Higher Connectivity of Tropicalizations

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joint work with:

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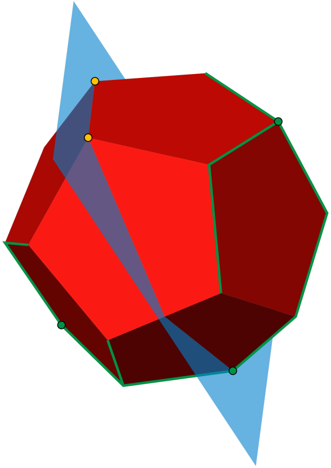
Combinatorial Algebra meets Algebraic Combinatorics

Inspiration

Balinski's Theorem (1961)
The edge graph of a d-dimensional polytope is d-connected, i.e. removing $d-1$ vertices and their incident edges does not disconnect the graph.

Question.
Is there higher connectivity for higher dimensional skeleta of a polytope? for tropicalizations of irreducible varieties?



picture from wikipedia

Tropicalization

Let X be a subvariety of of $(K^*)^n$ defined by an ideal $I \subset K[x_1, \ldots, x_n]$.

Let L be an algebraically closed field extension of K with a non-trivial non-Archimedean valuation.

Example. $K = \mathbb{C}$, $L = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$ = Puiseux series with coefficients in $\mathbb{C}$.
$\mathrm{val}(2t^{-\frac{1}{2}} - 3 + 5t^{\frac{1}{2}} + 7t + \cdots) = -\frac{1}{2}$.

Definition. The tropicalization is $\mathrm{trop}(X) = \overline{\{\mathrm{val}(x) : x \in X(L)\}} \subset \mathbb{R}^n$.

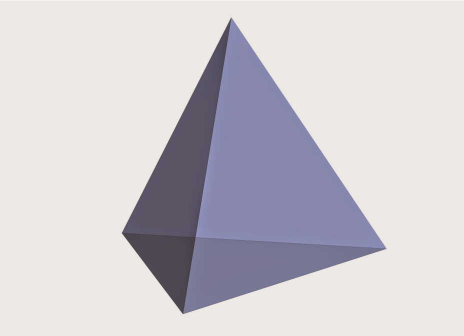
Note: $\mathrm{trop}(X \cup Y) = \mathrm{trop}(X) \cup \mathrm{trop}(Y)$, $\mathrm{trop}(X \cap Y) \subseteq \mathrm{trop}(X) \cap \mathrm{trop}(Y)$.

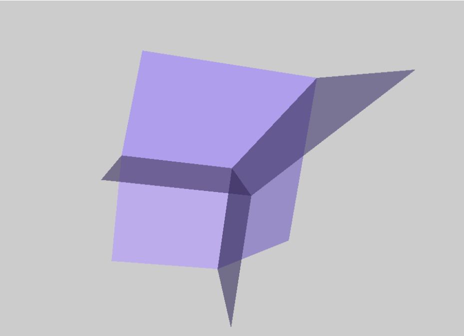
Examples:

- $\mathrm{trop}((K^*)^n) = \mathbb{R}^n$
- $\mathrm{trop}(V(xy - z^2)) = \{(a, b, c) \in \mathbb{R}^3 : a + b = 2c\}$
- $\mathrm{trop}(V(1 + x + y))$ = union of rays in directions $(0, 1), (1, 0), (-1, -1)$

Newton polytope of polynomial f := convex hull of exponent vectors

$\mathrm{trop}(V(f))$ = union of normal cones to edges of the Newton polytope of f





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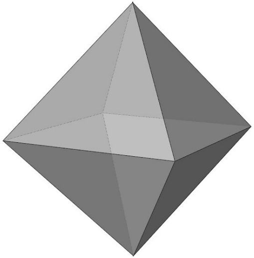
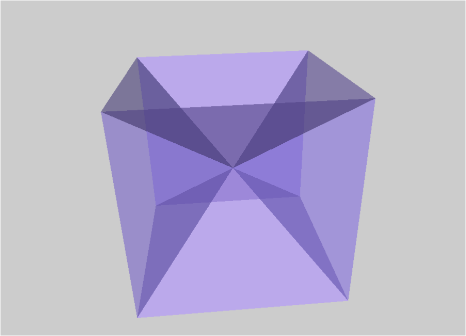
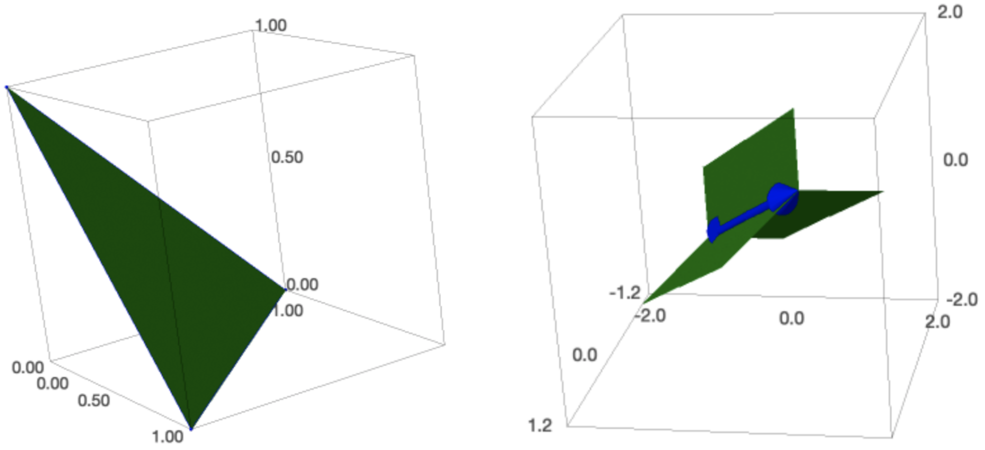






$$\mathrm{trop}(V(x+y+z))$$

The Fundamental Theorem of tropical geometry (Speyer–Sturmfels, Draisma, Payne, Gubler, ... 2003^+)

Let I be an ideal in $K[x_1, \ldots, x_n]$. The following subsets of $\mathbb{R}^n$ coincide.

1. $\operatorname{trop}(I) := \overline{\{\operatorname{val}(x) : x \in X(L)\}}$
2. $\bigcap_{f \in I} \operatorname{trop}(f)$
3. $\{w \in \mathbb{R}^n : in_w(I) \not\ni \text{monomial}\}$

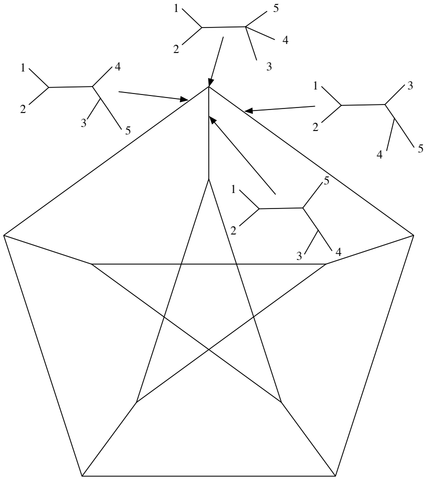
The Structure Theorem.

If X is an irreducible variety in (K^*) of dimension d, then $\operatorname{trop}(X)$

- is a pure d-dimensional polyhedral complex (Bieri–Groves 1984)
- is balanced
- is connected through codimension one (Einsiedler–Kapranov–Lind 2004, Bogart–Jensen–Speyer–Sturmfels–Thomas 2005, Cartwright–Payne 2012)

Tropicalization

- binomial ideal $\rightsquigarrow$ linear space
- principal ideal $\rightsquigarrow$ normal cones to edges of Newton polytope
- suppose $f_1, \ldots, f_r$ have the same Newton polytope P and generic coefficients $\langle f_1, \ldots, f_r \rangle \rightsquigarrow$ normal cones to r-dimensional faces P
- linear space $\rightsquigarrow$ Bergman fan of a matroid
- Grassmannian of lines $Gr(2, n) \rightsquigarrow$ space of phylogenetic trees on n taxa



Phylogenetic tree space on five taxa

Recall the **Structure Theorem**: If X is irreducible, then $\mathrm{trop}(X)$ is connected through codimension one.

Stronger Structure Theorem (Maclagan–Y 2019).
Let K be a field of characteristic 0 that is either algebraically closed, complete, or real closed with convex valuation ring.
Let X be a d-dimensional irreducible subvariety of $(K^*)^n$.
Then $\mathrm{trop}(X)$ **is** $d - \ell$ **connected through codimension one**, where d is the dimension of $\mathrm{trop}(X)$ and ℓ is the dimension of the lineality space of $\mathrm{trop}(X)$.
That is it is connected even after removing $d - \ell - 1$ closed maximal faces.

Remark. In general, higher connectivity depends not only on the ground set but also on the chosen polyhedral structure. This theorem is true for **every** polyhedral structure.

Corollary. Balinski's Theorem for rational polytopes.

- The complete normal fan of a polytope P is the tropicalization of the irreducible variety $(K^*)^d$.
- The theorem says it remains connected through codimension one after removing any $d-1$ maximal cones.
- Dualizing, we get that the graph of the polytope P remains connected after removing any $d-1$ vertices.

More generally, cones over **skeleta** of rational **polytopes** are tropicalizations of irreducible varieties, so we get higher connectivity for them too.

(My undergrad student Daniel Hathcock showed higher connectivity for non-rational polytopes as well.)

Even more generally, stable intersections of tropical hypersurfaces have higher-connectivity.

More combinatorial consequences

- The space of phylogenetic trees on n taxa is $n-3$ connected.
- For a rank r matroid M that is representable over a field of characteristic zero, the Bergman complex (order complex of lattice of flats) $r-1$-connected.

We also have a separate proof of connectivity for arbitrary (non-representable) matroids.

Proof of Higher Connectivity

- mod out the lineality space
- induction on dimension
- base case: dimension one, curves. This is the usual structure theorem.
- reduce dimension by slicing with generic rational hyperplanes (need the theorem below)
- subtle polyhedral arguments

Tropical Bertini Theorem. (Maclagan–Y 2019)
Let K be an algebraically closed valued field of **characteristic zero**, whose value group contains $\mathbb{Q}$. Let X be an **irreducible** variety over K of dimension $d \geq 2$. Then for a "generic" rational hyperplane H, **the intersection** $\mathrm{trop}(X) \cap H$ **is again the tropicalization of an irreducible variety**.

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A usual hyperplane = tropicalization of a hypersurface defined by a binomial.

It is **not** true that if an ideal I is irreducible, then $I + \langle x^a - cx^b \rangle$ is irreducible for "generic" exponents a, b and "generic" coefficient c.

For example, for $I = \langle x_1 - x_2^2 x_3^2 \rangle$, the ideal $I + \langle x_1^{a_1} x_2^{2a_2} x_3^{2a_3} - c \rangle$ is not irreducible for any tuple of integers $(a_1, a_2, a_3) \notin \mathbb{Z}(1, -1, -1)$.

But the statement is true after tropicalization.

To prove the Tropical Bertini, we used:

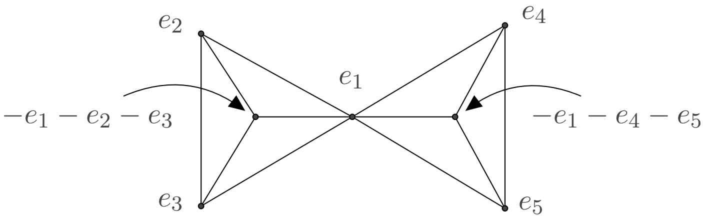
Toric Bertini Theorem (Fuchs, Mantova, Zannier 2018)

Let X be an irreducible quasiprojective variety of dimension d over an algebraically closed field of **characteristic zero**, and let $\pi : X \to (K^*)^d$ be a dominant map that is finite onto its image, satisfying the "pullback property".

Then there is a finite union $\mathcal{E}$ of proper subtori of $(K^*)^d$ such that, for every subtorus $T \subset (K^*)^d$ not contained in $\mathcal{E}$ and every point $p \in (K^*)^d$, the preimage $\pi^{-1}(p \cdot T)$ is an irreducible subvariety of X.

An obstruction to realizability in tropical geometry

If a polyhedral complex "looks like" the tropicalization of a variety, is it really the tropicalization of a variety?



This is two dimensional polyhedral fan, which is not 2-connected, so it is not the tropicalization of an **irreducible** variety.

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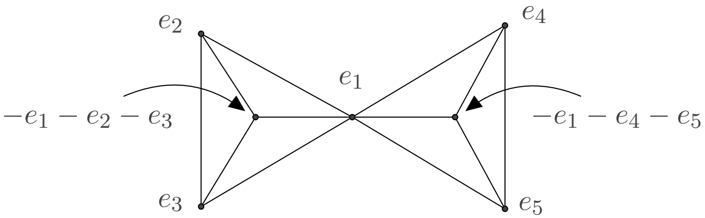

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References

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- Clemens Fuchs, Vincenzo Mantova, and Umberto Zannier, *On fewnomials, integral points, and a toric version of Bertini's theorem*, J. Amer. Math. Soc. 31 (2018), no. 1, 107-134.
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Thank you for your attention!