Math 3330 - Practice problems - Fall 2011.

The practice session is moved to 3:30 pm on Thursday.

1. Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0,1\}$, with x adjacent to y if x and y differ in exactly two positions. Determine the number of components of G.

2. Let G be a connected simple graph not having P_4 (path on 4 vertices) or C_3 (cycle on three vertices) as an induced subgraph. Prove that G is a complete bipartite graph.

3. Let $\overline{G}$ denote the complement of G. Prove that $\chi(G) + \chi(\overline{G}) \leq |V(G)| + 1$.

4. Prove that every $k-1$-regular k-critical graph is a complete graph or an odd cycle.

5. Let S be a set of size n. Define a graph G with vertex set $S \times S$ such that a vertex $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ if and only if $x_1 \neq x_2$ and $y_1 \neq y_2$. Determine $\chi(G)$ and $\chi(\overline{G})$.

6. Let G be a graph on n vertices $v_1, \dots, v_n$, such that v_i is adjacent to v_j iff $|i-j| \leq 3$. Is G planar or not?

Solution set

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1. We need to show that G has two connected components G_1 and G_2:

$$\begin{aligned} G_1 &= \{v : \text{ the number of zeros in } v \text{ is even}\}, \\ G_2 &= \{v : \text{ the number of zeros in } v \text{ is odd}\}. \end{aligned}$$

   Complete the proof!

2. First we will show that G has no odd cycles. Suppose not, i.e. suppose that G has an odd cycle. Let $C_k = \langle v_1, v_2, \ldots, v_k \rangle$ be the smallest odd cycle in G. Clearly $k > 4$, since G has no C_3. Now, we claim that C_k is an induced subgraph of G. Because if not, there exists $1 \leq i < j \leq k$ with $(i, j) \neq (1, n)$ and $|i - j| > 1$, such that v_i is adjacent to v_j in G. Then $\langle v_i, \ldots, v_j \rangle$ and $\langle v_{j+1}, \ldots, v_k, v_1, \ldots, v_i \rangle$ are both cycles in G, and one of them has to be odd. But this is a contradiction, since C_k is the smallest cycle of odd length. This implies that $\langle v_1, v_2, v_3, v_4 \rangle$ is an induced subgraph of G, which is a contradiction. Thus, G has no odd cycle, and it is a bipartite graph.

   Let U_1 and U_2 be a bipartite partition for G. Let $v \in U_1$ and $w \in U_2$. We claim that v is adjacent to w. Suppose not, i.e. suppose that v and w are not adjacent. Consider the shortest path $v, v_1, v_2, \ldots, v_m, w$ from v to w. This path has length at least 4. Moreover, it is induced, otherwise there exists a shorter path from v to w. Thus G has P_4 as an induced subgraph, which is a contradiction. Hence, G forms a complete bipartite graph.

3. We prove by induction.
   Induction hypothesis: First note that the condition holds for any graph on 3 vertices (check!).
   Induction hypothesis: Assume that for $k > 3$, and any graph G on k vertices, we have $\chi(G) + \chi(\overline{G}) \leq k + 1$.
   Induction step: Let G be a graph on $k+1$ vertices. Let v be a vertex in G. By induction hypothesis, $\chi(G - v) + \chi(\overline{G - v}) \leq k + 1$. Note that $\overline{G - v} = \overline{G} - v$. Now consider each of the following cases:
   Case 1: If $\chi(G) = \chi(G-v)$, then $\chi(G)+\chi(\overline{G}) \leq \chi(G-v)+\chi(\overline{G}-v)+1 \leq k+2$, and we are done.
   Case 2: If $\chi(\overline{G}) = \chi(\overline{G}-v)$, then $\chi(G)+\chi(\overline{G}) \leq \chi(G-v)+1+\chi(\overline{G}-v) \leq k+2$, and we are done.
   Case 3: Assume that $\chi(G) = \chi(G - v) + 1$ and $\chi(\overline{G}) = \chi(\overline{G} - v) + 1$. This

implies that $\deg_G(v) \geq \chi(G-v)$, otherwise we could have extended the $\chi(G-v)$-coloring of $G - v$ to G properly. Similarly, $\deg_{\overline{G}}(v) \geq \chi(\overline{G} - v)$. Thus

$$\chi(G - v) + \chi(\overline{G} - v) \leq \deg_G(v) + \deg_{\overline{G}}(v) = k.$$

But this implies that $\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + 1 + \chi(\overline{G} - v) + 1 \leq k + 2$. Note that in the above argument, we have used the result that deletion of a vertex of G decreases the chromatic number of G by at most 1.

4. Brook's Theorem! You need to make an argument regarding connectedness of G.

5. First note that the largest independent set in G has size n (why?). Thus $\chi(G) \geq \frac{|V(G)|}{\alpha(G)} = \frac{n^2}{n}$. Let $S = \{s_1, \ldots, s_n\}$. It is easy to see that $f : V_G \to \{1, \ldots, n\}$, $f((s_i, s_j)) = i$ is a proper coloring of G (why?). Thus $\chi(G) = n$. Also note that $u = (s_i, s_j)$ is adjacent to $v = (s_i', s_j')$ in $\overline{G}$ if and only if $s_i = s_i'$ or $s_j = s_j'$. Thus $\omega(\overline{G}) \geq n$, as the subgraph of $\overline{G}$ induced on $\{(s_1, s_i) : 1 \leq i \leq n\}$ forms a clique. So its chromatic number is at least n. Moreover, the coloring $g : V_G \to \{0, \ldots, n-1\}$, $g((s_i, s_j)) = i + j (\mod n)$ forms a proper coloring of $\overline{G}$ (why?), hence $\chi(\overline{G}) = n$.

6. Planar!