TECHNISCHE UNIVERSITÄT WIEN

An SL_4-web basis from hourglass plabic graphs

Stephan Pfannerer

Recipient of a DOC Fellowship of the Austrian Academy of Sciences
Institute of Discrete Mathematics and Geometry
TU Wien

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GOCC

Graduate Online Combinatorics Colloquium

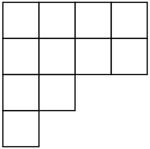
The GOCC is an online combinatorics seminar organized for and run by graduate students following the principle:

1. We are all learning
2. Everyone has something to contribute
3. No one has all the answers

- Starting again in February 2023
- GOCCcombinatorics@gmail.com

Joint work with Jessica Striker, Oliver Pechenik, Joshua Swanson and Christian Gaetz.

Standard Young tableaux



Standard Young tableaux

$\mathrm{SYT}(\lambda)$: set of all standard Young tableaux of shape $\lambda \vdash n$.

- Bijective filling: cells of $\lambda \to [n]$
- Increasing along rows and columns

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- Dimension of the *Specht module* S^λ: $|\mathrm{SYT}(\lambda)|$
- Let λ be a $r \times k$ rectangle, then

$$S^\lambda \cong \text{some invariant space of } \mathrm{SL}_r$$

Promotion

Schützenberger *promotion* pr : $\mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$:

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We want a *diagrammatic basis*. To obtain them we construct bijections between rectangular SYT and some diagrams intertwining promotion and rotation.

SL_2-webs: Non crossing perfect matchings

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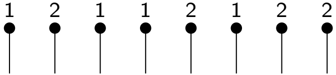

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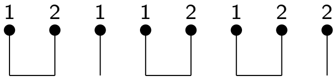

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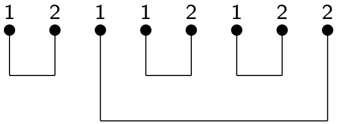

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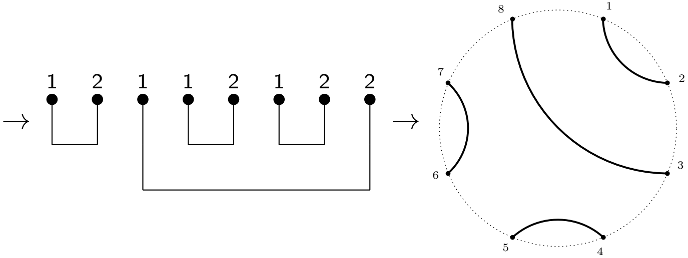

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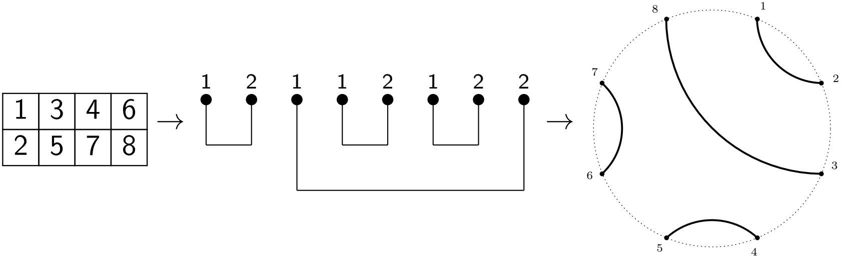

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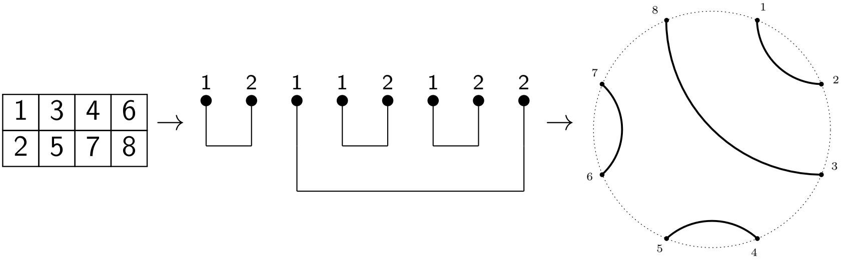

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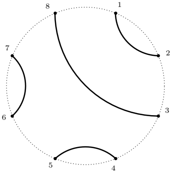
$s_3 \cdot$  $=$

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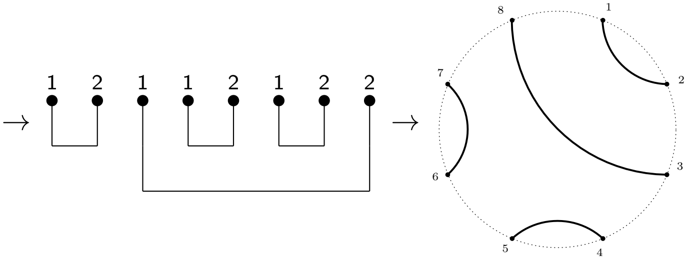

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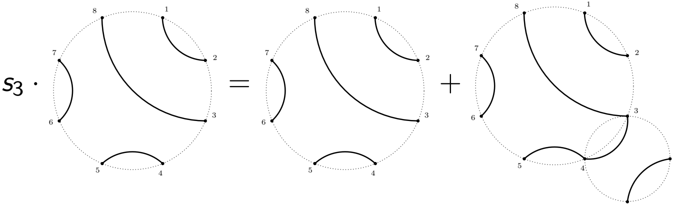

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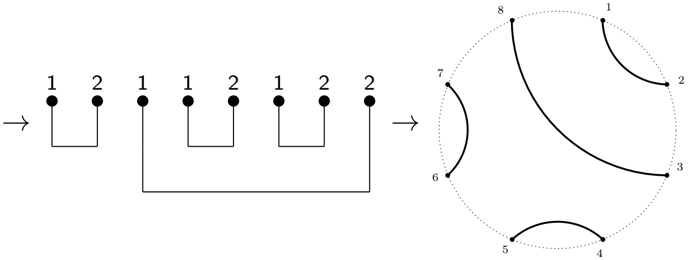

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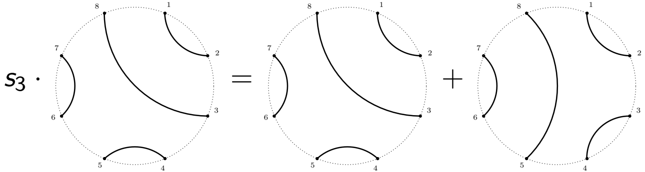

SL_3-webs

- *Khovanov, Kuperberg 1999:*
  $3 \times n$ SYT are in bijection with irreducible SL_3-webs, i.e. certain *plabic graphs*.

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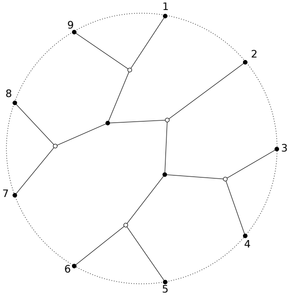

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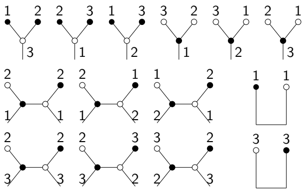

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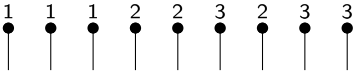

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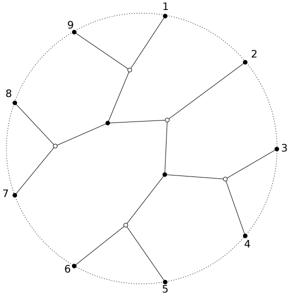

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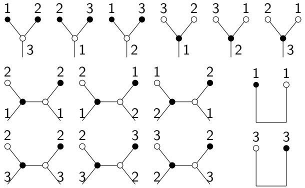

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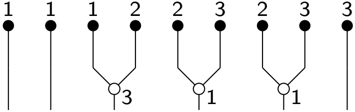

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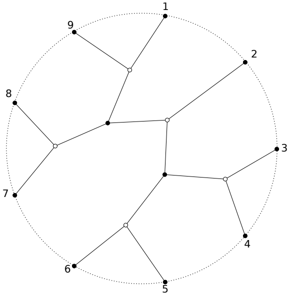

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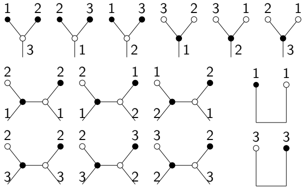

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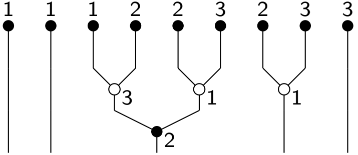

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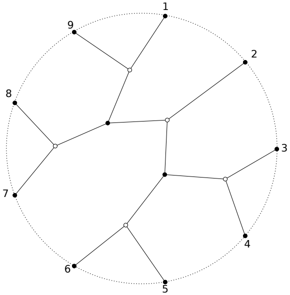

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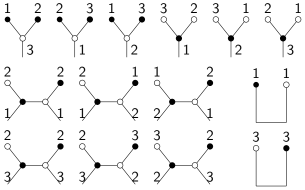

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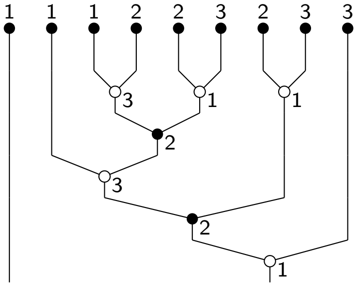

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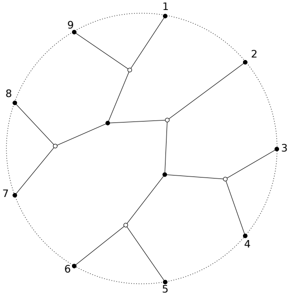

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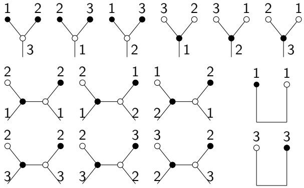

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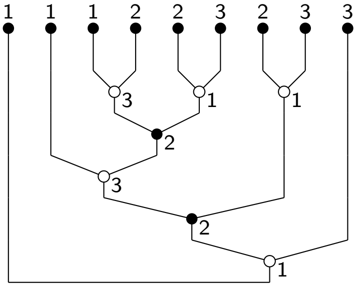

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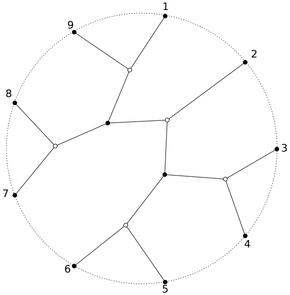

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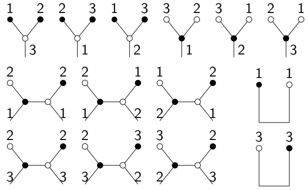

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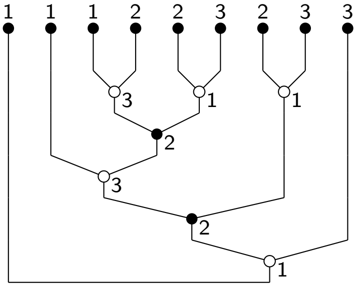

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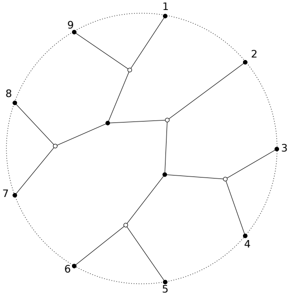
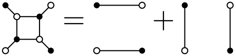


$\mathfrak{S}_n$ action

Attach "uncrossing" and reduce according to .

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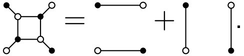
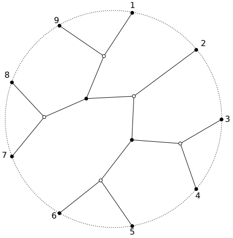
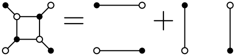
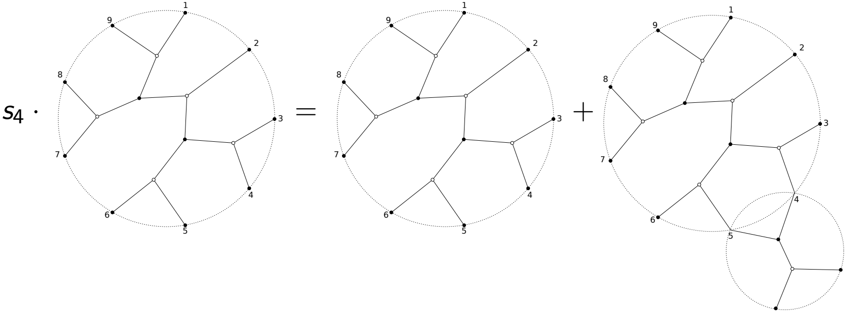
Attach "uncrossing" and reduce according to 

$s_4 \cdot$  $=$

Attach “uncrossing” and reduce according to  .



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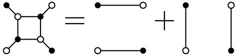
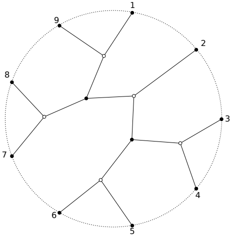
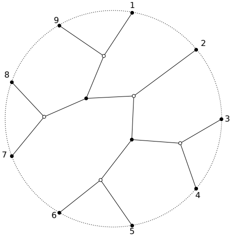
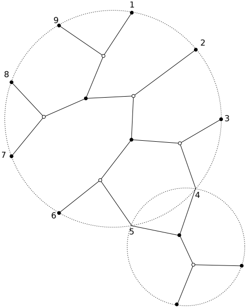
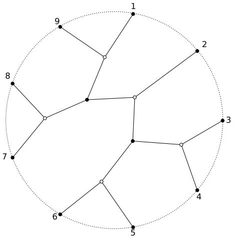
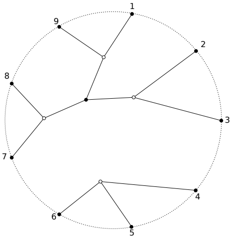
Attach "uncrossing" and reduce according to  .

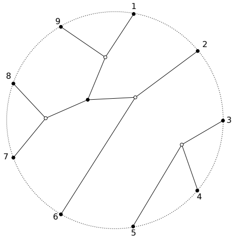
$s_4 \cdot$ 

$=$ 

$+$ 

$=$ 

$+$ 

$+$ 

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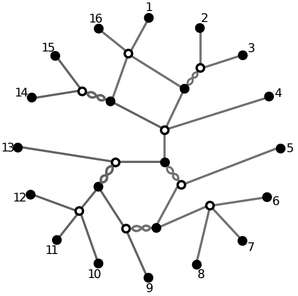

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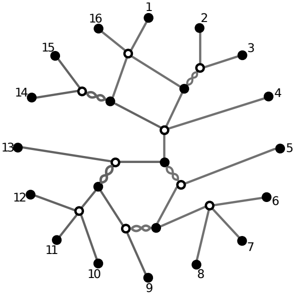

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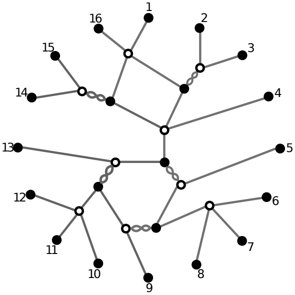
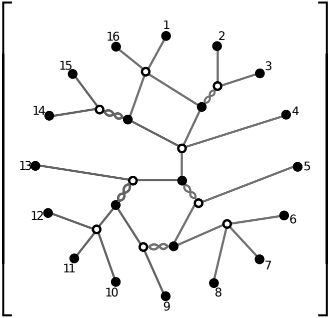


Beyond three rows

- Kuperberg '96: "The main *open problem* [...] is how to generalize them to higher rank."
- SL_r: *Spanning sets* and relations, but NOT a basis by Kim '03, Morrison '07, Fontaine '12, Cautis–Kamnitzer–Morrison '14, Fraser–Lam–Le '19
- SL_4: *Web basis* by Gaetz–Pechenik–P.–Striker–Swanson '23+
  - Over 100 growth rules and two families of infinitely many rules
  - Applying the rules in different order can give different graphs

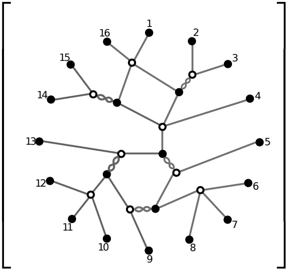
1	2	5	6
3	4	7	10
8	9	11	14
12	13	15	16

$\rightarrow$



Trip permutations

i-th rule of the road: take the *i*-th exit from the left (right) at an unifilled (filled) vertex.



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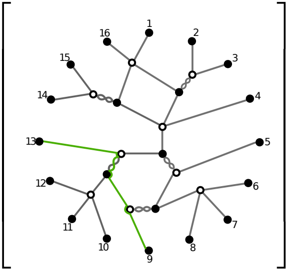

$\xrightarrow{\text{trip}_1}$ 4 3 14 10 9 7 8 16 13 11 12 $\underline{6}$ $\underline{5}$ 15 $\underline{2}$ $\underline{1}$

Trip permutations

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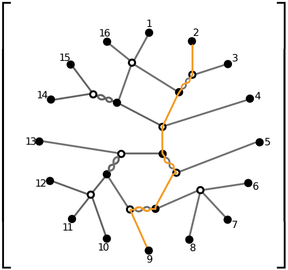
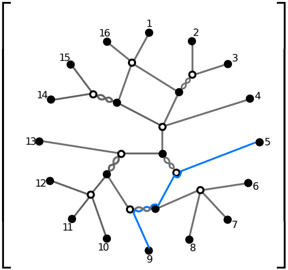


$\xrightarrow{\text{trip}_1}$ 4 3 14 10 9 7 8 16 13 11 12 $\underline{6}$ $\underline{5}$ 15 $\underline{2}$ $\underline{1}$

$\xrightarrow{\text{trip}_2}$ 14 9 16 15 11 8 13 $\underline{6}$ $\underline{2}$ 12 $\underline{5}$ $\underline{10}$ $\underline{7}$ $\underline{1}$ $\underline{4}$ $\underline{3}$

Trip permutations

i-th rule of the road: take the *i*-th exit from the left (right) at an unifilled (filled) vertex.



$\xrightarrow{\text{trip}_1}$ $4\ 3\ 14\ 10\ 9\ 7\ 8\ 16\ 13\ 11\ 12\ \underline{6}\ \underline{5}\ 15\ \underline{2}\ \underline{1}$

$\xrightarrow{\text{trip}_2}$ $14\ 9\ 16\ 15\ 11\ 8\ 13\ \underline{6}\ \underline{2}\ 12\ \underline{5}\ \underline{10}\ \underline{7}\ \underline{1}\ \underline{4}\ \underline{3}$

$\xrightarrow{\text{trip}_3}$ $16\ 15\ \underline{2}\ \underline{1}\ 13\ 12\ \underline{6}\ \underline{7}\ \underline{5}\ \underline{4}\ \underline{10}\ \underline{11}\ \underline{9}\ \underline{3}\ \underline{14}\ \underline{8}$

Trip permutations

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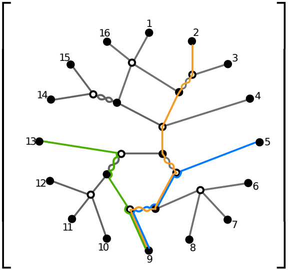

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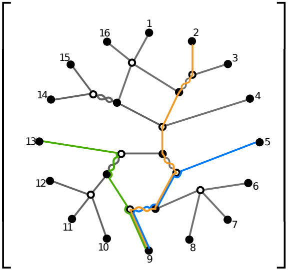

$\xrightarrow{\text{trip}_1}$ $4\ 3\ 14\ 10\ 9\ 7\ 8\ 16\ 13\ 11\ 12\ \underline{6}\ \underline{5}\ 15\ \underline{2}\ \underline{1}$

$\xrightarrow{\text{trip}_2}$ $14\ 9\ 16\ 15\ 11\ 8\ 13\ \underline{6}\ \underline{2}\ 12\ \underline{5}\ \underline{10}\ \underline{7}\ \underline{1}\ \underline{4}\ \underline{3}$

$\xrightarrow{\text{trip}_3}$ $16\ 15\ \underline{2}\ \underline{1}\ 13\ 12\ \underline{6}\ \underline{7}\ \underline{5}\ \underline{4}\ \underline{10}\ \underline{11}\ \underline{9}\ \underline{3}\ \underline{14}\ \underline{8}$

Anti-exceedances of a permutation π:
$\text{Aexc}(\pi) = \{i \mid \pi^{-1}(i) > i\}$

$\text{Aexc}(\text{trip}_i) = \{\text{Entries of first } i \text{ rows of the tableau}\}$

Trip permutations

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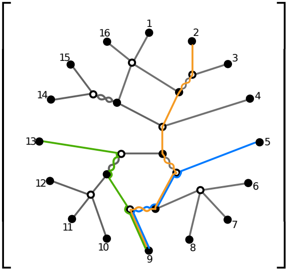

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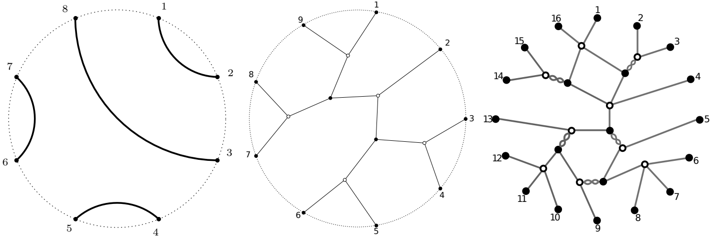
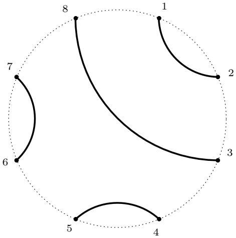
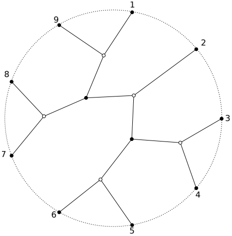
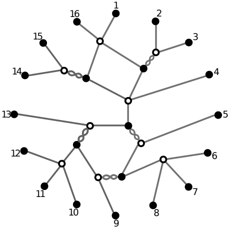
Further buzzwords



Further buzzwords







- Crystal graphs
- Statistical mechanics
- ASMs and plane partitions
- Quantum link invariants
- Cluster algebras
- Totally nonnegative Grassmannian

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Y	O	U		
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