Math 3330 - Solution to Assignment 5 - Fall 2011.

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1. Prove that every graph G has a vertex ordering relative to which the sequential vertex-coloring algorithm uses $\chi(G)$ colors.
   Solution: Let $f : V_G \to \{1, \ldots, \chi(G)\}$ be a proper coloring of G. For every $1 \leq i \leq \chi(G)$, let U_i be the set of vertices colored i. Consider a vertex ordering of G, say $v_1, v_2, \ldots, v_n$, in which for every $1 \leq i < j \leq \chi(G)$, every vertex in U_i is indexed smaller than every vertex in U_j.

   Claim: Let $g : V_G \to \{1, 2, \ldots\}$ be the coloring of G obtained by applying the sequential vertex-coloring algorithm to G with the above ordering. Then for every $1 \leq i \leq \chi(G)$ and every $v \in U_i$, $g(v) \leq i$.

   Proof of claim: We prove by induction. First note that for every vertex v in U_1, we have $g(v) = 1$.
   Induction hypothesis: Let $k < \chi(G)$ be fixed. Suppose that for every $1 \leq i \leq k$ and $v \in U_i$, $g(v) \leq i$.
   Induction step: Let $v \in U_{k+1}$ be given. Since U_{k+1} is an independent set, any neighbor of v that is indexed less than v is an element of $U_1 \cup \ldots \cup U_k$. Thus by induction hypothesis, any neighbor of v that is indexed less than v is colored from the set $\{1, \ldots, k\}$. Hence $g(v) \leq k + 1$.

   Therefore, g uses only $\chi(G)$ colors.

2. For every $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering $v_1, v_2, \ldots, v_n$ of its vertices with respect to which the sequential vertex-coloring algorithm uses $k + 1$ colors.
   Solution: We construct T_ks by induction.
   Basis of induction: For $k = 0$, let T_0 be an isolated vertex v_0. Then T_0 has maximum degree 0 and is colored by 1 color. For $k = 1$, let T_1 be the unique tree on two vertices, and v_1, v_2 be any ordering of the vertices. Clearly T_1 satisfies the required properties.
   Induction hypothesis: Assume that for every $0 \leq i \leq k$, one can construct a tree T_i with an ordering of vertices of T_i, say $v_1^i, \ldots, v_{n_i}^i$ such that $\delta_{\max}(T_i) = i$ and the sequential vertex-coloring algorithm uses $i + 1$ colors for T_i, and $v_{n_i}^i$ is colored $i + 1$.
   Induction step: Construct T_{k+1} in the following way: Let $T'_{k+1} = T_0 \cup T_1 \cup T_2 \cup \ldots \cup T_k$, where for every $0 \leq i \leq k$, T_i is constructed in induction hypothesis. Construct T_{k+1} by adding a new vertex w to T'_{k+1} and joining w to

$v_0, v^1_{n_1}, v^2_{n_2}, \ldots v^k_{n_k}$. Consider the ordering of vertices of T_{k+1}

$$v_0, v^1_1, \ldots, v^1_{n_1}, \ldots, v^i_1, \ldots, v^i_{n_i}, \ldots, v^k_1, \ldots, v^k_{n_k}.$$

Note that $\delta_{\max}(T_{k+1}) = k+1$. It is clear that the sequential vertex-coloring algorithm colors T_{k+1} with $k+2$ colors.

3. Prove that

   (i) Adding an edge to a graph increases the chromatic number by at most 1.

   (ii) Deleting a vertex from a graph decreases the chromatic number by at most 1.

   Solution of (i): Let u and v be vertices of a graph G. Consider a proper coloring f of vertices of G with colors $\{1, \ldots, \chi(G)\}$. Let $G' = G + e$ be the graph obtained by adding the edge $e := uv$ to G. Consider the coloring f' for G' as follows:

   $$f'(x) = \begin{cases} f(x) & x \neq u \\ \chi(G)+1 & x = u \end{cases}.$$

   It is very easy to see that f' is a proper coloring of G'. Thus $\chi(G') \leq \chi(G)+1$. Moreover, since G is a subgraph of G', we have $\chi(G) \leq \chi(G')$. Therefore, $\chi(G) \leq \chi(G') \leq \chi(G)+1$.

   Solution of (ii): Suppose not, i.e. suppose that there exists a graph G and a vertex $v \in V_G$ such that $\chi(G-v) \leq \chi(G)-2$. Consider a proper coloring f of vertices of $G-v$ with colors $\{1, \ldots, \chi(G)-2\}$. Define the coloring f' of G to be

   $$f'(x) = \begin{cases} f(x) & x \in V_G \setminus \{v\} \\ \chi(G)-1 & x = v \end{cases}.$$

   Clearly, f' is a proper coloring of vertices of G with colors $\{1, \ldots, \chi(G)-1\}$, which implies that $\chi(G) \leq \chi(G)-1$. But this is a contradiction.

4. Describe how to construct a connected graph with chromatic number c and independence number a, for arbitrary $a \geq 1$ and $c \geq 2$.
   Solution: Let $G' = H_1 \cup \ldots \cup H_a$, where each H_i is a complete graph on c vertices. Pick a vertex v_i from each H_i. Construct G from G' by adding the edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{a-1}, v_a)$ to G'. Note that since $c \geq 2$, it is easy to check that G has independence number a. It is also clear that G has chromatic number c.

5. Prove that every k-chromatic graph has at least $\frac{k(k-1)}{2}$ edges.
Solution: Consider a coloring of G with k colors. For each $1 \leq i \leq k$, let U_i denote the set of vertices of G that are colored i. Clearly each U_i is an independent set. Assume that there exist $i \neq j$ such that no edge in G has one endpoint in U_i and one in U_j. In that case, $U_i \cup U_j$ forms an independent set. Thus, we can obtain a proper $k-1$-coloring of G, which is a contradiction. Hence for every pair $i \neq j$, there exists an edge with one endpoint in U_i and one in U_j. Therefore, there are at least $\frac{k(k-1)}{2}$ edges in G.

6. Let G be a graph. We define the complement of G, denoted by $\overline{G}$, as follows:

$$V_{\overline{G}} = V_G, \quad \forall x, y \in V_G,\ x \text{ is adjacent to } y \text{ in } G \Leftrightarrow x \text{ is not adjacent to } y \text{ in } \overline{G}.$$

Construct a graph G with a vertex v such that $\chi(G-v) < \chi(G)$ and $\chi(\overline{G}-v) < \chi(\overline{G})$.
Solution: The cycle of length 5, C_5, with any vertex in it works. Because $\overline{C_5} = C_5$.