)^s$ has r Jordan blocks of order $q+1$ and $s-r$ Jordan blocks of order q.

Example If $n = 7$ and $s = 3$ we have $q = 2$ and $r = 1$ and we have

$$\begin{array}{rcl} J_7 & : & e_7 \to e_6 \to e_5 \to e_4 \to e_3 \to e_2 \to e_1 \to 0 \\ \\ (J_7)^3 & : & e_7 \to e_4 \to e_1 \to 0 \\ & & e_6 \to e_3 \to 0 \\ & & e_5 \to e_2 \to 0 \end{array}$$

that is $(J_7)^3$ has partition $(3, 2, 2)$.

Examples $(J_8)^3$ has partition $(3, 3, 2)$, $(J_9)^3$ has partition $(3, 3, 3)$.

Hence if $u_1 - u_t \leq 1$ B is a power of a regular nilpotent matrix. The next proposition shows that this is the only case in which B commutes with a regular nilpotent matrix.

Let $n_i \in \{1, \dots, t\}$, $i = 1, \dots, r_B$, be such that $n_1 = 1$, $u_{n_i} - u_{n_{i+1}-1} \leq 1$, $u_{n_i} - u_{n_{i+1}} > 1$ for $i = 1, \dots, r_B - 1$, $u_{n_{r_B}} - u_t \leq 1$. We have that r_B is the minimum possible l such that there exist partitions $P_1, \dots, P_l$ of natural numbers which together form P and such that the difference between any two numbers of P_i is less or equal than 1 for $i = 1, \dots, l$.

Examples If $P = (4, 4, 2, 1, 1)$ we have $r_B = 2$, if $P = (5, 4, 3, 1, 1)$ we have $r_B = 3$, if $P = (9, 7, 5, 1)$ we have $r_B = 4$.

Proposition 3 *There exists a non-empty open subset of $\mathcal{N}_B$ such that if A belongs to it we have* $\operatorname{rank} A = n - r_B$ *(that is A has r_B Jordan blocks).*

Example If $P = (7, 6, 6, 5, 4, 2)$ we have $r_B = 3$, hence any element of $\mathcal{N}_B$ has at least 3 Jordan blocks.

Let s_B be the maximum of the cardinalities of the subsets

$\{i_1, \dots, i_l\}$ of $\{1, \dots, t\}$ such that $i_1 < \dots < i_l$ and $u_{i_1} - u_{i_l} \le 1$.

Example If $P = (5,4,4,2,2)$ then $(5,4,4)$ is a subpartition of P such that the difference between its first number and its last number is less or equal than 1, moreover it has the maximum possible number of elements among all the subpartitions with this property; hence $s_B = 3$.

Examples If $P = (5,4,3,1,1)$ we have $s_B = 2$, if $P = (7,5,1)$ we have $s_B = 1$.

By Proposition 3 if $s_B > 1$ we have $Q(P) > P$; the next proposition claims that the partition of the s_B-th power of any element of $\mathcal{N}_B$ is less or equal than P.

Proposition 4 *For any* $A \in \mathcal{N}_B$ *and* $m \in \mathbb{N}$ *we have*

$$\operatorname{rank}(A^{s_B})^m \le \operatorname{rank} B^m.$$

Now we describe some partitions P such that the partition $Q(P)$ is easily found.

Let $\widetilde{u}_i = u_{n_i} + \dots + u_{n_{i+1}-1}$ for $i = 1, \dots, r_B - 1$ and let $\widetilde{u}_{r_B} = u_{n_{r_B}} + \dots + u_t$. Let $\widetilde{P} = (\widetilde{u}_1, \dots, \widetilde{u}_{r_B})$.

Examples If $P = (5,4,4,3,2,2)$ then $\widetilde{P} = (5+4+4, 3+2+2) = (13,7)$; if $P = (7,5,2,2,2,1)$ then $\widetilde{P} = (7, 2+2+2+1, 5) = (7,7,5)$.

Proposition 5 *If* $s_B = n_{i+1} - n_i = t + 1 - n_{r_B}$ *for* $i = 1, \dots, r_B - 1$ *we have* $Q(P) = \widetilde{P}$.

Proof. There exists $\widetilde{B} \in \mathcal{N}(n, K)$ with partition $\widetilde{P}$ such that $B = (\widetilde{B})^{s_B}$, hence $\widetilde{B} \in \mathcal{N}_B$. Then there exists a non-empty open subset of $\mathcal{N}_B$ such that if A belongs to it we have $\operatorname{rank}(A^{s_B})^m = \operatorname{rank} B^m$ for $m \in \mathbb{N}$, that is A^{s_B} has the same partition as B. Then A has partition $\widetilde{P}$.

As a particular case we get the following result.

Corollary 6 *If $s_B = 1$ we have $Q(P) = P$.*

Examples If $P = (5, 4, 4, 3, 2, 2)$ we have $Q(P) = (13, 7)$; if $P = (5, 5, 3, 3, 2, 1)$ we have $Q(P) = (10, 8, 1)$; if $P = (8, 5, 3, 1)$ then $Q(P) = P$.

In general it is not true that $Q(P) = \widetilde{P}$. In fact, it may happen that there exists $m \in \mathbb{N}$ such that $\operatorname{rank}\left(A^{s_B}\right)^m < \operatorname{rank} B^m$ for any $A \in \mathcal{N}_B$. For example, if

$$P \quad = \quad (u, \quad \underbrace{1, \ldots, 1}_{s})$$

it can be shown by a calculation that the maximum index of nilpotency of the elements of $\mathcal{N}_B$ is $\max\{u, s+2\}$. Then $Q(P) = (\max\{u, s+2\}, n - \max\{u, s+2\})$ (since $r_B = 2$), which is different from (u, s) if $s > u - 2$.

Examples If $P = (3, 1, 1, 1)$ we have $Q(P) = (5, 1)$; if $P = (7, 1, 1, 1, 1)$ we have $Q(P) = \widetilde{P} = (7, 4)$.

Another interesting open problem is to find for which partitions P the variety of all the pairs of commuting elements of $\mathcal{Z}_B$ is irreducible. This is obviously true if B is regular. Neubauer and Sethuraman ([10]) proved that it is irreducible if $\operatorname{rank} B \geq n - 2$.

References

[1] V. Baranovsky, The Variety of Pairs of Commuting Nilpotent Matrices is Irreducible, *Transform. Groups* **6** n. 1 (2001) 3-8.

[2] R. Basili, On the Irreducibility of Varieties of Commuting Matrices, *J. Pure Appl. Algebra* **149** n. 2 (2000) 107-120.

[3] R. Basili, On the irreducibility of commuting varieties of nilpotent matrices, *J. Algebra* **268** (2003) 58-80.

[4] J. Briançon, Description de $\mathrm{Hilb}^n\, \mathbb{C}\{x, y\}$, *Invent. Math.* **41** n. 1 (1977) 45-89.

[5] M. Gerstenhaber, On Dominance and Varieties of Commuting Matrices, *Ann. Math.* **73** (1961) 324-348.

[6] W. Hesselink, Singularities in the nilpotent scheme of a classical group, *Trans. Am. Math. Soc.* **222** (1976) 1-32.

[7] A.A. Iarrobino, Punctual Hilbert Schemes, *Mem. Amer. Math. Soc.* **10** n. 188 (1977).

[8] T.S. Motzkin and O. Taussky, Pairs of Matrices with Property L II, *Trans. Amer. Math. Soc.* **80** (1955) 387-401.

[9] H. Nakajima, Lectures on Hilbert Schemes of Points on Surfaces, University Lecture Series 18, Providence, RI: American Mathematical Society (1999).

[10] M.G. Neubauer and B.A. Sethuraman, Commuting Pairs in the Centralizer of 2-Regular Matrices, *J. Algebra* **214** (1999) 174-181.

[11] A. Premet, Nilpotent commuting varieties of reductive Lie algebras, Invent. Math. 154, No 3, 653-683 (2003).

[12] R.W. Richardson, Commuting Varieties of Semisimple Lie Algebras and Algebraic Groups, *Compositio Math.* **38** (1979)