Nova Scotia Math League

2015–2016

Game Two

PROBLEMS AND SOLUTIONS

Team Questions

1. In Sparkville, one out of three women is an electrician and two out of three electricians are women. If there are 330 women in Sparkville, how many electricians live in the town?

   Solution: Let W and E be the sets of women and electricians, respectively. We are given $|E \cap W| = |W|/3 = 2|E|/3$. So $|E| = |W|/2 = 165$.

   □

2. Jane walks 25% faster than her brother Jim. If Jim leaves for school 3 minutes before Jane, how long will it take Jane to catch up?

   Solution: Let Jim's speed be 1 unit per minute. They catch up after t minutes, where $\frac{5}{4}t = (t+3)$. Solve to get $t = 3 \cdot 4 = 12$ minutes.

   □

3. A bank machine only dispenses \$20 and \$50 notes. If John's daily withdrawal limit is \$1000, how many different (nonzero) amounts of money can he withdraw on any given day?

   Solution: This is equivalent to finding number of positive integers up to 100 expressible as $2x + 5y$. Note that 4 and 5 are expressible, and therefore so is everything larger, since any even number > 4 is equal to $4 + 2x$ for some x, and any odd number > 5 is equal to $5 + 5y$ for some y. Since 1 and 3 are not so expressible, the answer is $100 - 2 = 98$.

   □

4. A Grade 1 class sits in a circle. The teacher numbers the students $1, 2, 3, \ldots$ starting with Jane and going clockwise. After she is done, Ryan insists that he is number 1 and goes about renumbering everyone (again clockwise) starting with himself. In doing so, Jane's number changes to 10, and her sister Sara's number changes from 14 to 3.

   How many students are in the class?

   Solution: The increase in Jane's number plus the decrease in Sara's must be the number of students in the class: $(10-1) + (14-3) = 9 + 11 = 20$.

   □

5. The eight lines

$$\begin{array}{llll} y = x+1 & y = 2x+1 & y = 3x+1 & y = 4x+1 \\ y = x-1 & y = 2x-1 & y = 3x-1 & y = 4x-1 \end{array}$$

   are drawn in the (x, y)-plane. Find the number of distinct points of intersection.

   Solution: The four lines in the first row are concurrent at $(0, 1)$ and those in the second row are concurrent at $(0, -1)$. Each line in the first row intersects only three in the second row, since one is parallel. So there are $1 + 1 + 4 \cdot 3 = 14$ points of intersection.

   □

6. Three 6-sided dice — one red, one green and one blue — are thrown. Leting R, G and B denote the values showing, find the probability that $R \le G \le B$.

   Solution: It's straightforward to count valid triples (R, G, B) by hand: The result is that there are $1 + 2 + 3 + \cdots + i = \binom{i+1}{2}$ possibilities in which $R = 7 - i$. So the total number of valid triples is $\binom{7}{2} + \binom{6}{2} + \cdots + \binom{2}{2} = 56$, and the desired probability is $\frac{56}{216} = \frac{7}{27}$.

   A more robust and general solution is to let $R = 1 + r$, $G = R + g = 1 + r + g$ and $B = G + b = 1 + r + g + b$, where $r, g, b \ge 0$, and look for solutions to $r + g + b \le 5$. Introduce slack variable $s \ge 0$ to change this to the equation $r + g + b + s = 5$. This has $\binom{5+3}{3} = 56$ solutions, since it's the number of ways of distributed 5 identical objects amongst 4 people.

   $\square$

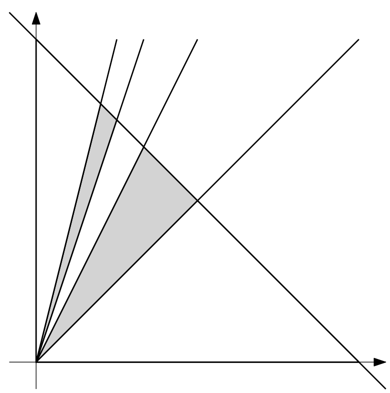
7. The polynomial $f(x) = x^3 + ax^2 + bx + c$ satisfies

   $$f(x+2) = x^3 + 7x + 5 \quad \text{for all } x.$$

   Determine $a + b + c$.

   Solution: We have $a + b + c = f(1) - 1$. But $f(1) = f(-1+2) = (-1)^3 + 7(-1) + 5 = -3$. So $f(1) = -4$. $\square$

8. The lines $y = x$, $y = 2x$, $y = 3x$ and $y = 4x$ intersect the line $x + y = 1$ to create the figure below.

   Find the area of the shaded region.

   Solution: The x-coordinates of the intersection points are $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$, and base of the big triangle has length 1. So the ratio of the shaded area to that of the full triangle is $(\frac{1}{4} - \frac{1}{5}) + (\frac{1}{2} - \frac{1}{3}) = \frac{13}{60}$. Since the full triangle has area $\frac{1}{2}$, the shaded area is $\frac{13}{120}$.

   Alternatively, use the fact that the area of the triangle with vertices $(0, 0)$, (a, b) and (c, d) is $\frac{1}{2}|ad - bc|$. (This comes from the determinant.)

   $\square$

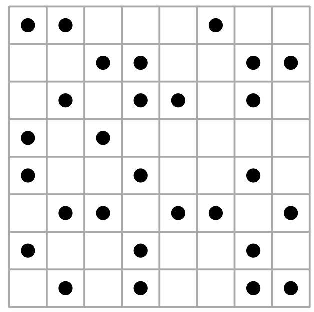
9. Alan, Betty, Clara, Doug, and Elizabeth go on an Easter egg hunt. Altogether the kids find 125 eggs. The three girls have a total of 60, with Elizabeth having more than Betty and Clara combined, and Clara having 5 more than Betty. Alan has more than Doug, but Doug has more than each of the girls.

   How many eggs does Clara have?

Solution: We have $E + B + C = 60$, with $E > B + C$. Thus $E \geq 31$. This forces $D \geq 32$, which forces $A \geq 33$. So $125 = A + B + C + D + E \geq 60 + 32 + 33 = 125$. Thus all inequalities are equalities. So $E = 31 \implies B + C = 29 \implies C = 17$.

$\square$

10. Kate has placed pennies on an 8×8 grid as shown below.



Kate now wishes to add or remove pennies to make every horizontal row the same. Find the least number of moves (additions and removals) she must perform to accomplish her goal.

Solution: Look at each column in turn. If a column has p_i pennies, where $p_i \leq 4$, then remove all the pennies from that column. Otherwise add $8 - p_i$ pennies. So the minimal number of moves is $\sum_i \min\{p_i, 8 - p_i\} = 4 + 4 + 3 + 3 + 2 + 2 + 3 + 3 = 24$.

$\square$

Pairs Relay

P-A. Chocolate chip cookies can be purchased in packs of 3 for \$2.50 or packs of 5 for \$3.50. Oatmeal cookies come only in packs of 6 for \$4.

Let A be the most cookies you can purchase for \$35. $\boxed{\text{Pass on A}}$

Solution: Trial and error leads to the correct solution very quickly.

Alternatively, we can analyse as follows: We should buy at most one pack of 3, as any two 3-packs should be traded for a 6-pack. And we should buy at most two 5-packs, as any three of them could be traded for two 6-packs and a 3-pack. So at least $35 - 2(3.5) - 2.5 = 25.5$ should be spent on 6-packs. Thus we must buy at least 7 of them. With 7 we can buy 52 cookies (seven 6-packs and two 5-packs). With 8 we can buy only 51 cookies (eight 6-packs and one 3-pack). So the answer is $\mathsf{A} = 52$. □

P-B. You will receive A.

For certain x and y we have $\mathsf{A} : 100 = (2x + y) : (4x + y)$.

Let $\mathsf{B} = \dfrac{x}{y}$. [*Hint:* This should be an integer.] $\boxed{\text{Pass on B}}$

Solution: Solve $\mathsf{A}/100 = (2\mathsf{B} + 1)/(4\mathsf{B} + 1)$ to get $\mathsf{B} = (100 - \mathsf{A})/(4\mathsf{A} - 200)$. With $\mathsf{A} = 52$ get $\mathsf{B} = 6$ □

P-C. You will receive B.

Compute:

$$\mathsf{C} = \frac{(\mathsf{B}^2 + 3\mathsf{B} + 2)(\mathsf{B}^2 - 4\mathsf{B} + 3)}{(\mathsf{B}^2 - 3\mathsf{B} + 2)(\mathsf{B}^2 - 2\mathsf{B} - 3)}.$$

$\boxed{\text{Pass on C}}$

Solution: Factor and simplify to get $\mathsf{C} = (\mathsf{B} + 2)/(\mathsf{B} - 2) = 2$. □

P-D. You will receive C.

When the line $y = \mathsf{C}x + \mathsf{D}$ is shifted to the left by D units and shifted down by C units, the resulting line has equation $y = \mathsf{C}x + 2\mathsf{D}$.

Find D. $\boxed{\text{Done!}}$

Solution: The given translations map $y = \mathsf{C}x + \mathsf{D}$ to $y = \mathsf{C}(x + \mathsf{D}) + \mathsf{D} - \mathsf{C} = \mathsf{C}x + (\mathsf{D} - \mathsf{C} + \mathsf{DC})$. Therefore $2\mathsf{D} = \mathsf{D} - \mathsf{C} + \mathsf{DC}$, so $\mathsf{D} = \mathsf{C}/(\mathsf{C} - 1)$. With $\mathsf{C} = 2$ get $\mathsf{D} = 2$. □

Individual Relay

I-A. Let A be the number of ways the letters of the word YELL can be arranged so that the two L's are not adjacent.

$\boxed{\text{Pass on A}}$

Solution: The words can be of the form $L * L*$, $*L * L$, or $L * *L$. In each case, there are two ways of filling in the stars. So $\mathsf{A} = 6$ possibilities overall.

Alternatively, Arrange all 4 letters in $4!/2!$ ways, and subtract the $3!$ ways in which the L's appear together.) □

I-B. You will receive A.

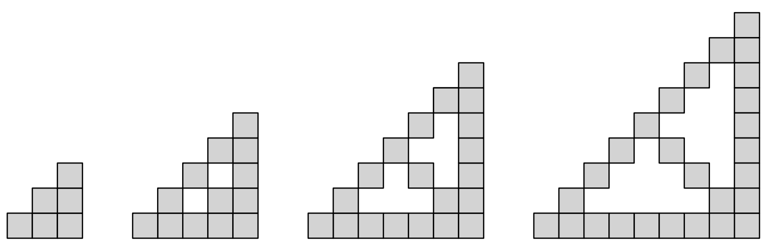
The average of a list of A numbers is $\mathsf{A} - 1$. Let B be the average after the numbers A and 2A are appended to the list.

$\boxed{\text{Pass on B}}$

Solution: Let S be the sum of the original list. Then $S/\mathsf{A} = \mathsf{A} - 1 \implies S = \mathsf{A}^2 - \mathsf{A}$. Thus $\mathsf{B} = (S + \mathsf{A} + 2\mathsf{A})/(\mathsf{A} + 2) = (\mathsf{A}^2 + 2\mathsf{A})/(\mathsf{A} + 2) = \mathsf{A}$. □

I-C. You will receive B.

Let C be the number of squares in the B-th figure of this sequence:



$\boxed{\text{Pass on C}}$

Solution: The B-th diagram has perimeter $3(2\mathsf{B} - 1) + 3 = 6\mathsf{B}$ and diagonal $\mathsf{B} - 1$, giving a total of $7\mathsf{B} - 1$ squares. With $\mathsf{B} = 6$ get $\mathsf{C} = 41$. □

I-D. You will receive C.

Solution X is 15% alcohol by volume and Solution Y is C% alcohol by volume. Let D be the number of litres of Solution X that must be mixed with 5L of Solution Y to result in a 25% alcohol solution.

$\boxed{\text{Done!}}$

Solution: Solve $(5\mathsf{C} + 15\mathsf{D})/(5 + \mathsf{D}) = 25$ to get $\mathsf{D} = (\mathsf{C} - 25)/2$. With $\mathsf{C} = 41$ get $\mathsf{D} = 8$. □