Math 3330 - Solution set for Assignment 1 - Fall 2011.

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1. Let $n \geq 2$, and G be a simple graph with n vertices. Prove that there are vertices v and w of G such that $\deg(v) = \deg(w)$.

   Solution: Let G be a simple graph with n vertices. Then there are n possible values for degrees of vertices of G, namely $0, 1, \ldots, n-1$. Note that a vertex of degree $n-1$ must be joined to every other vertex in G. Hence the degree values 0 and $n-1$ cannot occur simultaneously in G. Thus there are $n-1$ possible degree values for n vertices of G, which by pigeonhole principle implies that there are vertices v and w of G such that $\deg(v) = \deg(w)$.

2. Without using the recursive method of Theorem 3 of Lecture 3 and the corollary thereafter, determine whether the following sequences are graphic or not:

   (i) $\langle 4, 2, 2, 1, 0, 0\rangle$.
   (ii) $\langle 2, 2, 2, 2\rangle$.
   (iii) $\langle 4, 3, 2, 1, 0\rangle$.
   (iv) $\langle 4, 4, 4, 4, 3, 3, 3, 3\rangle$.
   (v) $\langle 3, 2, 2, 1, 0\rangle$.

   Solution of Part (i): $\langle 4, 2, 2, 1, 0, 0\rangle$ is not a graphic sequence, because the number of odd terms in the sequence is odd. Recall that every graph has an even number of vertices of odd degrees.

   Solution of Part (ii): $\langle 2, 2, 2, 2\rangle$ is graphic, because it is the degree sequence of the cycle of length 4.

   Solution of Part (iii): Using Problem 1, one can observe that $\langle 4, 3, 2, 1, 0\rangle$ is not graphic.

   Solution of Part (iv): $\langle 4, 4, 4, 4, 3, 3, 3, 3\rangle$ is graphic, because it is the degree sequence of the graph G defined as below.

$$\begin{aligned} V_G &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \\ E_G &= \{\{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_3, v_7\} \\ &\quad \{v_3, v_8\}, \{v_3, v_5\}, \{v_4, v_8\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_1, v_2\}, \{v_3, v_4\}\}. \end{aligned}$$

   Solution of Part (v): $\langle 3, 2, 2, 1, 0\rangle$ is graphic, because it is the degree sequence of the graph G defined as below

$$\begin{aligned} V_G &= \{v_1, v_2, v_3, v_4, v_5\}, \\ E_G &= \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_3\}\}. \end{aligned}$$

3. Using the recursive method of Theorem 3 of Lecture 3 and the corollary thereafter, determine whether the following sequences are graphic or not:

   (i) $\langle 7,7,6,5,4,4,3,2\rangle$.

   (ii) $\langle 4,4,3,3,3,3,2,2,2\rangle$.

   (iii) $\langle 5,5,4,3,2,2,2,1\rangle$.

   (iv) $\langle 5,5,4,4,2,2,1,1\rangle$.

   Solution of Part (i): Applying the recursive method of the above mentioned theorem, we get

   $$\langle 7,7,6,5,4,4,3,2\rangle \to \langle 6,5,4,3,3,2,1\rangle \to \langle 4,3,2,2,1,0\rangle \to \langle 2,1,1,0,0\rangle \to \langle 0,0,0,0\rangle.$$

   Clearly $\langle 0,0,0,0\rangle$ is a graphic sequence, hence $\langle 7,7,6,5,4,4,3,2\rangle$ is graphic too.

   Solution of Part (ii): Applying the recursive method of the above mentioned theorem, we get

   $$\begin{aligned}\langle 4,4,3,3,3,3,2,2,2\rangle &\to \langle 3,2,2,2,3,2,2,2\rangle \to \langle 3,3,2,2,2,2,2,2\rangle \to \langle 2,1,1,2,2,2,2\rangle\\ &\to \langle 2,2,2,2,2,1,1\rangle \to \langle 1,1,2,2,1,1\rangle \to \langle 2,2,1,1,1,1\rangle\\ &\to \langle 1,0,1,1,1\rangle \to \langle 1,1,1,1,0\rangle \to \langle 0,1,1,0\rangle\\ &\to \langle 1,1,0,0\rangle \to \langle 0,0,0\rangle.\end{aligned}$$

   Clearly $\langle 0,0,0\rangle$ is a graphic sequence, hence $\langle 4,4,3,3,3,3,2,2,2\rangle$ is graphic too.

   Solution of Part (iii): Applying the recursive method of the above mentioned theorem, we get

   $$\begin{aligned}\langle 5,5,4,3,2,2,2,1\rangle &\to \langle 4,3,2,1,1,2,1\rangle \to \langle 4,3,2,2,1,1,1\rangle \to \langle 2,1,1,0,1,1\rangle\\ &\to \langle 2,1,1,1,1,0\rangle \to \langle 0,0,1,1,0\rangle \to \langle 1,1,0,0,0\rangle \to \langle 0,0,0,0\rangle.\end{aligned}$$

   Clearly $\langle 0,0,0,0\rangle$ is a graphic sequence, hence $\langle 5,5,4,3,2,2,2,1\rangle$ is graphic too.

   Solution of Part (iv): Applying the recursive method of the above mentioned theorem, we get

   $$\begin{aligned}\langle 5,5,4,4,2,2,1,1\rangle &\to \langle 4,3,3,1,1,1,1\rangle \to \langle 2,2,0,0,1,1\rangle \to \langle 2,2,1,1,0,0\rangle\\ &\to \langle 1,0,1,0,0\rangle \to \langle 1,1,0,0,0\rangle \to \langle 0,0,0,0\rangle.\end{aligned}$$

   Clearly $\langle 0,0,0,0\rangle$ is a graphic sequence, hence $\langle 5,5,4,4,2,2,1,1\rangle$ is graphic too.

4. For each of the following sequences list all the graphs that realize the sequence:

   (i) $\langle 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\rangle$.

   (ii) $\langle 2, 2, 2, 2, 2, 2, 2, 1, 1\rangle$.

   (iii) $\langle 4, 4, 4, 4, 4\rangle$.

   Solution of Part (i): Only one graph G has he degree sequence $\langle 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\rangle$, because if we have a vertex v of degree 10 in a graph of 11 vertices, then v is adjacent to every other vertex. The graph G is defined as

$$\begin{aligned} V_G &= \{v_0, v_1, \ldots, v_{10}\}, \\ E_G &= \{\{v_0, v_i\} : 1 \leq i \leq 10\}. \end{aligned}$$

   Solution of Part (ii): One can use induction to show that the only connected graph with the degree sequence $\langle 2, 2, \ldots, 2, 1, 1\rangle$ (with n 2's) is the path of length $n+1$. Similarly the only connected graph with the degree sequence $\langle 2, 2, \ldots, 2\rangle$ (with n 2's) is the cycle of length n. Moreover, if $\langle 2, 2, \ldots, 2, 1, 1\rangle$ is the degree sequence of a graph, both vertices of degree one appear in the same connected component. Hence the list of all the graphs that realize $\langle 2, 2, 2, 2, 2, 2, 2, 1, 1\rangle$ is

   - P_8,
   - a graph with two connected components P_5 and C_3,
   - a graph with two connected components P_4 and C_4,
   - a graph with two connected components P_3 and C_5,
   - a graph with two connected components P_2 and C_6,
   - a graph with two connected components P_1 and C_7,
   - a graph with three connected components P_2, C_3, and C_3,
   - a graph with three connected components P_1, C_3, and C_4,

   where for every positive integer n, P_n denotes a path of length n and C_n denotes a cycle of length n.

   Solution of Part (iii): The only graph that realizes $\langle 4, 4, 4, 4, 4\rangle$ is the complete graph on 5 vertices.

5. Prove or disprove: There exists a simple graph G with 13 vertices, 31 edges, three vertices of degree one, and seven vertices of degree four.

Solution: Suppose a graph G with the above description exists. Let $V_G = \{v_1, v_2, \ldots, v_{13}\}$ be the vertex set of G, and assume that

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = 1, \quad \text{and} \quad \deg(v_4) = \ldots = \deg(v_{10}) = 4.$$

By the degree-sum theorem, we know that $3+4\times 7+\deg(v_{11})+\deg(v_{12})+\deg(v_{13}) = 2 \times 31$, hence $\deg(v_{11}) + \deg(v_{12}) + \deg(v_{13}) = 31$. Construct the new graph H by removing v_1, v_2, v_3, and the three edges that are incident on these vertices from G. Let α_1, α_2, and α_3 denote the degrees of v_{11}, v_{12}, and v_{13} in H. Clearly $\alpha_i \leq 9$ for every $1 \leq i \leq 3$. Hence $\alpha_1 + \alpha_2 + \alpha_3 \leq 27$. Note that

$$\deg(v_{11}) + \deg(v_{12}) + \deg(v_{13}) \leq \alpha_1 + \alpha_2 + \alpha_3 + 3 \leq 30,$$

since there are at most three edges from $\{v_1, v_2, v_3\}$ to $\{v_{11}, v_{12}, v_{13}\}$. But this is a contradiction, because $\deg(v_{11}) + \deg(v_{12}) + \deg(v_{13}) = 31$.