# Math 3330 - Solutions of Assignment 6 - Fall 2011.

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1. In each case, either prove the statement or disprove it by providing a counterexample.

   (i) Subdividing an edge $e$ in a graph $G$ causes the edge-chromatic number to increase by at most 1.

   (ii) Subdividing an edge $e$ in a graph $G$ causes the edge-chromatic number to decrease by at most 1.

   **Solution of Part (i):** The statement is false. For a counter example, let $G$ be a cycle of length 3. Clearly, $\chi'(G) = 3$, but subdividing any edge in $G$ will create a 4-cycle which has edge-chromatic number 2.

   **Solution of Part (ii):** The statement is false. For a counter example, let $G$ be a single edge. Clearly, $\chi'(G) = 1$, but subdividing the edge in $G$ will create a path of length 2 which has edge-chromatic number 2.

2. In each of the following cases, determine if the graph is planar or not. If the graph is planar, draw its embedding with no edge-crossings. If not, provide a proof.

   (i) Peterson graph.

   (ii) $K_{3,3} - e$, where $e$ is any edge in the complete bipartite graph $K_{3,3}$.

   (iii) $K_8 - K_{3,3}$, where $K_8$ is a complete graph on 8 vertices.

   (iv) $K_{6,6} - C_{12}$, where $k_{6,6}$ is a complete bipartite graph with partitions of size 6, and $C_{12}$ is a cycle of length 12.

   **Solution of part (i):** Not planar. A detailed explanation is provided in a separate file.

   **Solution of Part (ii):** Planar. A detailed explanation is provided in a separate file.

   **Solution of Part (iii):** Not planar. A detailed explanation is provided in a separate file.

   **Solution of Part (iv):** Not planar. A detailed explanation is provided in a separate file.

3. Let $G$ be a minimal non-planar graph, i.e. $G$ is non-planar, but every subgraph of $G$ is planar.

(i) Prove that $G$ is connected.

(ii) Prove that $G$ does not have any cut-vertices.

**Solution of Part (i):** Suppose not, i.e. suppose that $G$ is not connected. Let $G_1$, $G_2, \dots, G_m$ be the connected components of $G$. Since $G$ is minimal, each $G_i$ is planar. Thus $G$ is planar as well. But this is a contradiction.

**Solution of Part (ii):** Suppose $G$ has a cut vertex $v$. Then $G - v$ has at least two components $G_1$ and $G_2$. The graphs $G_1 + v$ and $G_2 + v$ are planar.

First note that $G_1 + v$ has a planar drawing (i.e. drawing on the plane with no edge-crossing) with $v$ on its most outer face. To observe this, it is enough to embed (without any edge-crossings) $G_1 + v$ on the surface of a sphere. Then, consider a face with $v$ on its border, and consider the stereographic projection of the sphere onto the plane through any point inside that face (Understanding this part of the proof is optional).

Thus $G$ is planar, because we can embed the planar drawing of $G_1 + v$ obtained above inside a "face" of a planar drawing of $G_2 + v$ that has $v$ on its boundary, and merge the vertex $v$ of $G_1 + v$ and $G_2 + v$ to get a planar drawing of $G$.

4. Prove that every planar graph decomposes into two bipartite graphs.

**Solution:** Let $G$ be a planar graph. By the four-color theorem, there exists a proper coloring $f : V_G \to \{1, 2, 3, 4\}$ of vertices of $G$. For each $1 \leq i \leq 4$, define,

$$U_i := \{v \in V_G : f(v) = i\}.$$

Clearly, each $U_i$ forms an independent set of $G$. Now define the bipartite subgraphs $G_1$ and $G_2$ of $G$ as

$$\begin{aligned}
V_{G_1} &= V_1 \cup V_2,\ V_1 = U_1 \cup U_2 \ \text{ and } \ V_2 = U_3 \cup U_4, \\
E_{G_1} &= \{xy \in E_G : x \in V_1 \text{ and } y \in V_2\}, \\
V_{G_2} &= W_1 \cup W_2,\ W_1 = U_1 \cup U_3 \ \text{ and } \ W_2 = U_2 \cup U_4, \\
E_{G_2} &= \{xy \in E_G : x \in U_1 \text{ and } y \in U_2\} \cup \{xy \in E_G : x \in U_3 \text{ and } y \in U_4\}.
\end{aligned}$$

It is not hard to see that $G_1$ and $G_2$ are bipartite subgraphs of $G$ which decompose $G$ (check!).

5. Give an explicit edge-coloring to prove that $\chi'(K_{r,s}) = \delta_{\max}(K_{r,s})$.

**Solution:** Without loss of generality, assume that $r \leq s$. Let $U_1 = \{v_1, \dots, v_s\}$ and $U_2 = \{u_1, \dots u_r\}$ be the bipartite partition of $K_{r,s}$. Note that $\deg(u_i) = s$ and $\deg(v_j) = r$. Consider the coloring

$$f : E_G \to \{1, \dots, s\}, \quad f(v_i u_j) = i + j \pmod{s}.$$

Note that for each $u_i$, all edges adjacent to $u_i$ are colored differently, because $i + j_1 \neq i + j_2 \pmod{s}$ if $1 \leq j_1 \neq j_2 \leq s$. Similarly, all edges adjacent to $v_j$ are colored differently for each $j$. Hence $f$ is a proper edge coloring of $G$.

6. Prove that for any tree $T$ we have $\chi'(T) = \delta_{\max}(T)$.

   **Solution:** We prove by induction on the number of vertices of $T$.
   **Basis of induction:** This is true for any tree on 2 or 3 vertices.
   **Induction hypothesis:** Suppose that for every tree $T$ on $n \geq 4$ vertices, $\chi'(T) = \delta_{\max}(T)$.
   **Induction step:** Let $T$ be a tree on $n+1$ vertices. Let $v$ be a leaf of $T$. Then $T - v$ is a tree on $n$ vertices. Thus by induction hypothesis, $\chi'(T - v) = \delta_{\max}(T - v)$. Since $\delta_{\max}(T - v) \leq \delta_{\max}(T)$, there exists a proper edge-coloring of $T - v$ with $\delta_{\max}(T)$ colors. The tree $T$ has one vertex $v$ and one edge $e$ more than $T - v$. Clearly $e$ has at most $\delta_{\max}(T) - 1$ neighbors. Thus we can extend the above coloring to an edge-coloring of $T$ with $\delta_{\max}(T)$ colors.