Nova Scotia Math League

2013–2014

Game Three

SOLUTIONS

Team Question Solutions

1. Let N and M be the integers whose decimal expansions consist of 100 and 98 ones, respectively, so that $N = 100M + 11$. We wish to know the remainder when N is divided by 4. But 100 is divisible by 4, so the number we seek is simply the remainder when 11 is divided by 4, namely 3.

2. Since $\triangle CDE$ is equilateral and $ABCD$ is a square we have $\angle DCE = 60°$ and $\angle DCB = 90°$. Therefore $\angle ECB = 30°$. Furthermore, $|CE| = |CB|$, so $\triangle CEB$ is isosceles with $\angle CEB = \angle CBE$. Thus $\angle CBE = \frac{1}{2}(180 - \angle ECB) = 75°$ and $\angle ABE = 90 - 75 = 15°$. By symmetry we also have $\angle EAB = 15°$, so that $\angle AEB = 180 - 2 \cdot 15 = 150°$.

3. Notice that

$$\begin{aligned} x^2 - 2014x = y^2 - 2014y \quad &\iff \quad x^2 - y^2 = 2014x - 2014y \\ &\iff \quad (x-y)(x+y) = 2014(x-y). \end{aligned}$$

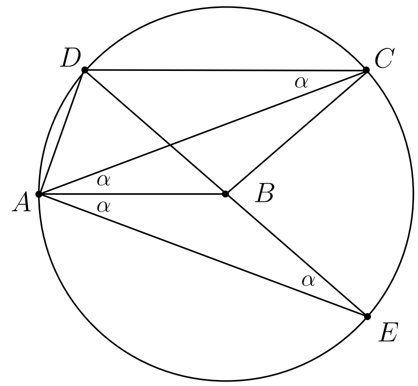
   But since $x \neq y$, we can divide the last equation by $x - y \neq 0$ to get $x + y = 2014$.

   Alternative Solution: Since the value of $x + y$ is apparently dependent only on the identity $x^2 - 2014x = y^2 - 2014y$, we can set $y = 0$ and solve for x to get $x = 0$ or $x = 2014$. Since $x \neq y$ we choose $x = 2014$, and thus $x + y = 2014 + 0 = 2014$.

4. The minute and hour hands are aligned at noon. The hour hand advances at a rate of $\frac{360}{12} = 30°$ per hour, while the minute hand advances at a rate of $360°$ per hour. Since 12 minutes is $\frac{1}{5}$ of an hour, the angle between the two hands at 12:12pm is $\frac{360}{5} - \frac{30}{5} = 66°$.

5. The word MISSISSIPPI consists of one M, two P's, four I's, and four S's. Any palindromic rearrangement must therefore be of the form wMw^{-1}, where w is a 5-letter word consisting of one P, two I's and two S's, and where w^{-1} is the reverse of w. (For example, ISISPMPSISI has w =ISISP and w^{-1} =PSISI.) There are $\frac{5!}{2!\,2!} = 30$ such words w, so this is the number of palindromic rearrangements.

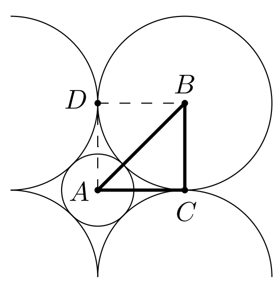
6. Since $|BA| = |BC| = |BD|$, the circle centred at B with radius $|BA|$ passes through A, C, and D. Extend DB to meet this circle at E, as shown in the diagram below.



For convenience, let $\angle CAB = \alpha$. Since $DC||AB$, we have $\angle ACD = \alpha$. But $\angle ACD$ and $\angle AED$ are both subtended by arc AD, so we also have $\angle AED = \alpha$. Moreover, since $|AB|$ and $|BE|$ are radii, $\triangle ABE$ is isosceles and thus $\angle BAE = \angle DEA = \alpha$. Therefore $\angle DAE = \angle DAC + 2\alpha$. But DE is a diameter, so $\angle DAE = 90°$. With $\alpha = 20°$, this yields $\angle DAC = 50°$.

Note: There are many ways of solving this problem, all of which amount to "chasing angles" through a diagram. It is certainly not necessary to use properties of the circle, although this is often a very good approach whenever a problem involves a few points equidistant from one point.

7. Let the radius of the large circles be r, and let the radius of the small circle be 1. Let A and B be the centres of the small circle and the upper-right large circle, respectively. Let C and D be the points of tangency between the large circles as indicated in the figure below.



Note that $\angle ACB = \angle ADB = 90°$, since these angles are formed by radii meeting tangents. Thus $ACDB$ is a square, with sides of length r. In particular, $\triangle ABC$ is a 45-45-90 triangle, with legs of length r and hypotenuse $1 + r$. It follows that $1 + r = r\sqrt{2}$, so that $r = 1/(\sqrt{2} - 1) = \sqrt{2} + 1$. The desired ratio is therefore $r : 1 = (1 + \sqrt{2}) : 1$.

8. Notice that the length of the ant's path is one less than the number of grid points that it touches along the path. But when the ant reaches $(10, 0)$, it will have hit every point in the square $\{(x, y) : -9 \le x, y \le 9\}$, as well as the points $(10, y)$ for $-9 \le y \le 0$. This is a total of $(2 \cdot 9 + 1)^2 + 10 = 371$ points, so the distance travelled is 370.

Alternative Solution: One can also easily add up the distances around each successive square "loop", to get a distance of $4(2+4+6+\cdots+18)=360$ to the point $(-9,-9)$. Another 10 steps are then needed to reach $(10,0)$.

9. For any integer n, exactly one number from $\{1,2,3,4,5,6\}$ can be added to n to obtain a multiple of 6. So regardless of the sum of the first two dice, there will be exactly *one* outcome for the third die that will make the sum of the three dice divisible by 6. The desired probability is therefore $\frac{1}{6}$.

10. Regard the grid as the set of integer points $\mathcal{S}=\{(x,y) : 0\le x,y\le 4\}$. The midpoint of the line joining $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is

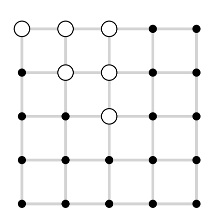
$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right),$$

which is a lattice point precisely when both x_1+x_2 and y_1+y_2 are divisible by 2. But x_1+x_2 is divisible by 2 if and only if x_1 and x_2 are both even or both odd, and the same same statement holds for y_1 and y_2. So we are looking for the number of pairs $\{P,Q\}$ where *both* P and Q are of the form (even, even), or (odd, odd), or (even, odd), or (odd, even).

There are $3^2=9$ points in $\mathcal{S}$ of the form (even, even). These are the points (x,y) with $x,y\in\{0,2,4\}$. Similarly there are $2^2=4$ points $(x,y)\in\mathcal{S}$ of the form (odd, odd), namely those with $x,y\in\{1,3\}$. Finally there are $2\cdot 3=6$ points of the form (even, odd), and the same number of the form (odd, even).

Thus there are total of $\binom{9}{2}+\binom{4}{2}+2\binom{6}{2}=72$ possible pairs $\{P,Q\}$.

Alternative Solution: The configuration is small enough that all possibilities can be counted by hand. As usual, the key is to do so in an organized manner. One efficient method is to count, for each lattice point M, the number of segments PQ whose midpoint is M. The symmetry of the grid allows one to perform this computation for only the 6 points M highlighted in the figure below.



Pairs Relay Solutions

P-A. A number is divisible by 15 if and only if it is divisible by 3 and also by 5. Every rearrangement of 5667 is divisible by 3, since the sum of the digits $5+6+6+7=24$ is divisible by 3. The rearrangements that are divisible by 5 are precisely those that end in a 5, namely 6675, 6765, and 7665. So there are $\mathtt{A}=3$ rearrangements divisible by 15.

P-B. The area of a circle is quadrupled when its radius (and hence diameter) is doubled. Hence we have $2\mathtt{B}=\mathtt{B}+\mathtt{A}$, or simply $\mathtt{B}=\mathtt{A}$. With $\mathtt{A}=3$ we have $\mathtt{B}=3$

P-C. If $\mathtt{B}$ numbers have an average of 6, then they sum to $6\mathtt{B}$. If 6 numbers have an average of $\mathtt{B}$, then they too sum to $6\mathtt{B}$. So the combined list contains $\mathtt{B}+6$ numbers that sum to $12\mathtt{B}$. The average of this list is $\mathtt{C}=12\mathtt{B}/(\mathtt{B}+6)$. With $\mathtt{B}=3$ we get $\mathtt{C}=4$

P-D. Let $|PQ|=1$. Then $|PQ|:|QR|=6:\mathtt{C}$ yields $|QR|=\mathtt{C}/6$. Thus

$$|PR|=|PQ|+|QR|=1+\frac{\mathtt{C}}{6}=\frac{\mathtt{C}+6}{6}.$$

We also have

$$|RS|=\frac{|QR|}{|PQ|}\cdot\frac{|RS|}{|QR|}=\frac{\mathtt{C}}{6}\cdot\frac{\mathtt{C}}{6}=\frac{\mathtt{C}^2}{36}.$$

Therefore

$$\mathtt{D}=\frac{|PR|}{|RS|}=\frac{6(\mathtt{C}+6)}{\mathtt{C}^2}.$$

With $\mathtt{C}=4$ we obtain $\mathtt{D}=\frac{15}{4}$.

Individual Relay Solutions

I-A. The entire square has area $4^2 = 16$, and the three non-shaded triangles have areas $\frac{1}{2}(4)(3) = 6$, $\frac{1}{2}(4)(2) = 4$, and $\frac{1}{2}(2)(1) = 1$. Therefore $\mathtt{A} = 16 - 6 - 4 - 1 = 5$.

I-B. The bottle of syrup has a volume of $\frac{1}{1+3} \cdot 2 = \frac{1}{2}$ litres. Mixing in the ratio 1 part syrup to $\mathtt{A}$ parts milk therefore produces $\mathtt{B} = \frac{1}{2}(1 + \mathtt{A})$ litres of chocolate milk. Letting $\mathtt{A} = 5$ gives $\mathtt{B} = 3$.

I-C. Notice that

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{x + 1 + (x - 1)}{x + 1 - (x - 1)} = \frac{2x}{2} = x.$$

Therefore $f(f(\mathtt{B})) = \mathtt{B}$, so we have

$$\mathtt{C} = f(f(f(\overbrace{f(f(\mathtt{B}))}^{=\mathtt{B}}))) = f(\overbrace{f(f(\mathtt{B}))}^{=\mathtt{B}}) = f(\mathtt{B}) = \frac{\mathtt{B} + 1}{\mathtt{B} - 1}.$$

Setting $\mathtt{B} = 3$ gives $\mathtt{C} = 2$.

I-D. Since the answer depends only on the fact that $x : y = 3 : \mathtt{C}$, we can simply set $x = 3$ and $y = \mathtt{C}$ to get $\mathtt{D} = \frac{3-\mathtt{C}}{3+\mathtt{C}}$. Setting $\mathtt{C} = 2$ gives $\mathtt{D} = \frac{1}{5}$.