Math 3330 - Assignment 2 - Fall 2011.

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1. Provide a simple graph as an example for each of the following cases. Provide an explanation in each case.

   (i) A walk that is not a trail.

   (ii) A trail that is not a path.

   (iii) A closed trail that is not a cycle.

   (iv) A nontrivial closed walk that does not contain any cycles.

   Solution of Part (i): Let K_5 be the complete graph on 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$. The walk $\langle v_1, v_2, v_4, v_1, v_2, v_3\rangle$ is not a trail.

   Solution of Part (ii): Let K_5 be the complete graph on 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$. The trail $\langle v_1, v_2, v_4, v_1, v_3\rangle$ is not a path.

   Solution of Part (iii): Let K_5 be the complete graph on 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$. The closed trail $\langle v_1, v_2, v_4, v_1, v_3, v_5, v_1\rangle$ is not a cycle.

   Solution of Part (iv): Let K_5 be the complete graph on 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$. The closed walk $\langle v_1, v_2, v_4, v_2, v_1\rangle$ does not contain any cycle.

2. Let $k \geq 1$ be an integer. Let G be a simple graph whose vertices all have degree at least k. Prove that

   (i) G contains a path of length k.

   (ii) If $k \geq 2$ then G contains a cycle of length at least k.

   Solution or Part (1): Consider a path of maximum length in G, say $P = \langle v_1, v_2, \ldots, v_l\rangle$.

   Claim: The vertex v_1 cannot be adjacent to any vertex in $V_G \setminus \{v_1, \ldots, v_l\}$.
   Proof of claim: Suppose not, i.e. suppose that there exists a vertex $w \in V_G \setminus \{v_1, \ldots, v_l\}$ such that v_1 is adjacent to w. Then the walk $P' = \langle w, v_1, \ldots, v_l\rangle$ is a path in G with length more than P. But this is a contradiction, since P is a path of maximum length in G.

   Hence v_1 can be adjacent only to $\{v_2, \ldots, v_l\}$, which means that $\deg(v_1) \leq l - 1$. On the other hand, we know that v_1 has degree at least k. Thus $k \leq l - 1 = \text{length(P)}$, which implies that P has length at least k.

   Solution or Part (2): Consider a path of maximum length in G, say $P = \langle v_1, v_2, \ldots, v_l\rangle$. By (i) we know that $l \geq k + 1$. As we explained above, the vertex v_1 cannot be adjacent to any vertex in $V_G \setminus \{v_1, \ldots, v_l\}$. Hence v_1 has to be adjacent to at least $k - 1$ vertices in $\{v_3, v_4, \ldots, v_l\}$, since the degree of v_1 is at least k. Therefore there is at least an index $k \leq i \leq l$ such that v_i is adjacent to v_1. The walk $\langle v_1, \ldots, v_i, v_1\rangle$ is a cycle of length $i \geq k$.

3. Let $n \geq 3$ be an integer. We define the complete bipartite graph $K_{n,n}$ on the vertex set
$$V_{K_{n,n}} = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\},$$
where $X = \{v_1, \ldots, v_n\}$ and $Y = \{u_1, \ldots, u_n\}$ form a bipartite partition for $K_{n,n}$. Moreover, for every $1 \leq i, j \leq n$, v_i is adjacent to u_j. Let x and y be two different nonadjacent vertices in $K_{n,n}$.

   (i) Find the number of all the paths from x to y of length 2.

   (ii) Find the number of all the paths from x to y of length 3.

   (iii) Find the number of all the paths from x to y of length 4.

   (iv) In general, for every positive integer k, find the number of paths from x to y in $K_{n,n}$ of length k.

Solution: Since x and y are two different nonadjacent vertices in $K_{n,n}$, they both should belong either to X or to Y. Without loss of generality assume that $x, y \in X$

Solution of Part (i): Any path of length 2 from x to y is of the form $\langle x, u_i, y\rangle$. Since they are n choices for u_i, the number of such paths is n.

Solution of Part (ii): A path starting from x of length 3 is of the form $\langle x, u_i, v_j, u_k\rangle$. Thus every path of length 3 starting at x will end in Y. Therefore the number of paths of length 3 from x to y is zero.

Solution of Part (iii): Any path of length 4 from x to y is of the form $\langle x, u_i, v_j, u_{i'}, y\rangle$, where $v_j \in X \setminus \{x, y\}$ and $i \neq i'$. Since every vertex in X is adjacent to every vertex in Y, they are n choices for u_i, $n-2$ choices for v_j, and $n-1$ choices for $v_{i'}$. Hence the number of such paths is $n(n-2)(n-1)$.

Solution of Part (iv): Let k be a positive integer. If k is odd, then the number of paths of length k from x to y is zero (as we explained in Part (ii)). Suppose k is even. Repeating the argument in (iii), we can see that the number of paths from x to y of length k is

$$[n(n-2)][(n-1)(n-3)]\ldots[(n-\frac{k}{2})(n-\frac{k}{2}+2)](n-\frac{k}{2}+1) = (n-\frac{k}{2}+1)\prod_{i=0}^{\frac{k}{2}-2}(n-i)(n-i-2).$$

4. Let P and Q be paths of maximum length in a connected simple graph G. Prove that P and Q have a common vertex.

Solution: Let $P = \langle p_1, \ldots, p_n\rangle$ and $Q = \langle q_1, \ldots, q_n\rangle$ be paths of maximum length. Suppose not, i.e. suppose that P and Q do not have a common vertex. Since G is connected, there is a path between every pair of vertices in G. For every $1 \leq i, j \leq n$, let $R_{i,j}$ denote a shortest path from p_i to q_j. Let R_{i_0,j_0} denote the path with the shortest length among $\{R_{i,j}\}_{1 \leq i,j \leq n}$. Since P and Q have no common vertices, R_{i_0,j_0}

has length at least 1.

Claim: R_{i_0,j_0} and P do not have a common vertex except p_{i_0}. Similarly, R_{i_0,j_0} and Q do not have a common vertex except q_{j_0}.
Proof of claim: We prove the claim for P. The statement for Q can be proved similarly. Towards a contradiction, suppose R_{i_0,j_0} and P have a vertex other than p_{i_0} in common. Let p_k be the last vertex on R_{i_0,j_0} that is in common with P. Then the subwalk of R_{i_0,j_0} from p_k to q_{j_0} has a shortest length than R_{i_0,j_0}. But this is a contradiction with the choice of R_{i_0,j_0}. Hence R_{i_0,j_0} and P do not have a common vertex except p_{i_0}.

Let $R_{i_0,j_0} = \langle p_{i_0}, v_1, \ldots, v_t, q_{j_0} \rangle$. Consider the following paths in each of the following cases:

- If $i_0, j_0 \leq \frac{n}{2}$, consider the path $\langle p_n, p_{n-1}, \ldots, p_{i_0}, v_1, \ldots, v_t, q_{j_0}, \ldots, q_n \rangle$.
- If $i_0 \leq \frac{n}{2}$ and $j_0 > \frac{n}{2}$, consider the path $\langle p_n, p_{n-1}, \ldots, p_{i_0}, v_1, \ldots, v_t, q_{j_0}, q_{j_0-1}, \ldots, q_1 \rangle$.
- If $i_0 > \frac{n}{2}$ and $j_0 \leq \frac{n}{2}$, consider the path $\langle p_1, p_2, \ldots, p_{i_0}, v_1, \ldots, v_t, q_{j_0}, \ldots, q_n \rangle$.
- If $i_0 > \frac{n}{2}$ and $j_0 > \frac{n}{2}$, consider the path $\langle p_1, p_2, \ldots, p_{i_0}, v_1, \ldots, v_t, q_{j_0}, q_{j_0-1}, \ldots, q_1 \rangle$.

Observe that in each of the above cases, we have a path of length more than P (or Q), since R_{i_0,j_0} is nontrivial. But this is a contradiction. So P and Q have a common vertex.

5. Prove that every closed walk W of odd length in a simple graph contains a cycle.
Hint: First show that if W does not contain any cycles then there exists an edge in W which repeats immediately.

Solution: We first prove the following claim:

Claim: If a closed walk W does not contain any cycles then there exists an edge in W which repeats immediately.
Proof of claim: Suppose not, i.e. there is no edge which repeats immediately. Consider all the nontrivial closed subwalks of W, and let W_0 be one with the minimum length. Since no edge in W repeats immediately, the length of W_0 is at least 3. It is easy to show that W_0 is a trail (because if an edge is repeated in W_0, we can find a shorter closed subwalk of W_0). Now by a theorem in the notes, any nontrivial closed trail contains a cycle. This implies that W contains a cycle, which is a contradiction.

Towards a contradiction, assume that $W = \langle v_1, \ldots, v_n, v_1 \rangle$ is a closed walk of odd length in a simple graph, and it does not contain a cycle. By the above claim, there is an edge e with endpoints v_i and v_{i+1} that repeats immediately, i.e. $\langle v_i, v_{i+1}, v_i \rangle$ is a subwalk of W (i.e. $v_{i+2} = v_i$). Consider the new walk $W' = \langle v_1, \ldots, v_i, v_{i+3} \ldots, v_n, v_1 \rangle$ constructed by removing the repeated edge from W. Clearly W' is a closed walk of odd length, and it does not contain any cycles (since W does not contain any cycles.) Hence we can repeat the above procedure, i.e. find an edge that is repeated immediately, and remove it. Repeating the above procedure, we finally get to a closed walk of length 3, which is clearly contained in W. It is easy to see that the only closed walk of length 3 in a simple graph is a triangle. But this is a contradiction, because we assumed that W does not contain any cycles.