A non-iterative rule for straightening fillings of Young diagrams

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Straightening

The ubiquity of Young diagrams and straightening

Fundamental combinatorial objects

- irreducible representations of the symmetric group S_n
- polynomial irreducible representations of the general linear group GL_N
- standard monomial basis for the space of sections of an ample line bundle on a flag variety/Schubert variety

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All of the above rely on a straightening process.

Young diagrams

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Defn. A partition is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k$.
Visualize a partition by its Young diagram λ, an upper left justified collection of boxes with λ_i boxes in row i.

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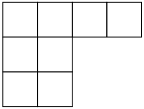

Fillings of Young diagrams

Defn.

- A filling of shape λ is an assignment of a value in $[n]$ to each box of λ
- A tableau is a filling such that values in columns increase strictly downwards
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Filling

1	3	4	2
2	2		
2	1		

Tableau

2	1	4	1
3	2		
4	3		

Semistandard Tableau

1	2	3	3
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Vector spaces composed of fillings

Fix a partition λ and content z.

The sets

$F(\lambda, z)$ is the set of fillings of shape λ and content z.

$\cup$

$S(\lambda, z)$ is the subset of semistandard tableau of shape λ and content z.

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The vector space

Let $\mathbb{C}^{F(\lambda,z)}$ be the complex vector space with basis $F(\lambda, z)$.

A subspace and its generators

The subspace

Let $A(\lambda, z)$ be subspace of $\mathbb{C}^{F(\lambda,z)}$ generated by

- Grassmannian sums: $E + F$ where E and F differ in a single column by a single transposition

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 4 \\ \hline 4 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline 4 \\ \cline{1-1} \end{array}$$

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- Plücker sums: $E - \sum F$ where for a fixed $E \in F(\lambda, z)$, j, m the F arise from E by swapping the top m elements in column $j+1$ with any m elements in column j

$m = 1,\ j = 1$

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The fundamental theorem and straightening

Theorem. [Young] The semistandard tableau in $S(\lambda, z)$ form a basis of the factor space $\mathbb{C}^{F(\lambda,z)}/A(\lambda, z)$.

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For λ with d boxes and $z = (1, \ldots, 1)$, can define an action of S_d on $\mathbb{C}^{F(\lambda,z)}/A(\lambda, z)$. This is the the irreducible S_d-representation associated to λ.

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- Prescribe a relation in $A(\lambda, z)$ that rewrites a given (non-semistandard) filling as a sum of other fillings that are smaller in some total order

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(1) Theoretical: Iterative methods give almost no control over the coefficients that arise. Even showing that a particular coefficient is nonzero is difficult.

(2) Computational: Straightening a filling with ~50 boxes can take hours on a computer. Difficult to optimize (or parallelize).

An example of a classical straightening algorithm

Let $n = 5$, $\lambda = (2, 2, 1)$, and $z = (1, 1, 1, 1, 1)$. We will straighten

1	2
4	3
5	

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simplify / reorder

Non-iterative Straightening

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Fix a partition λ, content z, and $F, S \in F(\lambda, z)$

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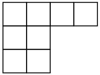

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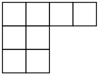

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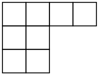

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- $sgn(\underline{\pi})$ equals the product of the signs of each permutation

Defn. The rearrangement coefficient of F w.r.t. S, $\mathcal{R}_{F,S}$, is the sum of signs of all $\underline{\pi} \in C(\lambda)$ s.t. $F_{\underline{\pi}}$ has the same content in each row as S.

Rearrangement coefficients - example

Example.

Let $\lambda = (4, 2, 2)$ and $z = (2, 2, 2, 2)$ with

$$F = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 4 & 1 \\ \hline 3 & 2 \\ \cline{1-2} 4 & 3 \\ \cline{1-2} \end{array} \qquad S = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 4 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} \end{array}$$

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We want to rearrange F into S. Forced to swap 2 and 4. Then swap 2 and 3. Then F and S have the same content in each row. The sign of this permutation is then 1.

$\mathcal{R}_{F,S} = 1.$

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$\mathcal{R}_{F,S} = 1.$

$\mathcal{R}_{S,F} = 0.$ Why?

Straightening in a new basis

Order and label the semistandard tableau in $S(\lambda, z)$ as

$$S_1 \succ S_2 \succ \cdots \succ S_{K_{\lambda,z}}$$

where $\succ$ is the row word order.

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The $\mathbb{D}$-basis.

We define a new basis of the factor space $\mathbb{C}^{F(\lambda,z)}/A(\lambda, z)$ by

$$\mathbb{D}_{S_i} = S_i - \sum_{j<i} \mathcal{R}_{S_i,S_j} \cdot \mathbb{D}_{S_j}$$

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- Straightening: Compute $\mathbb{D}$-basis. Compute (at most) $K_{\lambda,z}$ rearrangement coefficients.
- The $\mathbb{D}$-basis only depends on λ and z; straightening multiple fillings is efficient

An example of straightening via new method

Let $n = 5$, $\lambda = (2, 2, 1)$, and $z = (1, 1, 1, 1, 1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} \quad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array} \quad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} \quad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

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Let $n = 5$, $\lambda = (2,2,1)$, and $z = (1,1,1,1,1)$. The five semistandard tableau are

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with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

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$$\mathbb{D}_{S_1} := S_1$$

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$$\mathbb{D}_{S_5} := S_5 - \mathcal{R}_{S_5,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_5,S_2} \cdot \mathbb{D}_{S_2} - \mathcal{R}_{S_5,S_3} \cdot \mathbb{D}_{S_3} - \mathcal{R}_{S_5,S_4} \cdot \mathbb{D}_{S_4} = S_5 - S_1$$

An example of straightening via new method

Let $n = 5$, $\lambda = (2,2,1)$, and $z = (1,1,1,1,1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

Then

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An example of straightening via new method

Let $n = 5$, $\lambda = (2, 2, 1)$, and $z = (1, 1, 1, 1, 1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

Then

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We will now straighten

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}$$

An example of straightening via new method

Let $n = 5$, $\lambda = (2,2,1)$, and $z = (1,1,1,1,1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

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We will now straighten

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} = \mathcal{R}_{F,S_1} \cdot \mathbb{D}_{S_1} + \mathcal{R}_{F,S_2} \cdot \mathbb{D}_{S_2} + \mathcal{R}_{F,S_3} \cdot \mathbb{D}_{S_3} + \mathcal{R}_{F,S_4} \cdot \mathbb{D}_{S_4} + \mathcal{R}_{F,S_5} \cdot \mathbb{D}_{S_5}$$

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Let $n = 5$, $\lambda = (2,2,1)$, and $z = (1,1,1,1,1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

Then

$$\begin{aligned}
\mathbb{D}_{S_1} &:= S_1 \\
\mathbb{D}_{S_2} &:= S_2 - \mathcal{R}_{S_2,S_1} \cdot \mathbb{D}_{S_1} = S_2 \\
\mathbb{D}_{S_3} &:= S_3 - \mathcal{R}_{S_3,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_3,S_2} \cdot \mathbb{D}_{S_2} = S_3 \\
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\mathbb{D}_{S_5} &:= S_5 - \mathcal{R}_{S_5,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_5,S_2} \cdot \mathbb{D}_{S_2} - \mathcal{R}_{S_5,S_3} \cdot \mathbb{D}_{S_3} - \mathcal{R}_{S_5,S_4} \cdot \mathbb{D}_{S_4} = S_5 - S_1
\end{aligned}$$

We will now straighten

$$\begin{aligned}
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} &= \mathcal{R}_{F,S_1} \cdot \mathbb{D}_{S_1} + \mathcal{R}_{F,S_2} \cdot \mathbb{D}_{S_2} + \mathcal{R}_{F,S_3} \cdot \mathbb{D}_{S_3} + \mathcal{R}_{F,S_4} \cdot \mathbb{D}_{S_4} + \mathcal{R}_{F,S_5} \cdot \mathbb{D}_{S_5} \\
&= 0 \cdot \mathbb{D}_{S_1} + 1 \cdot \mathbb{D}_{S_2} - 1 \cdot \mathbb{D}_{S_3} - 1 \cdot \mathbb{D}_{S_4} + 1 \cdot \mathbb{D}_{S_5}
\end{aligned}$$

An example of straightening via new method

Let $n = 5$, $\lambda = (2,2,1)$, and $z = (1,1,1,1,1)$. The five semistandard tableau are

$$S_1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad S_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad S_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad S_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

with $S_1 \succ S_2 \succ S_3 \succ S_4 \succ S_5$.

Then

$$\mathbb{D}_{S_1} := S_1$$

$$\mathbb{D}_{S_2} := S_2 - \mathcal{R}_{S_2,S_1} \cdot \mathbb{D}_{S_1} = S_2$$

$$\mathbb{D}_{S_3} := S_3 - \mathcal{R}_{S_3,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_3,S_2} \cdot \mathbb{D}_{S_2} = S_3$$

$$\mathbb{D}_{S_4} := S_4 - \mathcal{R}_{S_4,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_4,S_2} \cdot \mathbb{D}_{S_2} - \mathcal{R}_{S_4,S_3} \cdot \mathbb{D}_{S_3} = S_4$$

$$\mathbb{D}_{S_5} := S_5 - \mathcal{R}_{S_5,S_1} \cdot \mathbb{D}_{S_1} - \mathcal{R}_{S_5,S_2} \cdot \mathbb{D}_{S_2} - \mathcal{R}_{S_5,S_3} \cdot \mathbb{D}_{S_3} - \mathcal{R}_{S_5,S_4} \cdot \mathbb{D}_{S_4} = S_5 - S_1$$

We will now straighten

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1}\end{array} = \mathcal{R}_{F,S_1} \cdot \mathbb{D}_{S_1} + \mathcal{R}_{F,S_2} \cdot \mathbb{D}_{S_2} + \mathcal{R}_{F,S_3} \cdot \mathbb{D}_{S_3} + \mathcal{R}_{F,S_4} \cdot \mathbb{D}_{S_4} + \mathcal{R}_{F,S_5} \cdot \mathbb{D}_{S_5}$$

$$= 0 \cdot \mathbb{D}_{S_1} + 1 \cdot \mathbb{D}_{S_2} - 1 \cdot \mathbb{D}_{S_3} - 1 \cdot \mathbb{D}_{S_4} + 1 \cdot \mathbb{D}_{S_5}$$

$$= S_5 - S_4 - S_3 + S_2 - S_1$$

Closing thoughts

Applications:Theory

Using this non-iterative formula we are able to extend a result of Lakshmibai-Gonciulea that proves that the leading term when straightening is nonzero.

Closing thoughts

Applications:Theory

Using this non-iterative formula we are able to extend a result of Lakshmibai-Gonciulea that proves that the leading term when straightening is nonzero.

Applications:Computational

Have implemented this algorithm in C. It seems to be several orders of magnitude faster than traditional straightening.

Currently being used to compute multiplicites of GL_n-irreps in the kernel of the Hadamard-Howe map (related to Foulkes conjecture) extending results of Cheung-Ikenmeyer-Mkrtchyan.

Thank you!