NORMAL IDEALS OF GRADED RINGS

Sara Faridi

Mathematics Department, University of Michigan, Ann Arbor, MI 48109
email: faridi@math.lsa.umich.edu

Abstract. For a graded domain $R = k[X_0, ..., X_m]/J$ over an arbitrary domain k, it is shown that the ideal generated by elements of degree $\geq mA$, where A is the least common multiple of the weights of the X_i, is a normal ideal.

1. Introduction

In the theory of resolution of singularities, one wishes to be able to blow up a singular variety along a closed subscheme and obtain a smooth variety birational to the original one. One question that comes up is the existence of such resolutions; it is known for varieties over fields of characteristic zero [H], and is conjectured in the prime characteristic case. Another question is describing the closed subschemes that give smooth blowups. Translating this problem into the language of algebra, one is interested in ideals I of a ring R such that $\operatorname{Proj} R[It]$ is a smooth variety, where $R[It] = \oplus_{n \in \mathbf{N}} I^n t^n$ is the Rees ring of R along I. This leads one to focus on the Rees ring and its properties. In particular one would like to know when the Rees ring is normal. If R is a normal domain, $R[It]$ is normal if and only if I is a normal ideal, where by *normal ideal* we mean an ideal all of whose positive powers are integrally closed. Normal ideals have been studied in different cases, and in some special cases necessary and sufficient conditions have been given for an ideal to be normal; see for example [G], [O], and [HS].

Key words and phrases. Rees Ring, Normal Ideal.

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In general given a ring, finding an ideal whose blowup is regular has proven to be a difficult problem. The same is true even for finding normal ideals. In this paper we will construct a normal ideal I for a general graded domain $R = k[X_0, ..., X_m]/J$ which depends only on the weights of the variables $X_0, ..., X_m$, and is thus very simple to construct. We will begin by recalling some necessary definitions. For more on the construction of Rees rings see [E].

Let R be any $\mathbf{N}$-graded domain, which is a quotient of a polynomial ring $k[X_0, ..., X_m]$ modulo a homogeneous ideal J, where $k = R_0$ is an arbitrary domain and $X_0, ..., X_m$ are variables of positive weights $A_0, ..., A_m$. In practice, k is usually a field. Let $\mathbf{m} = (X_0, ..., X_m)$ be the irrelevant ideal of R. By abuse of notation, by $X_0, ..., X_m$ we will mean the images of the variables $X_0, ..., X_m$ in R. Throughout this paper $R_{\geq \alpha}$ refers to the ideal of R generated by the elements of degree at least α in the graded ring R.

We begin our search for normal ideals by reviewing some basic facts about them.

Definition 1. For Noetherian domains $S \subseteq T$ the *integral closure* of S in T, denoted by $\overline{S}$, is defined as all $x \in T$ that satisfy an equation of the form $x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$ where $a_i \in S$, for $i = 1, ..., n$.

Definition 2. For a Noetherian domain S and an ideal J of S, the *integral closure* of J in S, denoted by $\overline{J}$, is defined as

all $x \in S$ that satisfy an equation of the form $x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$ where $a_i \in J^i$, $i = 1, ..., n$; or equivalently,

all $x \in S$ for which there exists a nonzero $c \in S$ such that $cx^n \in J^n$ for all positive integers n (see [Ho]).

Note. These definitions also hold in the nondomain case; we just have to choose c not in any minimal prime of S in the second statement.

Here are a few well-known facts that we shall use:

Theorem 1. *If S is a normal domain, then $S[It]$ is normal iff I^n is integrally closed for every positive integer n.*

Theorem 2. *In an $\mathbf{N}$-graded domain $(S, \mathbf{n})$, for any positive integer α, the $\mathbf{n}$-primary ideal $I = S_{\geq \alpha}$ is integrally closed.*

Using Theorems 1 and 2, we will be looking for an ($\mathbf{m}$-primary) ideal I of the form $R_{\geq \alpha}$, with the property that $I^n = R_{\geq n\alpha}$ for all integers $n \geq 1$. Note that from Lemma (2.1.6) in [EGA] one can deduce that such an α always exists. Theorem 3 gives an explicit value for α.

2. Main Theorem

Theorem 3. *Let R be a graded domain, which is a quotient of a polynomial ring $k[X_0, ..., X_m]$ modulo a homogeneous ideal J, where k is an arbitrary domain and $X_0, ..., X_m$ are variables of positive weights $A_0, ..., A_m$. Let A be the least common multiple of $A_0, ..., A_m$. Then the ideal $I = R_{\geq mA}$ is a normal ideal. In particular, if R is normal, the Rees ring $R[It]$ is normal.*

We will show that for all positive integers p, $I^p = R_{\geq pmA}$. By Theorem 2, this will complete the proof.

Inductive Step. *Let R, I, and A be as in Theorem 3. For $p \geq 2$, if $I^{p-1} = R_{\geq (p-1)mA}$ then $I^p = R_{\geq pmA}$.*

It is obvious that $I^p \subseteq R_{\geq pmA}$. We have to show that the other inclusion holds. Also, observe that elements in $R_{\geq pmA}$ are sums of monomials in the X_i with coefficients in k, whose degrees are larger than or equal to pmA, and therefore we notice that $R_{\geq pmA}$ is generated by such monomials. Hence it is enough to show that every monomial of $k[X_0, ..., X_m]$ which lies in $R_{\geq pmA}$ also lies in I^p.

Lemma. *With notation as above, let $X_0^{c_0}...X_m^{c_m} \in R_{\geq pmA}$, where $c_0, ..., c_m$ are nonnegative integers, and let $a_0, ..., a_m$ be positive integers such that $a_i A_i = A$ for all i. Assume $I^{p-1} = R_{\geq (p-1)mA}$. Fix n such that $1 \leq n \leq m$, and suppose that $a_i \leq c_i$ for $0 \leq i < n$. For each i smaller than n, let k_i be the unique positive integer such that $k_i a_i \leq c_i < (k_i + 1)a_i$. Then:*
1) if $k_0 + ... + k_{n-1} \leq m - 1$, then for some a_j $(n \leq j \leq m)$, $a_j \leq c_j$;
2) if $k_0 + ... + k_{n-1} \geq m$, then $X_0^{c_0}...X_m^{c_m} \in I^p$.

Proof of Lemma. 1) If $c_i < a_i$ for all $n \leq i \leq m$, then

$$\begin{aligned} pmA &\leq c_0 A_0 + ... + c_m A_m \\ &< (k_0 + 1)a_0 A_0 + ... + (k_{n-1} + 1)a_{n-1} A_{n-1} + a_n A_n + ... + a_m A_m \\ &= (k_0 + ... + k_{n-1})A + nA + (m - n + 1)A \\ &\leq (m - 1)A + nA + (m - n + 1)A \\ &= 2mA. \end{aligned}$$

It follows that $p < 2$, contrary to the hypothesis.

2) Choose nonnegative integers $s_0, ..., s_{n-1}$ such that $s_i \leq k_i$ for all $0 \leq i \leq n-1$, and $s_0 + ... + s_{n-1} = m$. We see that

$$X_0^{c_0} ... X_m^{c_m} = X_0^{s_0 a_0} ... X_{n-1}^{s_{n-1} a_{n-1}} \cdot X_0^{c_0 - s_0 a_0} ... X_{n-1}^{c_{n-1} - s_{n-1} a_{n-1}} X_n^{c_n} ... X_m^{c_m}.$$

Now

$$\begin{aligned} &\deg X_0^{s_0 a_0} ... X_{n-1}^{s_{n-1} a_{n-1}} \\ &= s_0 A + ... + s_{n-1} A \\ &= mA, \end{aligned}$$

which implies that $X_0^{s_0 a_0} ... X_{n-1}^{s_{n-1} a_{n-1}} \in I$. On the other hand

$$\begin{aligned} &\deg X_0^{c_0 - s_0 a_0} ... X_{n-1}^{c_{n-1} - s_{n-1} a_{n-1}} X_n^{c_n} ... X_m^{c_m} \\ &= \deg X_0^{c_0} ... X_m^{c_m} - mA \\ &\geq pmA - mA \\ &= (p-1)mA. \end{aligned}$$

Therefore $X_0^{c_0 - s_0 a_0} ... X_{n-1}^{c_{n-1} - s_{n-1} a_{n-1}} X_n^{c_n} ... X_m^{c_m} \in I^{p-1}$, so $X_0^{c_0} ... X_m^{c_m} \in I^p$. ■

Proof of Inductive Step. We have $X_0^{c_0} ... X_m^{c_m} \in R_{\geq pmA}$, where $c_0, ..., c_m$ are nonnegative integers, and we want to show that it lies in I^p as well. Let $a_0, ..., a_m$ be positive integers such that $a_i A_i = A$ for all i.

If $c_i \geq m a_i$ for any i, say if $c_0 \geq m a_0$, then we will get

$$X_0^{c_0} ... X_m^{c_m} = X_0^{m a_0} \cdot X_0^{c_0 - m a_0} ... X_m^{c_m}.$$

Now, $\deg X_0^{m a_0} = m a_0 A_0 = mA$, and so $X_0^{m a_0} \in I$. On the other hand,

$$\deg X_0^{c_0 - m a_0} ... X_m^{c_m} = \deg X_0^{c_0} ... X_m^{c_m} - mA \geq pmA - mA = (p-1)mA.$$

So $X_0^{c_0 - m a_0} ... X_m^{c_m} \in I^{p-1}$ by our assumption. Therefore $X_0^{c_0} ... X_m^{c_m} \in I^p$, and we are done.

So let us look at the case where $c_i < m a_i$ for all i.

Now, we cannot have $c_i < a_i$ for all i, because in that case we get

$$pmA \leq c_0A_0 + ... + c_mA_m < a_0A_0 + ... + a_mA_m = (m+1)A,$$

and so

$$pm < m+1 \Rightarrow p < 1 + \frac{1}{m} \leq 2,$$

which is a contradiction.

So without loss of generality let $c_0 \geq a_0$, and let k_0 be the positive integer $(0 < k_0 < m)$ such that $k_0a_0 \leq c_0 < (k_0+1)a_0$.

Hereafter, we proceed by induction: Having chosen $k_0, ..., k_{n-1}$, if $k_0 + ... + k_{n-1} \geq m$, then $X_0^{c_0}...X_m^{c_m} \in I^p$ by the above lemma, and hence we are done.

However, if $k_0 + ... + k_{n-1} \leq m-1$, by part 1 of the lemma there exists a $j \geq n$ for which $a_j \leq c_j$. We can without loss of generality assume that $j = n$, and repeat the same cycle. Thus we are reduced to considering the case where $a_i \leq c_i$ for $i = 0, ..., m$. Let the $k_0, ..., k_m$ be the unique positive integers for which $k_ia_i \leq c_i < (k_i+1)a_i$, $(0 \leq i \leq m)$. We can easily see that $k_0 + ... + k_m \geq m$, because otherwise

$$\begin{aligned}
pmA &\leq c_0A_0 + ... + c_mA_m \\
&< (k_0+1)a_0A_0 + ...(k_m+1)a_mA_m \\
&= (k_0 + ... + k_m)A + (m+1)A \\
&\leq (m-1)A + (m+1)A \\
&= 2mA,
\end{aligned}$$

which implies that $p < 2$, contrary to our assumption. It follows then from part 2 of the same lemma that $X_0^{c_0}...X_m^{c_m} \in I^p$. This completes the proof of the inductive step. ■

Conclusion of Proof of Theorem 3. We need I^p to be integrally closed for all $p \geq 1$. By Theorem 2, it suffices to have $I^p = R_{\geq pmA}$ for $p \geq 1$. We prove this by induction. The case $p = 1$ is the definition of I. If $I^k = R_{\geq kmA}$ for all $1 \leq k \leq p-1$, then from the inductive step it follows that $I^p = R_{\geq pmA}$. If R is normal, it follows from Theorem 1 that $R[It]$ is a normal ring. ■

3. Examples

Example 1. Let k be a field, and $R = k[x,y,z]/(x^2+y^3-z^5)$, where x, y, z have weights 15,10, and 6, respectively. the least common multiple of these variables is 30, and therefore by theorem 3,

$$I = R_{\geq 60} = (x^4, x^3y^2, x^3yz, x^3z^3, x^2y^3, x^2y^2z^2, x^2yz^4, x^2z^5, xy^5, \\ xy^4z, xy^3z^3, xy^2z^5, xyz^6, xz^8, z^{10}, yz^9, y^2z^7, y^3z^5, y^4z^4, y^5z^2, y^6)$$

is a normal ideal for this ring.

Example 2. In general, if k is a field, and $R = k[x,y,z]/(x^a+y^b+z^c)$ is a domain, where the variables x, y, z have weights bc, ac, and ab, respectively, the ideal $I = R_{\geq 2abc}$ will be a normal ideal of R.

Example 3. For the polynomial ring $R = k[x,y,z]$ over a field k, one can find many normal ideals by assigning different weights to the variables x, y and z. For example, if we set $\deg x = 1$, $\deg y = 1$ and $\deg z = 2$ we find that the ideal $I = R_{\geq 4} = (x^4, x^3y, x^2y^2, x^2z, xy^3, xyz, y^4, y^2z, z^2)$ is a normal ideal.

Remark 1. The method described in Example 3 above for finding normal ideals of polynomial rings is not as interesting in the case of two variables. It is a theorem due to Zariski (see [Z]), that the set of integrally closed ideals in a regular local ring of dimension two is closed under multiplication. Therefore, in the ring $R = k[x,y]$, all integrally closed ideals are normal. In particular, all ideals of the form $I = R_{\geq \alpha}$, where α is a positive integer, are normal (all powers of I are integrally closed).

Remark 2. Looking at Theorems 1 and 2, it is natural to wonder whether for an $\mathbf{N}$-graded normal ring R and all positive integers α, one can say that $\overline{(R_{\geq \alpha})^n} = R_{\geq n\alpha}$ for $n \in \mathbf{N}$. The answer in general is no. For example, consider the ring $R = k[x,y]$ where k is a field, $\deg x = 2$, and $\deg y = 3$. Let $I = (R_{\geq 7})^3$. We will check if $\overline{I}$ is equal to $R_{\geq 21}$. Since $R_{\geq 7} = (x^4, yx^2, xy^2, y^3)$, we can calculate $I = (x^{12}, x^{10}y, x^8y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, y^9)$. If Γ is the set of pairs (a,b) corresponding to generators x^ay^b of I, we plot the points in Γ on the $\mathbf{R}^2$ plane, and we see that there are no pairs of positive integers (c,d) in the convex hull of the region $\Gamma + \mathbf{R}^{2+}$, besides those in $\Gamma + \mathbf{R}^{2+}$ itself. This implies that I is an integrally closed ideal (see [E]). On the other hand $x^{11} \in R_{\geq 21}$, but x^{11} does not belong to $I = \overline{I}$. Therefore $R_{\geq 21}$ cannot be the integral closure of I. (Note that it also follows from Zariski's theorem mentioned in Remark 1 that I is an integrally closed ideal.)

Remark 3. One might ask if all ideals of the form $I = R_{\geq\alpha}$, $\alpha \in \mathbf{N}$, in a normal graded ring are normal. The answer is no. A counterexample is the ring $R = k[x,y,z]/(x^2 + y^3 z + z^4)$, where x, y and z have degrees 2, 1 and 1, respectively. The ideal $I = R_{\geq 1} = (x,y,z)$ is integrally closed, but $I^2 = (x^2, xy, xz, y^2, yz, z^2)$ is not: $(x)^2 + (y^2)(yz) + (z^2)^2 = 0$, and hence $x \in \overline{I^2}$, but x is not in I^2. This, by the way, is an example of a ring whose test ideal is not normal, given by Hara and Smith in [HSm]. For more on test ideals and tight closure theory in general, see [HH].

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