Nova Scotia Math League

2011–2012

Game One

SOLUTIONS

Team Question Solutions

1. Clearly there's nothing special about the number 2011 in this context. The question is really asking about the relative sizes of n^{n+1} and $(n+1)^n$.

   Before digging in to any deep analysis it's helpful to have some empirical evidence at hand. This will let us make a (very educated) guess. For $n = 1, 2, 3, 4$ the computations involve only simple mental arithmetic:

n	n^{n+1}	$(n+1)^n$
1	1	2
2	8	9
3	81	64
4	1024	625

   For $n = 5$ the numbers get uncomfortably large, but we can easily make an estimate: We have $5^6 = 5^2 \cdot 5^4 = 25 \cdot 625$, whereas $6^5 = 6^2 \cdot 6^3 = 36 \cdot 216 = 12 \cdot 648$. So

   $$\frac{5^6}{6^5} = \frac{25 \cdot 625}{12 \cdot 648} \approx 2.$$

   That is, 5^6 is roughly twice as large as 6^5.

   The evidence above strongly suggests that n^{n+1} is larger than $(n+1)^n$, for $n > 2$, with the gap between these numbers getting ever larger as n increases. Therefore we conjecture that 2010^{2011} is bigger than 2011^{2010}.

   Note: Our conjecture that $n^(n+1) > (n+1)^n$ for $n > 2$ can be proved without much difficulty. By the binomial theorem, we have

   $$\begin{aligned}(n+1)^n &= n^n + \binom{n}{1}n^{n-1} + \binom{n}{2}n^{n-2} + \binom{n}{3}n^{n-3} + \cdots + \binom{n}{n-1}n + 1 \\ &= 2n^n + \sum_{k=2}^{n}\binom{n}{k}n^{n-k}.\end{aligned}$$

   But

   $$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} < \frac{n^k}{2}$$

   for all $k \geq 2$, so each of the $n-1$ terms in the summation above is less than than $n^n/2$. So we have

   $$(n+1)^n < 2n^n + (n-1)\frac{n^n}{2} = \frac{n^n(n+3)}{2}. \qquad (*)$$

   For $n \geq 3$ we have $(n+3)/2 \geq n$, and therefore the above inequality gives

   $$(n+1)^n < n^{n+1},$$

as desired.

In fact, our estimate $(*)$ yields something much stronger: Since $(n+3)/2 \approx n/2$ for large n, we find that $(n+1)^n$ is at most roughly *one half* of n^{n+1}. But we can be much more precise! The ratio between the ratio between n^{n+1} and $(n+1)^n$ is

$$\frac{n^{n+1}}{(n+1)^n} = \frac{n}{(1+\frac{1}{n})^n}.$$

Now recall that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, where $e \approx 2.718$ is Euler's constant. Thus n^{n+1} is approximately n/e times as large as $(n+1)^n$, for large n. The fact that e lies between 2 and 3 explains why $2^3 < 3^2$ but $3^4 > 4^3$.

2. The desired area is equal to the total area of the four semicircles, plus the area of the rectangle, less the area of the circumscribing circle.

   The two semicircles with diameter 3 together form a circle with area $\pi(\frac{3}{2})^2 = 9\pi/4$, while the semicircles with diameter 4 have total area $\pi(2)^2 = 4$. The area of the rectangle is $3 \cdot 4 = 12$.

   By Pythagorean theorem, the circumscribing circle has diameter $\sqrt{3^2+4^2} = 5$, so its area is $\pi(\frac{5}{2})^2 = 25\pi/4$.

   Thus the desired area is
   $$\frac{9\pi}{4} + 4\pi + 12 - \frac{25\pi}{4} = 12.$$

   Note: The fact that the shaded area equals the area of the original rectangle is no coincidence: it will always be so, independent of the dimensions of the rectangle. (Proof: Run the calculation above with 3 and 4 replaced by a and b.) This simple but pleasing result is a variation on the *lune of Hippocrates*.

3. For real numbers a and b, we can have $a^b = 1$ only if $a = \pm 1$ or $b = 0$. Therefore $x^{x^2+2011x+2012} = 1$ requires either $x = \pm 1$ or $x^2 + 2011x + 2012 = 0$. A quick check shows that both $x = \pm 1$ satisfy the equation. Moreover, the quadratic $x^2 + 2011x + 2012$ has a positive discriminant (namely $2011^2 - 4 \cdot 2011 > 0$), so it has two distinct real roots. Either of these values of x must be solutions to the original equation, since $x^0 = 1$ for every $x \in \mathbb{R}$. Therefore there are 4 solutions in total.

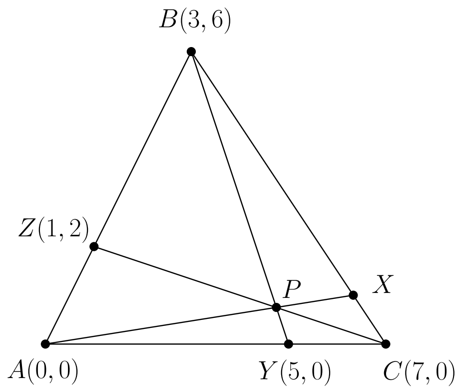
   Note: Clearly $a^0 = 1$ and $1^b = 1$ for all $a, b \in \mathbb{R}$. However, while $(-1)^b = 1$ is *possible*, we know this isn't true for all b. For instance, it holds for $b = 2$ and $b = \frac{2}{3}$, but fails for $b = 3$ and $b = \frac{1}{3}$. This begs a question: For exactly what values of b do we have $(-1)^b = 1$?

   The usual high school laws of exponents show that $(-1)^b = 1$ whenever b is a fraction of the form p/q, where p is even and q is odd. But what about $(-1)^{\sqrt{2}}$? Is this equal to

1? Or -1? Or something else? More generally, what does it mean to have an *irrational* exponent? We're happy to write "3^π", and our calculator will assign a value to this expression, but what does it mean? Food for thought!

(As a tease: It turns out that $(-1)^{\sqrt{2}}$ has infinitely many equally valid meanings! However, without further instruction, a mathematician would assume that this notation stands for the single complex number $\cos(\pi\sqrt{2}) + i\sin(\pi\sqrt{2})$, where $i = \sqrt{-1}$.)

4. Label the various points as in the diagram below.



Line ZC has equation $y = -\frac{1}{3}(x-7)$ while BY has equation $y = -3(x-5)$. These lines meet at P, so we solve

$$\{y = -\tfrac{1}{3}(x-y), \quad y = -3(x-5)\}$$

to find that $P = (\frac{19}{4}, \frac{3}{4})$. Thus AP has equation $y = \frac{3}{19}x$. But X lies on AP and also on line BC, which has equation $y = -\frac{3}{2}(x-7)$. So we solve

$$\{y = \tfrac{3}{19}x, \quad y = -\tfrac{3}{2}(x-7)\}$$

to find the coordinates of X, namely $(\frac{19}{3}, 1)$.

Alternative solution: Label the diagram as above. Since AX, BY, and CZ are concurrent at P, Ceva's Theorem gives $\frac{|BZ|}{|ZA|}\frac{|AY|}{|YC|}\frac{|CX|}{|XB|} = 1$. But $\frac{|BZ|}{|ZA|} = 2$ and $\frac{|AY|}{|YC|} = \frac{5}{2}$, so $\frac{|CX|}{|XB|} = \frac{1}{5}$. Thus $X = \frac{1}{6}(3,6) + \frac{5}{6}(7,0) = (\frac{19}{3}, 1)$.

5. Since α, β, γ satisfy $2x^3 + 2x^2 - 3x - 1 = 0$, their reciprocals $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ must satisfy

$$2\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) - 1 = 0.$$

Multiply by x^3 to find that these same values of x are roots of the cubic polynomial

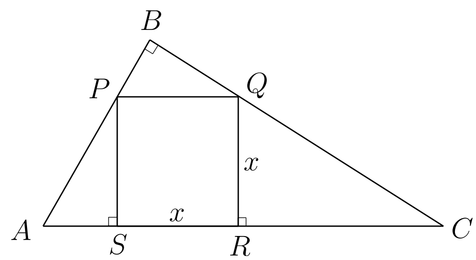
$$2 + 2x - 3x^2 - x^3 = 0.$$

(Any cubic polynomial with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$ must be a constant multiple of this one.)

Alternative solution: Factor the given polynomial as $(x-1)(2x^2+4x+1)$. Clearly $x = 1$ is a root, and the quadratic formula also provides roots $x = \frac{1}{2}(2 \pm \sqrt{2})$. The reciprocals of these roots are 1 and $2 \pm \sqrt{2}$. So the desired cubic is

$$(x-1)(x-2+\sqrt{2})(x-2-\sqrt{2}) = (x-1)(x^2+4x+2) = x^3+3x^2-2x-2.$$

6. Label the corners of the square as indicated below and let $x = |QR| = |PS|$ be the side length of the square.



Then $\triangle QRC$ and $\triangle ASP$ are similar to $\triangle ABC$, so we have

$$\frac{|QR|}{|RC|} = \frac{|AS|}{|SP|} = \frac{|AB|}{|BC|}.$$

That is,

$$\frac{x}{|RC|} = \frac{|AS|}{x} = \frac{1}{2}.$$

So we have $|RC| = 2x$ and $|AS| = x/2$. But

$$|AS| + |SR| + |RC| = |AC| = \sqrt{2^2+1^2} = \sqrt{5}$$

by Pythagorean theorem applied to $\triangle ABC$. Thus

$$\frac{x}{2} + x + 2x = \sqrt{5}.$$

This gives $x = 2\sqrt{5}/7$, so that the area of the square is $x^2 = 20/49$.

7. Let the common sum of the rows, columns, and diagonals be S and let the entry in the central square be x. Note that the sum of all entries in the the square must be $3S$, since we can think of this as adding together the 3 rows, each of whose entries sum to S. Thus

$$3S = 5+17+29+47+59+71+89+101+113 = 531 \implies S = 177.$$

Similarly, adding together the two diagonals and the central row gives $3S$. But we must obtain the same result by adding the first and third row together along with 3 copies of the central square. Thus

$$3S = 2S + 3x$$

It follows that $3x = S = 177$, so $x = 59$.

Note: This method shows that the central square in *any* 3×3 magic square must be $\frac{1}{9}$ of the sum all entries. That is, the central entry is the average of all 9 entries.

The particular magic square considered in this question was discovered by Rudolf Ondrejka. What makes it particularly interesting is that all of its entries are prime numbers! It can be completed as follows:

71	5	101
89	59	29
17	113	47

8. Each possible distribution of gifts corresponds with a sequence of four J's and four M's in some order. For instance, $JJMJJMMM$ corresponds with Santa giving two presents to Jenny, then giving one to Mike, then two to Jenny, then three to Mike. The total number of ways the presents could be distributed is the number of such sequences, which is $\binom{8}{4} = 70$. (Of 8 positions in the sequence, choose 4 in which to put the J's.)

   We need to know how many of these sequences have the property that, as we read from left to right, the number of M's we encounter up to any point is no greater than the number of J's we have encountered to that point. This is a fairly restrictive condition, so we should be able to count these "by hand" with a little bit of organized thinking.

   Clearly any such sequence must start with a J and end with an M. There are 5 sequences that begin with JM (i.e. are of the form $JM \cdots M$), namely:

   $JMJJJMMM$ $\quad$ $JMJJMJMM$ $\quad$ $JMJJMMJM$ $\quad$ $JMJMJJMM$ $\quad$ $JMJMJMJM$

   There are also 5 sequences that begin with JJM:

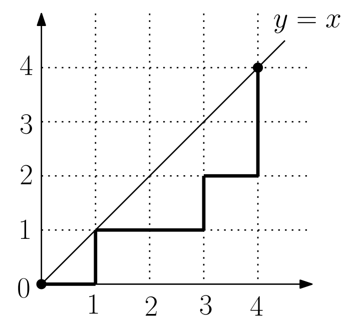
   $JJMJJMMM$ $\quad$ $JJMJMJMM$ $\quad$ $JJMJMMJM$ $\quad$ $JJMMJJMM$ $\quad$ $JJMMJMJM$

   There are only 3 sequences beginning with $JJJM$:

   $JJJMJMMM$ $\quad$ $JJJMMJMM$ $\quad$ $JJJMMMJM$

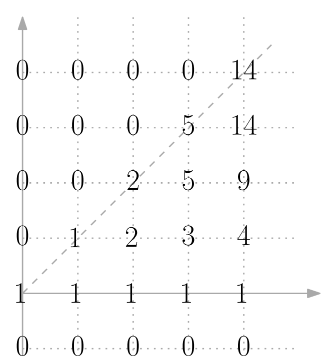
   And finally there is 1 sequence beginning with $JJJJM$, namely $JJJJMMMM$. So, altogether there are 14 valid sequences out of 70 possible sequences, resulting in a probability of $\frac{14}{70} = \frac{1}{5}$.

Alternative solution: A more elegant approach is to notice that every distribution of gifts corresponds with a path in the integer grid, as follows: Starting at $(0,0)$, we move right one unit whenever Jenny is given a gift, and up one unit whenever Mike is given a gift. After all the gifts are given, we find ourselves at $(4,4)$. Moreover, Jenny stayed ahead of Mike the whole time if and only our path never ventured above the line $y = x$. For instance, the distribution $JMJJMJMM$ corresponds with the path shown below:



Let $N(a,b)$ be the number of paths from $(0,0)$ to (a,b) that do not go above the line $y = x$. Then the problem comes down to determining $N(4,4)$.

This can be done quickly by recording the values $N(a,b)$ on the lattice itself in the following recursive manner: First, it is sensible to let $N(0,0) = 1$, so we write a "1" at the origin. Clearly $N(a,b) = 0$ if the point (a,b) is above $y = x$ or below the x-axis, so we write "0" at all such lattice points. Now we repeatedly use the fact that $N(a,b) = N(a-1,b) + N(a,b-1)$, which is clearly true since a path can only get to (a,b) by stepping right from $(a-1,b)$ or up from $(a,b-1)$. Thus, on the grid, we obtain the value at point by adding the numbers one unit left and one unit down (provided both of these are available). Doing so yields the following array:



So we see that $N(4,4) = 14$, as before. Notice that this method allows us to (almost instantly) compute $N(5,5) = 42$ by filling in a few more numbers on our grid.

Note: If, instead, Santa had n presents each for Jenny and Mike, then the probability of Jenny staying ahead of Mike turns out to be $\frac{1}{n+1}$. To see this requires a some tools for handling the recursive nature of this problem. But the necessary techniques are completely elementary and worthy of investigation! Your starting point should be to look into *Catalan numbers*. Briefly, the n-th Catalan number is given by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. This is the number of paths from $(0,0)$ to (n,n) that do not go above $y = x$. (Using the notation above, we have, $C_n = N(n,n)$.) The Catalan numbers are remarkably ubiquitous in mathematics, in that they count a wide variety of interesting objects. For instance, there are C_n ways of dissecting a regular n-gon into triangles by drawing diagonals.

9. Clearly any pass-code can either involve either 0, 1, or 2 instances of buttons being pressed two at a time. There are clearly 5! possible pass-codes with 0 simultaneous pressings. There are $\binom{5}{2} \cdot 4!$ codes that involve one pair of simultaneous buttons: Choose the pair of buttons in $\binom{5}{2}$ ways, and now treat this as a single button to leave 4 buttons that can be arranged in 4! ways. Finally, there are $\binom{5}{4} \cdot 3 \cdot 3!$ codes that involve two pairs of simultaneous buttons, seen as follows: First choose 4 buttons of 5 to constitute the pairs. Then pair these up in any of 3 ways (e.g. $\{A, B, C, D\}$ can be paired as $\{AB, CD\}$, $\{AC, BD\}$ or $\{AD, BC\}$). Now treat the pairs as single buttons, so we are left with 3 buttons that can be arranged in 3! ways.

   Altogether, there are
$$5! + \binom{5}{2} \cdot 4! + \binom{5}{4} \cdot 3 \cdot 3! = 450$$
   possible pass-codes.

10. The given rules for creating the (two) children of a given fraction are easily reverted to yield a rule that produces the (unique) parent of a given fraction. In particular, the parent of a/b is $a/(b-a)$ if $a < b$, and $(a-b)/b$ if $b < a$. (We can only have $a = b$ except when $a = b = 1$.)

    Thus to determine the level of $\frac{1001}{2011}$, we simply count how many times we must apply this rule before we obtain $\frac{1}{1}$, the root of the tree. It's best to think of the parent rule as "subtract small from big". That is, if the numerator is bigger than the denominator, then subtract the denominator from the numerator; otherwise, do the reverse.

    Applying the rule successively yields the following chain of fractions:
$$\frac{1001}{2011} \to \frac{1001}{1010} \to \overbrace{\frac{1001}{9} \to \frac{992}{9} \to \frac{983}{9} \to \cdots \to \frac{2}{9}}^{\lfloor 1001/9 \rfloor = 111 \text{ iterations}} \to \frac{2}{7} \to \frac{2}{5} \to \frac{2}{3} \to \frac{2}{1} \to \frac{1}{1}$$
    We applied the "parent rule" a total of $2 + 111 + 5 = 118$ times to get from $\frac{1001}{2011}$ to $\frac{1}{1}$. It follows that $\frac{1001}{2011}$ is at level 119 in the tree.

Note: This "tree of fractions" is known as the *Calkin-Wilf tree*. It has an absolutely fascinating property: Every positive fraction (in lowest terms) appears once, and only once, in the tree! This is stunning, because it allows us to easily *list* all of the fractions, as one might do in a phone directory. (For instance, start at the bottom of the tree and work upwards, reading left to right.) Being able to list the fractions is somewhat counter-intuitive, since on the number line there are an infinite number of fractions "between" any two given fractions!

The Calkin-Wilf tree is a slight modification of the *Stern-Brocot tree*, which is itself (very) closely related to the *Farey sequence*. These objects have many other fascinating properties aside from the one mentioned above.

Pairs Relay Answer Key

A. 4

B. -4

C. 10

D. 5

Individual Relay Answer Key

A. 5

B. 8

C. 5

D. 12