q,t - Fuß - Catalan numbers for finite reflection groups

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Overview

Finite reflection groups

q, t-Fuß-Catalan numbers for real reflection groups

Algebraic Combinatorics – the extended Shi arrangement

Combinatorial Algebra – rational Cherednik algebras

q, t-Fuß-Catalan numbers for complex reflection groups

Finite real reflection groups

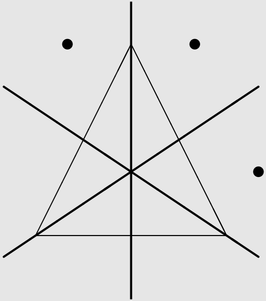
Finite real reflection groups

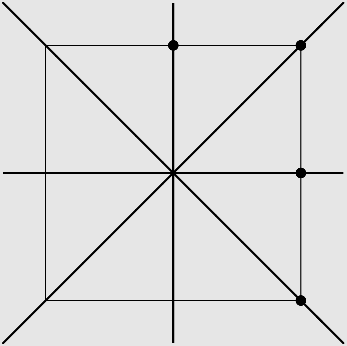
Let V be a finite-dimensional real vector space.

- A **(finite) real reflection group**

$$W = \langle t_1, \ldots, t_\ell \rangle \subseteq \mathrm{O}(V)$$

is a **finite group generated by reflections**.





Irreducible real reflection groups

- The following list of **root systems** determine (up to isomorphisms) the **irreducible finite real reflection groups**:

$$
\begin{aligned}
A_{n-1} &\qquad (\textbf{symmetric group}),\\
B_n &\qquad (\textbf{group of signed permutations}),\\
D_n &\qquad (\textbf{group of even-signed permutations}),\\
I_2(k) &\qquad (\textbf{dihedral group} \text{ of order } 2k) \text{ and}\\
H_3, H_4, F_4, E_6, E_7, E_8 &\qquad (\textbf{exceptional groups}).
\end{aligned}
$$

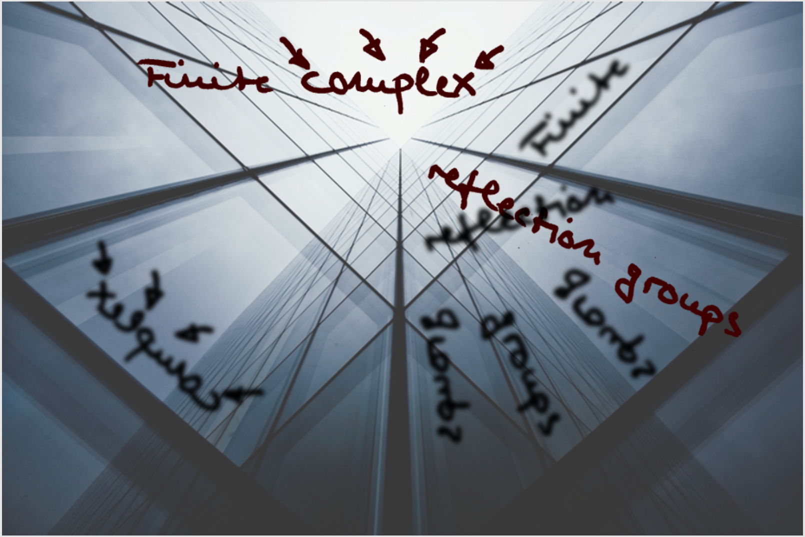
The symmetric group

The most classical example of a reflection group is the **symmetric group** $\mathcal{S}_n$ of all **permutations** of n letters.

$$123 = (), \quad 132 = (23), \quad 213 = (12),$$
$$231 = (123), \quad 312 = (132), \quad 321 = (13).$$

This group can be seen as the **reflection group** of type A_{n-1}:

$$\begin{array}{rcl} \textit{transposition} & \stackrel{\sim}{\longleftrightarrow} & \textit{reflection} \\ (i,j) & = & e_i \leftrightarrow e_j \\ \textit{simple transposition} & \stackrel{\sim}{\longleftrightarrow} & \textit{simple reflection} \\ (i,i+1) & = & e_i \leftrightarrow e_{i+1}. \end{array}$$


Finite complex
reflection groups

Finite complex reflection groups

Let V be a finite-dimensional **complex** vector space.

- A **complex reflection** $s \in \mathrm{U}(V)$
  i. has finite order and
  ii. its fixed-point space has codimension 1.
- A **(finite) complex reflection group**

$$W = \langle t_1, \ldots, t_\ell \rangle \subseteq O(V)$$

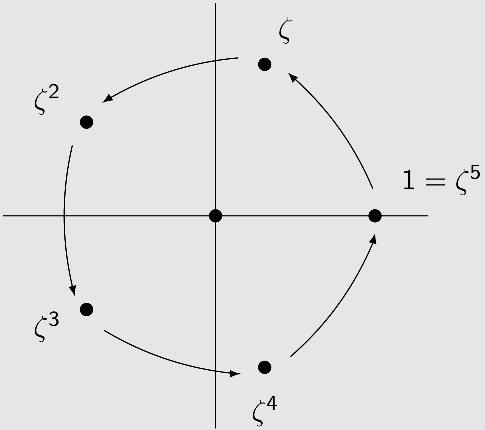
  is a **finite group generated by complex reflections**.

Irreducible complex reflection groups are determined by the following types:

$$\begin{array}{ll} G(m,p,n) & \text{with } p|m \text{ of order } m^n n!/p, \\ G_4 - G_{37} & \text{34 exceptional types.} \end{array}$$

(Shephard–Todd, Chevalley, 1950's)

The cyclic group



- It acts on $\mathbb{C}$ by multiplication of a **primitive root of unity** ζ.

Reflections ⟷ Reflecting hyperplanes

- In **real reflection groups**, any reflection t has **order two** and there is a 1 : 1-correspondence

$$\left\{ \text{ reflections } \right\} \;\stackrel{\sim}{\longleftrightarrow}\; \left\{ \text{ reflecting hyperplanes } \right\}$$
$$t \;\leftrightarrow\; H_\alpha .$$

- In **complex reflection groups**, any reflection t has **order** $k \geq 2$ and there is a correspondence

$$\left\{ \text{ reflections } \right\} \;\stackrel{\sim}{\longleftrightarrow}\; \left\{ \text{ reflecting hyperplanes } \right\}$$
$$t, t^2, \ldots, t^{k-1} \;\leftrightarrow\; H_\alpha .$$

Alternating polynomials

For a **permutation** $\sigma \in \mathcal{S}_n$, define a **diagonal action** on $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ by $\sigma(x_i) := x_{\sigma(i)}, \sigma(y_i) := y_{\sigma(i)}$.
E.g.,

$$231(2x_1x_2y_2^2y_3) = 2x_2x_3y_3^2y_1.$$

A polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called

- **invariant** if $\sigma(f) = f$,
- **alternating** if $\sigma(f) = \text{sgn}(\sigma)\, f$.

Example

$x_1y_2 + x_2y_1$ is **invariant**, $x_1y_2 - x_2y_1$ is **alternating**.

Alternating polynomials – Generalization

Let W be a real reflection group acting on V. The **contragredient action** of W on $V^* = \mathrm{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$

This gives an action of W on the symmetric algebra $S(V^*) = \mathbb{C}[\mathbf{x}]$ and 'doubling up' this action gives a **diagonal action** on $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[V \oplus V]$.

A polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called

- **invariant** if $\omega(f) = f$,
- **alternating** if $\omega(f) = \det(\omega)\, f$.

q,t – Fuß – Catalan numbers

q, t-Fuß-Catalan numbers as a bigraded Hilbert series

Let W be a reflection group now acting on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ and let

$$\mathcal{A} := \langle \text{ alternating polynomials } \rangle \trianglelefteq \mathbb{C}[\mathbf{x}, \mathbf{y}].$$

Define the W-module $M^{(m)}(W)$ to be **minimal generating space** of the **ideal** $\mathcal{A}^m$,

$$M^{(m)}(W) \;\; := \;\; \mathcal{A}^m / \langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m \; \cong \; \mathbb{C}\mathcal{B},$$

where $\mathcal{B}$ is any **homogeneous minimal generating set for** $\mathcal{A}^m$.

$M^{(m)}(W)$ sits inside a larger W-module $DR^{(m)}(W)$ as its **isotypic component**,

$$M^{(m)}(W) \cong \mathbf{e}_{\det}\big(DR^{(m)}(W)\big).$$

q, t-Fuß-Catalan numbers as a bigraded Hilbert series

Definition

For any real reflection group W, define **q, t-Fuß-Catalan numbers** to be the bigraded Hilbert series of $M^{(m)}(W)$,

$$\begin{aligned}\mathrm{Cat}^{(m)}(W;q,t) &:= \mathcal{H}(M^{(m)}(W);q,t)\\ &= \sum_{f\in\mathcal{B}} q^{\deg_{\mathbf{x}}(f)}t^{\deg_{\mathbf{y}}(f)}.\end{aligned}$$

- $\mathrm{Cat}^{(m)}(W;q,t)$ is a **symmetric polynomial** in q and t,
- it reduces in type A_{n-1} to the classical q, t-Fuß-Catalan numbers,

  $$\mathrm{Cat}^{(m)}(\mathcal{S}_n;q,t) = \mathrm{Cat}_n^{(m)}(q,t)$$

  introduced by Haiman in the 1990's.

Example: $\mathrm{Cat}^{(1)}(\mathcal{S}_3; q, t)$

For $W = \mathcal{S}_3$, one can show that

$$M^{(1)}(\mathcal{S}_3) = \mathbb{C}\left\{ \begin{array}{c} \Delta_{\{(1,0),(2,0)\}}, \Delta_{\{(1,0),(1,1)\}}, \Delta_{\{(0,1),(1,1)\}}, \\ \Delta_{\{(0,1),(0,2)\}}, \Delta_{\{(1,0),(0,1)\}} \end{array} \right\},$$

where

$$\Delta_{\{(i_1,j_1),(i_2,j_2)\}}(\mathbf{x},\mathbf{y}) := \det \begin{pmatrix} 1 & 1 & 1 \\ x_1^{i_1} y_1^{j_1} & x_2^{i_1} y_2^{j_1} & x_3^{i_1} y_3^{j_1} \\ x_1^{i_2} y_1^{j_2} & x_2^{i_2} y_2^{j_2} & x_3^{i_2} y_3^{j_2} \end{pmatrix}$$

is the **generalized Vandermonde determinant**. This gives

$$\begin{aligned} \mathrm{Cat}^{(1)}(\mathcal{S}_3; q, t) &= \mathcal{H}(M^{(1)}(\mathcal{S}_3); q, t) \\ &= q^3 + q^2 t + q t^2 + t^3 + qt. \end{aligned}$$

A conjectured formula for the dimension of $M^{(m)}(W)$

Computations of the **dimensions** of $M^{(m)}(W)$ were the first motivation for further investigations:

Conjecture

Let W be a real reflection group. Then

$$\mathrm{Cat}^{(m)}(W;1,1) = \prod_{i=1}^{\ell} \frac{d_i + mh}{d_i},$$

where

- ℓ is the **rank** of W,
- h is the **Coxeter number** and
- $d_1, \ldots, d_\ell$ are its **degrees**.

Fuß-Catalan numbers

- These numbers, called **Fuß-Catalan numbers**, count several combinatorial objects, e.g.,
  - **positive regions in the generalized Shi arrangement** (Athanasiadis, Postnikov),
  - **m-divisible non-crossing partitions** (Armstrong, Bessis, Reiner),
  - **facets in the generalized Cluster complex** (Fomin, Reading, Zelevinsky).
- They reduce for $m = 1$ to the well-known Catalan numbers associated to real reflection groups:

A_{n-1}	B_n	D_n
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n}{n} - \binom{2(n-1)}{n-1}$

$I_2(k)$	H_3	H_4	F_4	E_6	E_7	E_8
$k+2$	32	280	105	833	4160	25080

The classical q, t-Fuß-Catalan numbers

In type A, $\mathrm{Cat}^{(m)}(\mathcal{S}_n; q, t)$ occurred within the past 15 years in various fields of mathematics:

- Hilbert series of **space of diagonal coinvariants** (Haiman),
- complicated rational function in the context of **modified Macdonald polynomials** (Garsia, Haiman),
- Hilbert series of some cohomology module in the theory of **Hilbert schemes of points in the plane** (Haiman),
- they have a conjectured combinatorial interpretation in terms of two statistics on partitions fitting inside the partition $\mu := ((n-1)m, \ldots, 2m, m)$,

$$\mathrm{Cat}_n^{(m)}(q, t) = \sum_{\lambda \subseteq \mu} q^{\mathrm{area}(\lambda)} t^{\mathrm{bounce}(\lambda)}.$$

  - Proved for $m = 1$ (Garsia, Haglund) and for $t = 1$ (Haiman).

⌊Combinatorics⌉

The extended Shi arrangement

The extended Shi arrangement

Let W be a crystallographic reflection group.

$\mathrm{Shi}^{(m)}(W)$ is defined to be the collection of (translates of the reflecting) hyperplanes in V given by

$$\left\{H_\alpha^k : \alpha \in \Phi^+, -m < k \leq m\right\},$$

where $H_\alpha^k = \{x \in V : (x, \alpha) = k\}$.

- A **region of** $\mathrm{Shi}^{(m)}(W)$ is a **connected component** of its complement.

Remark

Coxeter arrangement $\subseteq$ **extended Shi arrangement**.

The extended Shi arrangement

Let W be a crystallographic reflection group.

Theorem (Yoshinaga)

The number of **regions** *in* $\mathrm{Shi}^{(m)}(W)$ *is given by*

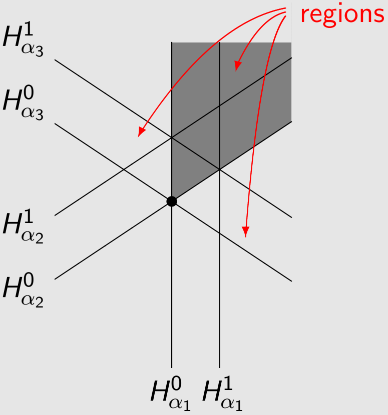
$$(mh+1)^{\ell}.$$

Theorem (Athanasiadis)

The number of **positive regions** *in* $\mathrm{Shi}^{(m)}(W)$ *– regions which lie in the* **fundamental chamber** *of the associated Coxeter arrangement – is given by*

$$\prod_{i=1}^{\ell} \frac{d_i + mh}{d_i}.$$

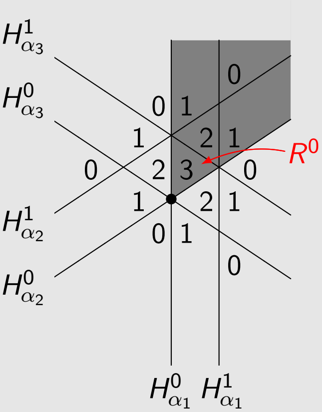
Example: $\mathrm{Shi}^{(1)}(A_2)$



$$\left|\{\text{ regions }\}\right| = 16 = (1\cdot 3+1)^2,$$

$$\left|\{\text{ positive regions }\}\right| = 5 = \frac{5\cdot 6}{2\cdot 3}.$$

Example: $\mathrm{Shi}^{(1)}(A_2)$ and $\mathrm{Cat}^{(1)}(A_2; q)$



$$\mathrm{Cat}^{(1)}(A_2; q) = \sum q^{\mathrm{coh}(R)} = 1 + 2q + q^2 + q^3.$$

Specialization $t = 1$.

Conjecture

Let W be a crystallographic reflection group. Then

$$\mathrm{Cat}^{(m)}(W; q, 1) = \sum q^{\mathrm{coh}(R)},$$

where the sum ranges over all regions of $\mathrm{Shi}^{(m)}(\Phi)$ which lie in the **fundamental chamber** of the associated Coxeter arrangement and where coh denotes the **coheight statistic**.

- The conjecture is known to be true for type A,
- was validated by computations for several types.

Specialization $t = q^{-1}$.

Conjecture

Let W be a reflection group. Then

$$\mathrm{Cat}^{(m)}(W; q, q^{-1}) = q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q},$$

where

- N is the number of reflections in W,
- $[n]_q := q^{n-1} + \ldots + q + 1$ is the usual q-analogue of n.

The dihedral groups

Theorem

Let W be the dihedral group of type $I_2(k)$. Then

$$\mathrm{Cat}^{(m)}(W;q,t) = \sum_{j=0}^{m} q^{m-j} t^{m-j} [jk+1]_{q,t},$$

where

$$[n]_{q,t} := q^{n-1} + q^{n-2}t + \ldots + qt^{n-2} + t^{n-1}.$$

Theorem

All shown conjectures hold for the dihedral groups.

Rational Cherednik algebras

Let W be a real reflection group. The **rational Cherednik algebra**

$$\mathsf{H}_c = \mathsf{H}_c(W)$$

is an **associative algebra** generated by

$$V, V^*, W,$$

subject to **defining relations** depending on a **rational parameter** c, such that

- the polynomial rings $\mathbb{C}[V], \mathbb{C}[V^*]$ and
- the group algebra $\mathbb{C}W$

are subalgebras of H_c.

A simple H_c-module

For $c = m + \frac{1}{h}$ there exists a unique simple H_c-module L which carries a natural filtration.

Theorem (Berest, Etingof, Ginzburg)

Let W be a real reflection group. Then

$$\begin{aligned} \mathcal{H}(\mathrm{gr}(L); q) &= q^{-mN}[mh+1]_q^{\ell}, \\ \mathcal{H}(\mathbf{e}(\mathrm{gr}(L)); q) &= q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q}, \end{aligned}$$

where $\mathrm{gr}(L)$ is the associated **graded module** *of L and where $\mathbf{e}(\mathrm{gr}(L))$ is its* **trivial component**.

The connection between L and the space of generalized diagonal coinvariants

Theorem

Let W be a real reflection group and let

$$DR^{(m)}(W) = \mathcal{A}^{m-1}/\mathcal{I}\mathcal{A}^{m-1}$$

be the **generalized diagonal coinvariants** *graded by degree in* $\mathbf{x}$ *minus degree in* $\mathbf{y}$*. Then there exists a natural* **surjection of graded modules***,*

$$DR^{(m)}(W) \otimes \det \twoheadrightarrow \mathrm{gr}(L),$$

where det *denotes the* **determinantal representation***.*

Remark

This theorem generalizes a theorem by Gordon, who proved the $m = 1$ case, following mainly his approach.

The connection between L and the space of generalized diagonal coinvariants

Conjecture (Haiman)

The W-stable kernel of the surjection in the previous theorem does not contain a copy of the trivial representation.

Corollary

If the previous conjecture holds, then

$$M^{(m)}(W) \cong \mathbf{e}(\mathrm{gr}(L))$$

as graded modules. In particular,

$$\mathrm{Cat}(W; q, q^{-1}) = q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q}.$$

Finite "complex" reflection groups

The cyclic group

The cyclic group $\mathcal{C}_k = \langle \zeta \rangle$ would act on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \;\; \zeta^b \cdot y^b = \zeta^{a+b} \cdot x^a y^b.$$

This would give

$$\begin{aligned}
\mathbb{C}[x,y]^{\mathcal{C}_k} &= \operatorname{span}\left\{x^a y^b : a + b \equiv 0 \mod k\right\}, \\
\mathbb{C}[x,y]^{\det} &= \operatorname{span}\left\{x^a y^b : a + b \equiv 1 \mod k\right\} \\
&= x\mathbb{C}[x,y]^{\mathcal{C}_k} + y\mathbb{C}[x,y]^{\mathcal{C}_k}, \\
\mathbb{C}[x,y]^{\det^{-1}} &= \operatorname{span}\left\{x^a y^b : a + b \equiv k-1 \mod k\right\} \\
&= \sum_{i+j=k-1} x^i y^j \mathbb{C}[x,y]^{\mathcal{C}_k}.
\end{aligned}$$

The cyclic group

$$\mathbb{C}[x,y]^{\det} = x\mathbb{C}[x,y]^{\mathcal{C}_k} + y\mathbb{C}[x,y]^{\mathcal{C}_k},$$

$$\mathbb{C}[x,y]^{\det^{-1}} = \sum_{i+j=k-1} x^i y^j \mathbb{C}[x,y]^{\mathcal{C}_k}.$$

We would have two possible choices to define q,t-Catalan numbers for the cyclic group $\mathcal{C}_k$:

$$\mathrm{Cat}^{(1)}(\mathcal{C}_k;q,t) = q+t \quad \textbf{or}$$
$$\mathrm{Cat}^{(1)}(\mathcal{C}_k;q,t) = q^{k-1} + q^{k-2}t + \ldots + qt^{k-2} + t^{k-1}.$$

- **Both choices would be in contradiction to the previously shown conjectures!**

Alternating polynomials – Generalization

Let W be a real reflection group acting on V.
The **contragredient action** of W on $V^* = \mathrm{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$

This gives an action of W on the symmetric algebra $S(V^*) = \mathbb{C}[\mathbf{x}]$ and 'doubling up' this action gives a **diagonal action** on

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[V \oplus V].$$

A polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called

- **invariant** if $\omega(f) = f$ for all $\omega \in W$,
- **alternating** if $\omega(f) = \det(\omega)\, f$ for all $\omega \in W$.

Alternating polynomials – Generalization

Let W be a complex reflection group acting on V.
The **contragredient action** of W on $V^* = \mathrm{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$

This gives an action of W on the symmetric algebra $S(V^*) = \mathbb{C}[\mathbf{x}]$ and 'doubling up' this action gives a **diagonal action** on

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[V \oplus V^*].$$

A polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called

- **invariant** if $\omega(f) = f$ for all $\omega \in W$,
- **alternating** if $\omega(f) = \det(\omega) f$ for all $\omega \in W$.

The cyclic group

The cyclic group $\mathcal{C}_k = \langle \zeta \rangle$ would act on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \;\; \zeta^b \cdot y^b = \zeta^{a+b} \cdot x^a y^b.$$

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The cyclic group $\mathcal{C}_k = \langle \zeta \rangle$ acts on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \ \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$

The cyclic group

The cyclic group $\mathcal{C}_k = \langle\zeta\rangle$ acts on $\mathbb{C}[x,y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \ \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$

This gives

$$\mathbb{C}[x,y]^{\mathcal{C}_k} = \operatorname{span}\left\{x^a y^b : a \equiv b \mod k\right\},$$

$$\begin{aligned}\mathbb{C}[x,y]^{\det} &= \operatorname{span}\left\{x^a y^b : a+1 \equiv b \mod k\right\}\\ &= x\mathbb{C}[x,y]^{\mathcal{C}_k} + y^{k-1}\mathbb{C}[x,y]^{\mathcal{C}_k},\end{aligned}$$

$$\begin{aligned}\mathbb{C}[x,y]^{\det^{-1}} &= \operatorname{span}\left\{x^a y^b : a \equiv b+1 \mod k\right\}\\ &= x^{k-1}\mathbb{C}[x,y]^{\mathcal{C}_k} + y\mathbb{C}[x,y]^{\mathcal{C}_k}.\end{aligned}$$

The cyclic group

The cyclic group $\mathcal{C}_k = \langle \zeta \rangle$ acts on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \ \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$

$$\mathbb{C}[x, y]^{\det} = x\mathbb{C}[x, y]^{\mathcal{C}_k} + y^{k-1}\mathbb{C}[x, y]^{\mathcal{C}_k},$$

$$\mathbb{C}[x, y]^{\det^{-1}} = x^{k-1}\mathbb{C}[x, y]^{\mathcal{C}_k} + y\mathbb{C}[x, y]^{\mathcal{C}_k}.$$

Now, we have (beside interchanging the roles of q and t) only the following choice:

$$\mathrm{Cat}^{(1)}(\mathcal{C}_k; q, t) := q + t^{k-1}.$$

The cyclic group

The cyclic group $\mathcal{C}_k = \langle \zeta \rangle$ acts on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \;\; \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$

Now, we have (beside interchanging the roles of x and y) only the following choice:

$$\mathrm{Cat}^{(1)}(\mathcal{C}_k; q, t) \quad := \quad q + t^{k-1}.$$

- The q-power equals the number of reflecting hyperplanes $N^* = 1$ and the t-power equal the number of reflections $N = k - 1$. We have

$$\begin{aligned} q^N \, \mathrm{Cat}^{(1)}(\mathcal{C}_k; q, q^{-1}) \quad &= \quad q^{N^*} \, \mathrm{Cat}^{(1)}(\mathcal{C}_k; q^{-1}, q) \\ &= \quad 1 + q^k = [2k]_q / [k]_q. \end{aligned}$$

Final conjecture

Conjecture

Let W be a well-generated complex reflection group. Then

$$q^{mN}\,\mathrm{Cat}^{(m)}(W;q,q^{-1}) = q^{mN^*}\,\mathrm{Cat}^{(m)}(W;q^{-1},q) = \prod_{i=1}^{\ell} \frac{[d_i+mh]_q}{[d_i]_q},$$

where

- N is the number of reflections in W and where
- N^* is the number of reflecting hyperplanes.