Combinatorial Hopf Algebras II: The wilderness of Hopf algebras

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Recall Hopf Algebra Def

Start with a Bialgebra

Product

$$\mu : H \otimes H \to H$$

Coproduct

$$\Delta : H \to H \otimes H$$

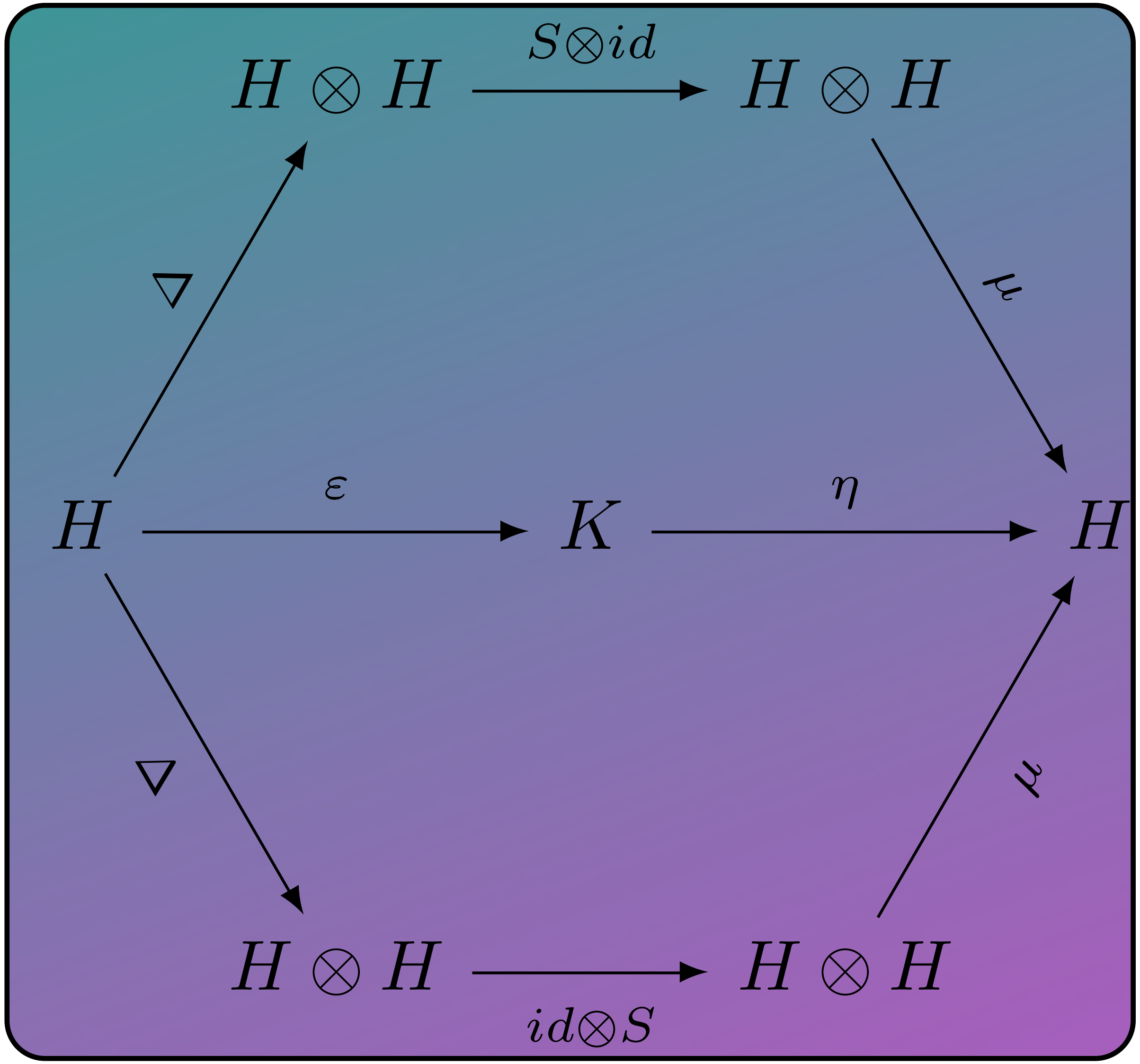
with unit and counit

$$\eta : K \to H$$

$$\varepsilon : H \to K$$

and an antipode map

$$S : H \to H$$



This diagram commutes

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start with the usual group algebra

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$$\Delta\mu(g \otimes h) = gh \otimes gh = (\mu \otimes \mu)(id \otimes \tau \otimes id)(\Delta \otimes \Delta)(g \otimes h)$$

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counit $\epsilon(g) = 1$

antipode $S(g) = g^{-1}$

unit $u(1) = e$

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$$u(1) = e$$

$$\mu(S \otimes id)\Delta(g) = \mu(S \otimes id)(g \otimes g) = g^{-1}g = e$$

$$u\epsilon(g) = u(1) = e$$

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But this is not a good example fo a combintorial Hopf algebra

Sym - symmetric functions in an arbitary/infinite number of variables

$$Sym = \mathbb{Q}\left[p_1, p_2, p_3, \ldots\right]$$

product inherited from polynomials in p's

Coproduct $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$

$$\Delta(p_\lambda) = \Delta(p_{\lambda_1})\Delta(p_{\lambda_2})\cdots\Delta(p_{\lambda_{\ell(\lambda)}})$$

$$S(p_k) = -p_k$$

$$\mu(S \otimes id)\Delta(f) = 0$$

if f is homogeneous of degree > 0

Sym - symmetric functions in an arbitary/infinite number of variables

$$Sym = \mathcal{L}\{m_\lambda : \lambda \vdash n\}$$

$$m_\lambda = \sum_{(i_1, i_2, \ldots, i_\ell)} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$$

product inherited from the polynomial ring

coproduct comes from replacing one set of alphabets by two X->left Y->right tensor

$$\Delta(m_\lambda) = \sum_{\mu \uplus \nu = \lambda} m_\mu \otimes m_\nu$$

QSym - Quasisymmetric functions

'the' Hopf algebra of compositions which is commutative and non-cocommutative

$$QSym_n = \mathcal{L}\{M_\alpha : \alpha \models n\} \qquad QSym = \bigoplus_{n\geq 0} QSym_n$$

$$M_\alpha = \sum_{i_1<i_2<\ldots<i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

$$\Delta(M_\alpha) = \sum_{k=0}^{\ell} M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes M_{(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell)}$$

Fundamental basis: $F_\alpha = \sum_{\alpha\leq\beta} M_\beta$

Aguiar-(N)Bergeron-Sottile - definition of CHA = graded + connected + has a multiplicative linear functional called a character

Result: every CHA has a morphism to QSym

$$QSym = \bigoplus_{n \geq 0} QSym_n \qquad QSym_n = \mathcal{L}\{F_\alpha : \alpha \models n\}$$

$$\zeta_Q(F_\alpha) = \begin{cases} 1 \ if \ \alpha = (n) \\ 0 \ otherwise \end{cases}$$

Combinatorial Hopf Algebra: I know it when I see it

FQSym $\mathfrak{S}Sym$

Malvenuto-Reutenauer Hopf algebra of permutations

$$Perm = \bigoplus_{n\geq 0} Perm_n \qquad Perm_n = \mathcal{L}\{F_\sigma : \sigma \in \mathfrak{S}_n\}$$

$$\mu : Perm_n \otimes Perm_m \to Perm_{n+m}$$

$$F_\sigma F_\tau = \sum_{\gamma=\sigma \sqcup\!\sqcup (\tau\uparrow_{+k})} F_\gamma$$

$$\Delta : Perm_n \to \bigoplus_{k=0}^{n} Perm_k \otimes Perm_{n+k}$$

$$\Delta(F_\sigma) = \sum_{k=0}^{n} F_{st(\sigma_1\sigma_2\cdots\sigma_k)} \otimes F_{st(\sigma_{k+1}\cdots\sigma_n)}$$

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$$Perm_n \to QSym_n$$

$$\Theta(F_\sigma) = F_{D(\sigma)}$$

I know it when I see it definition

What makes it a combinatorial Hopf algebra?

1 Graded and connected (degree 0 had dimension 1)

2 Follows recognizable structure of combinatorial objects

3 Generalizes structures we observe in Sym, QSym such as product, coproduct, internal (co)product (Kronecker product)*, composition (plethysm)*, fundamental basis*

4 Freely generated

5 Realized as a subalgebra* of $k[X_\infty]$ or $k\langle X_\infty\rangle$

6 Has a basis for which product and coproduct expand positively

The symmetric functions are isomorphic to the representation ring of the symmetric groups

product $\longleftrightarrow$ induction

coproduct $\longleftrightarrow$ restriction

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NSym is isomorphic to the representation ring of
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Krob-Thibon '97

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Peak $\longleftrightarrow$ Hecke-Clifford algebra at q=0

Bergeron(N)-Hivert-Thibon '03

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Bergeron(N)-Huilan Li ‘06: When does this happen?

Why look at CHAs?

Generalize structures of Sym and usually there are morphisms from CHAs to/from Sym

Example of open problem in Sym:

explain internal (Kronecker) product coefficients with combinatorics

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Generalize structures of Sym and usually there are morphisms from CHAs to/from Sym

Example of open problem in Sym:

explain internal (Kronecker) product coefficients with combinatorics

current state of affairs is that we can only explain cases of these coefficients because of connections with NSym

Example of open problem in Sym II:

Show positivity results in symmetric functions

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- Element 1: QSym (Gessel '84)

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LLT symmetric functions are Schur positive

Classical algebraic object of study:

$$\mathbb{Q}[x_1, x_2, \ldots, x_n] / < Sym^+ >$$

$$\mathbb{Q}[x_1, x_2, \ldots, x_n] / \langle p_1, p_2, \ldots, p_n \rangle$$
$$p_k = x_1^k + x_2^k + \cdots + x_n^k$$

Linear span of derivatives of the Vandermonde

$$\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

$$\mathcal{L}\{f(X_n) : g(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}) f(X_n) = 0 \quad \text{forall } g(X_n) \in Sym^{(n)}\}$$

Now that we have several spaces which seem to be analogues of Sym consider some analgous quotients

$$\mathbb{Q}[x_1, x_2, \ldots, x_n] / \left\langle QSym^+ \right\rangle$$

considered by Aval-Bergeron-Bergeron

graded space of dimension Calatan number

Now that we have several spaces which seem to be analogues of Sym consider some analgous quotients

$$Sym^{(n)} \subseteq QSym^{(n)}$$

Bergeron(F)-Reutenauer consider the quotient:

$$QSym^{(n)} / < Sym^{+} >$$

Conjecture : dimension is n!

Proven by Garsia-Wallach '03

Non commutative analogues

$$\mathbb{Q}\left\langle x_1, x_2, \ldots, x_n \right\rangle / \left\langle NCSym^+ \right\rangle$$

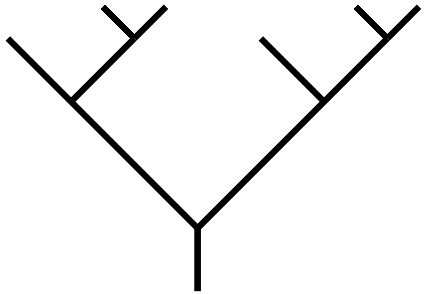
Considered by Bergeron(N)-Reutenauer-Rosas-Zabrocki

for the left and shuffle ideal

More recently by Bergeron(F)-Lauve

Still open: what happens with the two sided ideal?

q,t - enumeration of combinatorial objects



Novelli-Thibon '09

both of the following expressions are specializations of Hopf algebra of binary rooted trees

$$\sum_{\substack{(\sigma,\epsilon)|\mathrm{shape}\,(\mathcal{P}(\sigma))=T \\ \mathrm{sign}(i)=+1}} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\sigma,\epsilon)} = \prod_{s\in T} h_s(q,t)$$

$$h_s(q,t) := \frac{1}{1-q^n} \begin{cases} q^n - q^{n'}t & \textit{if } s \textit{ is the right son of its father,} \\ 1-q^{n'}t & \textit{otherwise.} \end{cases}$$

n *be the size of the subtree of root* s

n' *be the size of the left subtree of the previous one*