Math 3330 - Solution to Assignment 3 - Fall 2011.

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1. Prove or disprove each of the following statements:

   (i) Every graph with fewer edges than vertices has a component that is a tree.

   (ii) If a simple graph G has no cut-edge then every vertex of G has even degree.

   (iii) A graph is a forest if and only if every connected subgraph is an induced subgraph.

   Solution of Part (i): True! Let G be a graph such that $|E_G| < |V_G|$. Let $G_1, \ldots, G_k$ be the connected components of G. Towards a contradiction, assume that no connected component of G is a tree. Hence, for each $1 \leq i \leq k$, $|E_{G_i}| \geq |V_{G_i}|$. Thus,

   $$|E_G| = \sum_{i=1}^{k} |E_{G_i}| \geq \sum_{i=1}^{k} |V_{G_i}| = |V_G|,$$

   which is a contradiction. Hence there exists a component of G which is a tree.

   Remark: We used this fact: If G is a connected graph, then $|E_G| \geq |V_G| - 1$. Indeed, any connected graph contains a spanning tree (just apply any of the tree-growing algorithms we have), thus has at least $|V_G| - 1$ edges.

   Solution of Part (ii): False! The complete graph on 4 vertices has no cut-edge but the degree of each vertex is 3.

   Solution of Part (iii): $[\Rightarrow]$: Suppose G is a forest, and let H be a connected subgraph of G. Hence H is a subtree of G. Towards a contradiction, assume that H is not induced, i.e. there exist $v, u \in V_H$ such that v and u are joined through an edge e in G, and $e \notin E_H$. Therefore $H + e$ is a subgraph of G which has a cycle. But this is a contradiction since G is a forest.

   $[\Leftarrow]$: Suppose every connected subgraph of G is induced. Towards a contradiction assume that G has a cycle $\langle v_1, \ldots, v_k, v_1 \rangle$. Let e denote the edge joining v_1 and v_n. Let H be the subgraph of G induced on $\{v_1, \ldots, v_n\}$. Then the subgraph $H - e$ is clearly connected but not induced, which is a contradiction. Thus G has no cycles, i.e. it is a tree.

2. Suppose the average degree of the vertices of a connected graph is exactly 2. How many cycles does G have? Support your answer.

   Solution: Suppose G is a graph on n vertices $v_1, \ldots, v_n$. Suppose that $\frac{\deg(v_1)+\ldots+\deg(v_n)}{n} = 2$. Thus,

   $$\deg(v_1) + \ldots + \deg(v_n) = 2n,$$

which implies that the number of edges in G is n. The graph G has n vertices, thus it cannot be a tree. Thus G has at least one cycle, say C. Let e be an edge in C. Then the graph $G - e$ is connected, and has $n - 1$ edges, hence it is a tree. By a theorem in the notes, adding one edge e to the tree $G - e$ creates exactly one cycle. Thus G has exactly one cycle.

3. Prove that a graph G is a forest if and only if every induced subgraph of G contains a vertex of degree 0 or 1.

   Solution: [$\Rightarrow$]: Suppose G is a forest. Then every induced subgraph H of G is a forest too, and its connected components are trees. Hence H contains a vertex of degree 0 (if it has a trivial connected component) or 1 (if it has a nontrivial connected component).

   [$\Leftarrow$]: Let G be a graph in which every induced subgraph contains a vertex of degree 0 or 1. Towards a contradiction, suppose that G is not a forest, i.e. G has a cycle $\langle v_1, \ldots, v_k, v_1 \rangle$. Now in the subgraph of G induced on $\{v_1, \ldots, v_k\}$ every vertex has degree at least two, which is a contradiction. Thus G must be a forest.

4. Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

   Solution: [$\Leftarrow$]: Assume that $d_1 \geq d_2 \geq \ldots \geq d_n$ is a positive sequence that is the degree sequence of a tree T. Hence T has n vertices and $n - 1$ edges. Now by the degree-sum theorem, $\sum_{i=1}^{n} d_i = 2(n - 1) = 2n - 2$.

   [$\Rightarrow$]: We proceed by induction on n.

   Basis of induction: Let $n = 2$, and $d_1 \geq d_2 > 0$ be such that $d_1 + d_2 = 2$. Therefore, $d_1 = d_2 = 1$, and $d_1 \geq d_2$ is the degree sequence of the tree on two vertices.

   Induction hypothesis: Let $n > 2$. Suppose for every sequence $d_1 \geq \ldots \geq d_{n-1}$ of positive integers with $\sum_{i=1}^{n-1} = 2(n-2)$, there exists a tree on $n-1$ vertices with that degree sequence.

   Induction Step: Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be a sequence of positive integers such that $\sum_{i=1}^{n} d_i = 2(n - 1) = 2n - 2$. Clearly $d_n = 1$, because if not, $\sum_{i=1}^{n} d_i \geq 2n$. Let $k \in \{1, \ldots, n\}$ be such that $d_k > 1$ and $d_j = 1$ for all $j > k$. Consider the new sequence $d'_1 = d_1, \ldots, d'_{k-1} = d_{k-1}, d'_k = d_k - 1, d'_{k+1} = d_{k+1}, d'_{n-1} = d_{n-1}$. Then $d_1 + \ldots + d_{k-1} + (d_k - 1) + d_{k+1} + \ldots + d_{n-1} = 2n - 2 - 2 = 2(n-2)$, and by induction hypothesis there exists a tree with vertex set $\{v_1, \ldots, v_{n-1}\}$ such that $\deg(v_i) = d'_i$ for every $1 \leq i \leq n - 1$. Construct a new graph T' by adding a vertex w to T, and joining w to v_k. It is easy to see that T' is a tree, and $d_1 \geq d_2 \geq \ldots \geq d_n$ is the degree sequence of T'.

5. Let T be a tree in which all vertices adjacent to leaves have degree at least 3. Prove that there exists a pair (v_1, v_2) of leaves that have a common neighbor.

   Solution: Suppose not, i.e. assume that there exists a tree T on n vertices such that all the leaves have distinct neighbors. Let k denote the number of leaves of T. Therefore there are at least k vertices in T of degree 3 (these are the neighbors of the leaves). Note that the $n - 2k$ remaining vertices have degree at least 2, because they are not leaves. Thus

   $$\sum_{i=1}^{n} \deg(v_i) \geq k + 3k + 2(n - 2k) = 2n,$$

   which is a contradiction.