Nova Scotia Math League

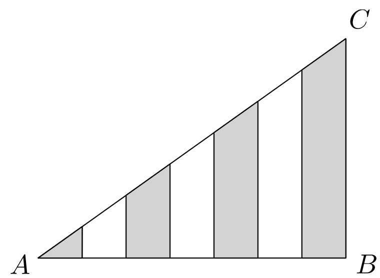
2014–2015

Game One

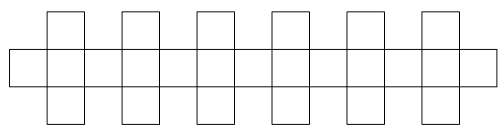
CONTEST AND SOLUTIONS

Team Questions

1. Right triangle ABC is divided into seven trapezoids of equal thickness, as shown below. Given that $|AB| = 4$ and $|BC| = 3$, determine the area of the shaded region.

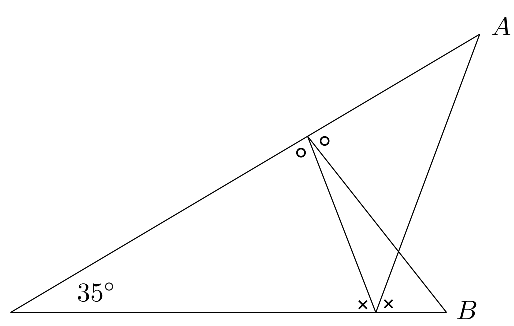


2. The shape below was created by pasting together 25 unit squares. When a similar shape is created with n squares, its perimeter is 100 units.
Determine n.



3. A broken photocopier can only make copies that are either 75%, 100%, or 160% the size of the original. However, various other output sizes can be achieved by making copies of copies. What is the minimum number of copies required to produce one that is 108% the original size?

4. In the figure below, the two angles marked with circles are equal, as are those marked with crosses. Determine the **sum** of the (acute) angles at A and B.



5. A old table of factorials contains the line

$$20! = 243290X008176640000,$$

where X represents a digit that cannot be read because the ink has smudged. Determine X.

Note: Recall that the *factorial* of a positive integer n is the product $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. For instance, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$.

6. Find all real numbers x such that

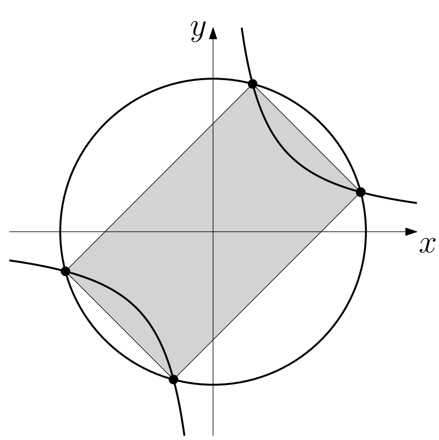
$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}}$$

7. Bob and Suzy each toss a fair coin 3 times in a row. What is the probability that Bob tosses fewer heads than Suzy?

8. Suppose $a = 1 + a^2$. Determine a^{60}.

9. How many different 4-digit numbers can be formed using only the digits 1, 2, and 3, if consecutive digits must differ by at most 1.

10. The circle $x^2 + y^2 = 36$ and the hyperbola $xy = 9$ intersect at exactly four points, as shown below. Find the area of the shaded region.

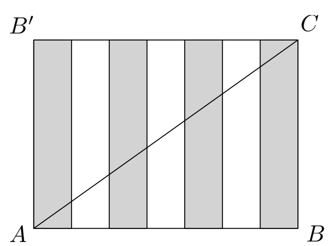


Team Question Answer Key

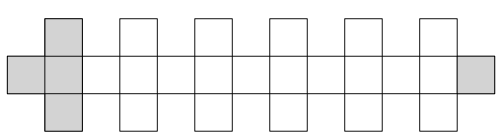
1. $\dfrac{24}{7}$
2. 49
3. 5
4. 75°
5. 2
6. $\dfrac{1 \pm \sqrt{5}}{2}$
7. $\dfrac{11}{32}$
8. 1
9. 41
10. $36\sqrt{3}$

Team Question Solutions

1. Complete $\triangle ABC$ to a rectangle $ABCB'$ and observe that the trapezoidal shaded regions naturally extend to rectangles. The shaded area within $\square ABCB'$ is clearly $\frac{4}{7}$ of its total area, or $\frac{4}{7} \cdot 3 \cdot 4 = \frac{48}{7}$. The shaded area within $\triangle ABC$ is half this much, or $\frac{24}{7}$.



2. Each such shape consists of several copies of a basic T-shape plus one additional square, both highlighted in the diagram below (which contains a total of 6 T-shapes).



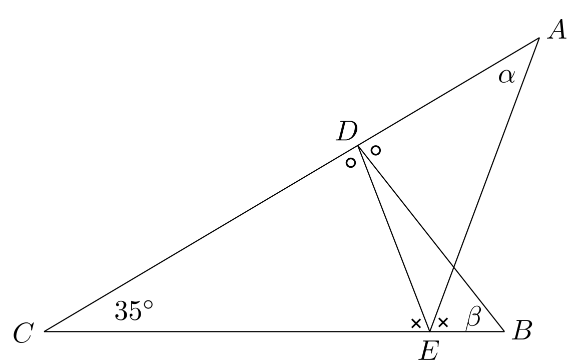
Each T-shape is composed of 4 squares and contributes 8 to the perimeter of the figure, except the first which contributes 9. The additional square further contributes 3 to the perimeter. So the figure composed of k T-shapes contains $4k+1$ squares and has perimeter $8(k-1)+9+3 = 8k+4$. Setting $8k+4 = 100$ yields $k = 12$, so $n = 4 \cdot 12 + 1 = 49$.

Alternative Solution: Viewed as a lattice polygon, the desired figure contains no internal lattice points and 100 boundary lattice points. Pick's Theorem immediately shows its area to be $0 + \frac{100}{2} - 1 = 49$.

3. Making a 100% copy is redundant, so we exclude these from consideration. Since $75\% = \frac{3}{4}$, $160\% = \frac{8}{5}$, and $108\% = \frac{27}{25}$, we wish to find the minimum value of $i+j$ such that
$$\frac{27}{25} = \left(\frac{3}{4}\right)^i \left(\frac{8}{5}\right)^j$$
Factoring into primes leaves us with
$$\frac{3^3}{5^2} = \frac{3^i}{2^{2i}} \cdot \frac{2^{3j}}{5^j},$$
which forces $i = 3$ and $j = 2$. Therefore the minimum number of copies is $3+2 = 5$.

Note: This analysis shows that the only *possible* way of obtaining a 108% copy from iterated 75% and 160% copies is by making 3 of the former and 2 of the latter (in any order).

4. Let points C, D, E be as indicated in the diagram below, and let the angles at A and B be α and β, respectively.



Considering the sums of the angles in triangles $\triangle CDE$, $\triangle AEC$, and $\triangle BCD$ gives
$$\circ + \times + 35^\circ = 180^\circ$$
$$\alpha + (180^\circ - \times) + 35^\circ = 180^\circ$$
$$\beta + (180^\circ - \circ) + 35^\circ = 180^\circ.$$
Summing these equations and simplifying yields $\alpha + \beta = 75^\circ$.

5. Certainly 20! is divisible by 9. But a number is divisible by 9 if and only if the sum of its digits is divisible by 9. Summing the digits of $243290X008176640000$ yields $52 + X$, and the only digit X that makes this sum divisible by 9 is $X = 2$.

   Note: One can quickly determine whether an integer is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 11, or 12 through simple manipulations of its digits. For instance, a number is divisible by 8 if and only if the number formed from its last 3 digits is divisible by 8; and a number is divisible by 11 if and only if the *alternating* sum of its digits is divisible by 11. (Example: 35201804 is divisible by 11 because $3-5+2-0+1-8+0-4 = -11$ is divisible by 11.) Deriving these "divisibility tests" is a standard exercise in elementary modular arithmetic.

6. The equation is of the form

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{a}}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b}}},$$

   where

$$a = x \qquad \text{and} \qquad b = 1+\frac{1}{1+\frac{1}{x}} = \frac{2x+1}{x+1}.$$

   Certainly the equation holds if and only if $a = b$. This leads to $x^2 - x - 1 = 0$, which has roots $x = \frac{1}{2}(1 \pm \sqrt{5})$.

   Note: One can also simplify each side of the given equation to get

$$\frac{3x+2}{2x+1} = \frac{8x+5}{5x+3},$$

   which again leads to the quadratic $x^2 - x - 1 = 0$ after cross multiplication. It is not an accident that the coefficients appearing in the simplified fractions are Fibonacci numbers (Can you see why?) And wherever the Fibonacci numbers arise, the *Golden Ratio* $\phi = \frac{1}{2}(1+\sqrt{5})$ is lurking close by.

   The Fibonacci numbers, of course, are generated by the recurrence relation $f_n = f_{n-1} + f_{n-2}$. It is well worth investigating the connections between (1) sequences defined via such relations, (2) roots of quadratic polynomials (such as the Golden Ratio), and (3) iterated fractions like those appearing in this problem, which are more commonly known as *continued fractions*.

   As a further teasing glimpse into the beautiful world of continued fractions, consider the following amazing identities:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} \qquad \text{and} \qquad \frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

7. Let B and S denote the number of heads tossed by Bob and Suzy, respectively. Since either $B = S$ or $B < S$ or $B > S$, we must have $P(B = S) + P(B < S) + P(B > S) = 1$. But clearly $P(B < S) = P(B > S)$, and therefore

$$P(B < S) = \frac{1 - P(B = S)}{2}.$$

To compute $P(B = S)$, note that there are 8 possible outcomes for either Bob or Suzy, namely $\{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$. So the probabilities that Bob (or Suzy) tosses 0,1, 2, or 3 heads are $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}$, and $\frac{1}{8}$, respectively. It follows that

$$P(B = S) = \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{3}{8} + \frac{1}{8} \cdot \frac{1}{8} = \frac{5}{16},$$

which yields $P(B < S) = \frac{1}{2}(1 - \frac{5}{16}) = \frac{11}{32}$.

Note: You might recognize the numerators $1, 3, 3, 1$ appearing in the above solution as entries in Pascal's triangle (*i.e.* binomial coefficients). What happens when Bob and Suzy toss the coin 4 times? or 5? or n? There are a couple lovely identities hiding in this problem, namely:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

and

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

These are just two examples of a myriad of identities involving binomial coefficients.

8. If $a = 1 + a^2$ then $a^2 = a - 1$. Multiply by a to get $a^3 = a^2 - a$, and replace a^2 with $a - 1$ to get $a^3 = -1$. Therefore $a^{60} = (a^3)^{20} = (-1)^{20} = 1$.

Alternative Solution: Rewrite $a = 1 + a^2$ as $a^2 - a + 1 = 0$. From the sum of cubes factorization $a^3 + 1 = (a + 1)(a^2 - a + 1)$, deduce that $a^3 + 1 = 0$. Thus $a^3 = -1$ as before.

9. For convenience, let us say a number is *good* if it consists only of digits 1, 2, and 3, with no consecutive digits differing by more than 1. Observe that every good n-digit is obtained by either

   (a) beginning with a good $(n-1)-$digit number N and appending either

$$\begin{cases} 1 \text{ or } 2 & \text{if } N \text{ ends in 1, or} \\ 1 \text{ or } 3 & \text{if } N \text{ ends in 2, or,} \\ 2 \text{ or } 3 & \text{if } N \text{ ends in 3,} \end{cases} \qquad \text{OR}$$

   (b) beginning with a good $(n-2)-$digit number and appending 22.

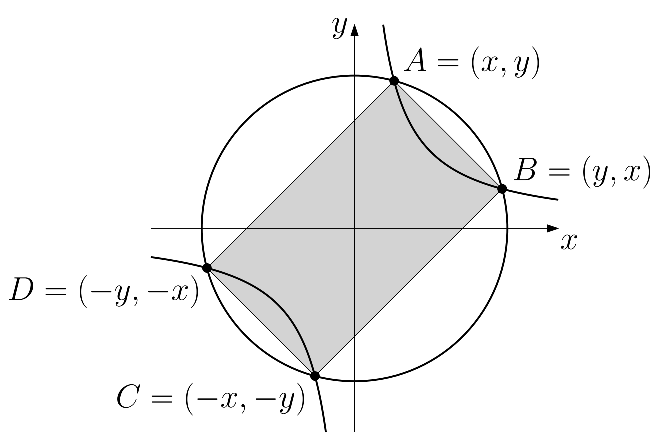
   Every good n-digit number is obtained uniquely in this manner. So if a_n is the number of good n-digit numbers then $a_n = 2a_{n-1} + a_{n-2}$. This recurrence relation is valid for $n \geq 2$ with the understanding that $a_0 = 1$, which accounts for the "empty" 0-digit number. Clearly $a_1 = 1$ (the possible numbers are 1, 2, and 3), so that

$$\begin{aligned} a_2 &= 2a_1 + a_0 = 2 \cdot 3 + 1 = 7 \\ a_3 &= 2a_2 + a_1 = 2 \cdot 7 + 3 = 17 \\ a_4 &= 2a_3 + a_2 = 2 \cdot 17 + 7 = 41. \end{aligned}$$

   That is, there are 41 good 4-digit numbers.

   Note: It is certainly not expected that any teams solve the problem in this manner. Given the time restrictions, it is much more reasonable to immediately begin hand-counting the possibilities, doing so in an organized manner to avoid overcounting and using symmetry wherever possible.

10. Suppose (x, y) satisfies both $x^2 + y^2 = 36$ and $xy = 9$. Observe that the points (y, x), $(-x, -y)$ and $(-y, -x)$ also satisfy both equations. So the general situation is as indicated in the diagram below.



Since AB and DC both have slope -1 while BC and AD have slope 1, $ABCD$ is a rectangle. Its area is therefore

$$\begin{aligned} |AB| \cdot |BC| &= \sqrt{(y-x)^2 + (x-y)^2}\sqrt{(-x-y)^2 + (-y-x)^2} \\ &= 2\sqrt{(x-y)^2(x+y)^2}. \end{aligned}$$

But $(x \pm y)^2 = x^2 + y^2 \pm 2xy = 36 \pm 2 \cdot 9$. So $ABCD$ has area $2\sqrt{18 \cdot 54} = 36\sqrt{3}$.

Alternative Solution: One could also substitute $y = 9/x$ into $x^2 + y^2 = 36$ and clear fractions to obtain the quartic $x^4 - 36x^2 + 81 = 0$. This is quadratic in x^2, so it may be solved with the quadratic formula to yield the coordinates of the intersection points A, B, C, D. The desired area may then be obtained by computing $|AB| \cdot |BC|$.

Pairs Relay

P-A. For certain numbers m and n we have the identity

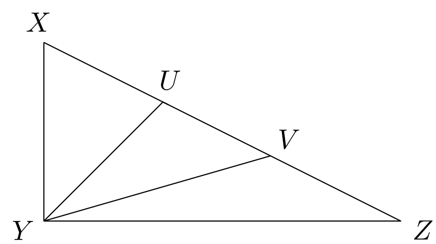
$$\frac{7x+19}{x^2+5x+6} = \frac{m}{x+2} + \frac{n}{x+3}.$$

Let $\mathsf{A} = mn$.

$\boxed{\text{Pass on } \texttt{A}}$

P-B. You will receive $\texttt{A}$.

Right triangle XYZ has legs $|XY| = 3$ and $|YZ| = \mathsf{A}$. The hypotenuse XZ is subdivided into 3 equal parts by points U and V.



Let B be the area of triangle UYV.

$\boxed{\text{Pass on } \texttt{B}}$

P-C. You will receive $\texttt{B}$.

Suppose X is 10% of Y, Y is 25% of Z, and W is B% of Z.

Let $\mathsf{C} = \dfrac{W}{X}$.

$\boxed{\text{Pass on } \texttt{C}}$

P-D. You will receive $\texttt{C}$.

Stations 1 through 9 are marked in clockwise order around a circle. Movement between stations is governed by the following rule: From station n you must move 2 stations clockwise if n is divisible by 3, and 1 station clockwise if not.

Begin at station C and let D be the final station after making 100 moves.

$\boxed{\text{Done!}}$

Pairs Relay Answer Key

A. 10

B. 5

C. 2

D. 8

Pairs Relay Solutions

P-A. We have

$$\frac{m}{x+2}+\frac{n}{x+3}=\frac{m(x+3)+n(x+2)}{(x+2)(x+3)}=\frac{(m+n)x+(3m+2n)}{x^2+5x+6}.$$

So the given identity holds when $\{m+n=7,\ 3m+2n=19\}$. This system has solution $(m,n)=(5,2)$, so $\mathsf{A}=mn=10$.

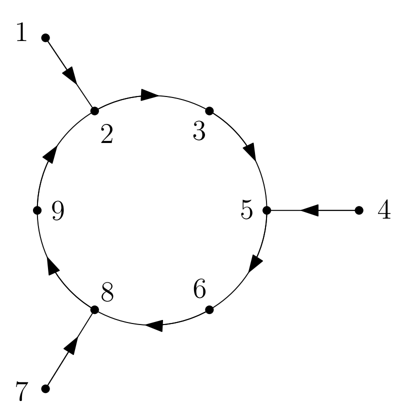
P-B. Triangles $\triangle XYU, \triangle UYV$, and $\triangle VYZ$ have equal bases $|XU|=|UV|=|VZ|$ and the same altitude, so the area of each is $1/3$ the area of $\triangle XYZ$. Thus $\mathsf{B}=\frac{1}{3}\cdot\frac{3\mathsf{A}}{2}=\mathsf{A}/2$. With $\mathsf{A}=10$ we have $\mathsf{B}=5$.

P-C. The given conditions yield $X=\frac{1}{10}Y$, $Y=\frac{1}{4}Z$, and $W=\frac{\mathsf{B}}{100}Z$. Therefore

$$\mathsf{C}=\frac{W}{X}=\frac{\frac{\mathsf{B}}{100}Z}{\frac{1}{10}\cdot\frac{1}{4}Z}=\frac{2}{5}\mathsf{B}$$

With $\mathsf{B}=5$, get $\mathsf{C}=2$.

P-D. The given rules are summarized by the following diagram, in which numbered dots represent stations and arrows indicate movement.



Note that regardless of the initial station, movement will inevitably fall into the repeating sequence 2–3–5–6–8–9–2–3–$\cdots$. The period of this sequence is 6, so if we begin at station $\mathsf{C} = 2$ then the terminal station after 100 moves is the one occurring $100 \bmod 6 = 4$ moves after 2, namely station $\mathsf{D} = 8$.

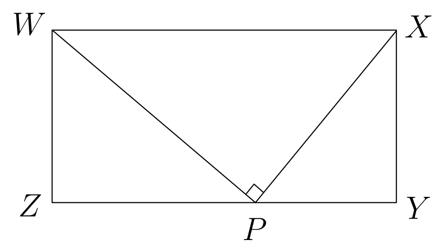
Individual Relay

I-A. Let A be the number of ways the letters of APPLE be rearranged so the first and last letters are consonants.

$\boxed{\text{Pass on } \mathtt{A}}$

I-B. You will receive $\mathtt{A}$.

Right triangle WPX is inscribed in rectangle $WXYZ$, which has sides $|WX| = \mathsf{A}$ and $|XY| = 8$.



Let B be the area of $\triangle WPX$. $\boxed{\text{Pass on } \mathtt{B}}$

I-C. You will receive $\mathtt{B}$.

Let C be the number of positive integers n that satisfy the inequality $n^2 + 1000 < \mathsf{B}n$.

$\boxed{\text{Pass on } \mathtt{C}}$

I-D. You will receive $\mathtt{C}$.

The degree measure of each interior angle of a regular polygon is an integer multiple of C.

Let D be the number of sides of this polygon. $\boxed{\text{Done!}}$

Individual Relay Answer Key

A. 18

B. 72

C. 35

D. 9

Individual Relay Solutions

I-A. Each possible arrangement is of the form $P\square\square\square P, P\square\square\square L$, or $L\square\square\square P$, where the squares represent any of the $3! = 6$ arrangements of the three remaining letters. (For instance, with $P\square\square\square P$, the squares are filled will all arrangements of the letters A, L, and E. Thus $\mathsf{A} = 3 \cdot 3! = 18$.

I-B. Triangle WPX has base $|WX|$ and altitude XY, so its area is $\mathsf{B} = \frac{1}{2}|WX||XY| = \frac{1}{2} \cdot \mathsf{A} \cdot 8 = 4\mathsf{A}$. With $\mathsf{A} = 18$, get $\mathsf{B} = 72$.

I-C. The number of positive integers n that satisfy this inequality is the number of positive integers strictly between the roots of the quadratic $n^2 - \mathsf{B}n + 1000 = 0$. The roots are $\frac{1}{2}(\mathsf{B} \pm \sqrt{\mathsf{B}^2 - 4000})$, which with $\mathsf{B} = 72$ gives $36 \pm \sqrt{296}$. Since $\sqrt{296}$ is between 17 and 18, there are $\mathsf{C} = 2 \cdot 17 + 1 = 35$ positive integers between the roots (namely $19, 20, 21, \ldots, 53$).

I-D. The degree sum of all interior angles in a regular n-gon is $180(n-2)$, as can be seen by dissecting the n-gon into $n-2$ triangles. So each interior angle of a regular n-gon is $180(n-2)/n$. We wish for this to be an integer multiple of $\mathsf{C} = 35$, so we require

$$\frac{180(n-2)}{n} = 35k$$

for some integer k. Divide by 5 to get $\frac{36(n-2)}{n} = 7k$, and note that since 7 is prime and does not divide 36 this forces $n-2$ to be divisible by 7. The smallest possible n is $n = 9$, which yields $k = 4$. Therefore $\mathsf{D} = 9$.

Note: A brief analysis shows that the only positive integers n and k satisfying $\frac{36(n-2)}{n} = 7k$ are $n = 9$ and $k = 4$. Thus the 9-gon is the *only* regular polygon whose interior angles are multiples of $35°$.