Nova Scotia Math League

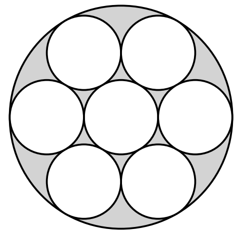
2015–2016

Game Three

PROBLEMS AND SOLUTIONS

Team Questions

1. Find the ratio of the shaded to unshaded area in the diagram below:



Solution: Say the small circle has radius 1, so the larger has radius 3. Then the desired ratio is $(9\pi - 7\pi) : 7\pi = 2 : 7$. □

2. John is driving on the highway at 100 km/h. He glances down and is quite pleased to notice that his odometer reads 56965 km, which is palindromic (*i.e.* it reads the same forwards and backwards).

   If John continues driving at the same speed, how many minutes will pass before his odometer again shows a palindrome?

Solution: The next palindrome is 57075, so John must travel $57075 - 56965 = 110$ km at 100 km/h, which takes $\frac{110}{100} \cdot 60 = 66$ minutes.

□

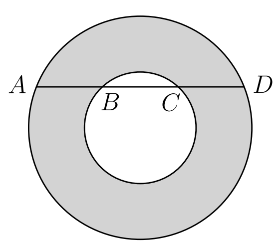
3. Recall that $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.

   Find the smallest n such that $n!$ is divisible by 1000.

Solution: We need $n!$ to be divisible by $2^3 5^3$, so the n is 15 (the third multiple of 5).

□

4. In the figure below, the two circles are concentric and $|AB| = |BC| = |CD| = 2$. Find the area of the shaded region.



Solution: Let r and R be the radii of the small and large circles. Let O be the centre of the circles and draw a perpendicular from O to meet AD at P. Then $|BP| = 1$ and $|AP| = 3$. Let $|OP| = x$. Get $x^2 + 1^2 = r^2$ and $x^2 + 3^2 = R^2$, so $R^2 - r^2 = 8$. The desired area is $\pi(R^2 - r^2) = 8\pi$.

Alternative Solution: If we assume that the question is well-posed (that is, it has an unambiguous answer), then there is a very clever trick for its solution. We simply observe that the answer to the problem must be independent of the exactly location of the chord AB. All that can possibly matter is the fact that $|AB| = |BC| = |CD| = 2$. So we can boldly assume that the smaller and larger circles are concentric, and AB is a diameter of the larger while BC is a diameter of the smaller. But the problem is trivial in this case, since the smaller circle has radius 1 and the larger has radius 3, meaning the difference of their areas is 8π.

(It must be emphasized that this approach only works because we *know* the problem is well-posed, appearing as it does on a contest. The proof that the answer is independent of the location of the chord is given in the first solution, where we see the area is independent of the value of x.)

□

5. Find the equation of the line that is obtained by reflecting the line $y = 1$ in the line $y = 2x$.

Solution: The lines meet at $(\frac{1}{2}, 1)$ and the desired line passes through this point. The line $y = -\frac{1}{2}x$ is perpendicular to $y = 2x$. It meets $y = 1$ at $(-2, 1)$, so by symmetry it meets the desired line at $(2, -1)$. (Draw a diagram.) The line through $(\frac{1}{2}, 1)$ and $(-2, 1)$ has equation $y = -\frac{4}{3}x + \frac{5}{3}$.

□

6. A broken watch gains 8 minutes per hour. If it is set to the correct time at noon, what is the real time when the watch reads 4:15pm?

Solution: The watch indicates 68 minutes for every 60 that really pass. So if x real minutes pass, then the watch will indicate y, where $\frac{60}{68} = \frac{x}{y}$. Thus $x = 15y/17$. At 4:15 we have $y = 4 \cdot 60 + 15 = 255$, so $x = 15(255)/17 = 225$. Thus the real time is 3:45pm.

□

7. Alan, Bob, and Carl go to Las Vegas to gamble, each bringing a different amount of money. If Alan or Bob were to double their money then the group's total would increase by 25% and 40%, respectively.

What would be the percent increase if Carl were to triple his money?

Solution: Say they brought X altogether. Then Alan and Bob brought 25% and 40% of X, respectively, meaning Carl accounted for 35% of the initial money. If he triples his share, the group will be up by 70%.

□

8. A square and a rectangle have perimeter 8, but the rectangle has only $\frac{7}{8}$ the area of the square. Find the length of the diagonal of the rectangle.

Solution: Say the rectangle has sides x and y. Then $x + y = 8/2 = 4$ and $xy = \frac{7}{8} \cdot 2^2 = \frac{7}{2}$. The desired diagonal is then

$$\sqrt{x^2 + y^2} = \sqrt{(x + y)^2 - 2xy} = \sqrt{4^2 - 2 \cdot \frac{7}{2}} = 3.$$

□

9. Find the sum of the digits of $(100000001)^5$.

Solution: Write this as $(10^6 + 1)^5$ and expand with binomial theorem to see that the answer is the same as the sum of the digits in the 5th row of Pascal's triangle, which is 14.

□

10. An ant paces along the x-axis at a constant rate of one unit per second. He begins at $x = 0$ and his path takes him one unit forward, then two back, then three forward, etc.

    How many times does the ant step on the point $x = 10$ in the first five minutes of his walk?

    Solution: The ant changes directions at $t = 1, 1+2, 1+2+3, \ldots$, at which times he is at points $x = 1, -1, 2, -2, 3, -3, \ldots$. Note that $1 + 2 + 3 + \cdots + n = 300 \iff n(n+1) = 600 \iff n = 24$. So after 5 minutes (300 seconds) the ant is at $x = -12$, meaning he has stepped on $x = 10$ exactly 5 times (one initial visit to $x = 10$, and then twice for each swing between ± 11 and ± 12).

    □

Pairs Relay

P-A. One litre of wine is poured into five litres of water. Five litres of this solution is then mixed with one litre of wine.

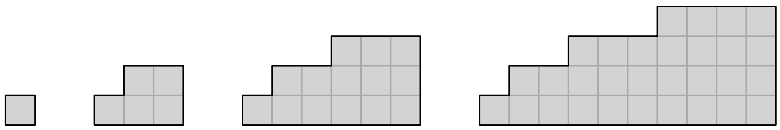
Let A be the ratio of wine to water in the final solution. **Pass on A**

Solution: The final solution contains $5 \cdot \frac{1}{6} + 1 = \frac{11}{6}$ litres of wine and $5 \cdot \frac{5}{6} = \frac{25}{6}$ litres of water. So the ratio is $\mathsf{A} = 11 : 25$.

□

P-B. You will receive A. Let $n = 25\mathsf{A}$, which should be an integer.

Let B be the perimeter of the n-th shape in the following series:



(Each of the small squares in the figure is 1×1.) **Pass on B**

Solution: The n-th shape has perimeter $2n + 2(1 + 2 + 3 + \cdots + n) = n^2 + 3n$. With $n = 25(11/25) = 11$ get $\mathsf{B} = 154$. □

P-C. You will receive B.

Let C be the smallest positive integer such that $1 + 2 + 3 + \cdots + \mathsf{C}$ is larger than B.

Pass on C

Solution: We need $\mathsf{C}(\mathsf{C}+1)/2 > \mathsf{B}$, which is equivalent to $C(C+1) > 2B$. Since C is an integer, $C = \left\lfloor \sqrt{2B} \right\rfloor$. With $\mathsf{B} = 154$ get $\mathsf{C} = \left\lfloor \sqrt{308} \right\rfloor = 18$.

□

P-D. You will receive C.

Alan and Bill are in a race, each running at constant speed. At 1:00pm, Alan is C metres ahead of Bill, and at 1:15pm he has tripled that lead. Alan finishes the race at 2:00pm.

Let D be the number of metres by which Alan beats Bill. **Done!**

Solution: Every 15 minutes, Alan increases his lead by 2C metres. So in the 45 minutes between 1:15 and 2:00 his lead increases from 3C to $3\mathsf{C} + 3(2\mathsf{C}) = 9\mathsf{C}$. With $\mathsf{C} = 18$ get $\mathsf{D} = 162$.

□

Individual Relay

I-A. The sum of the lengths of all edges of a cube is 24.

Let A be the surface area of this cube. **Pass on A**

Solution: Each side of the cube has length $24/12 = 2$, so the surface area is $\mathsf{A} = 6 \cdot 2^2 = 24$. □

I-B. You will receive A.

Evaluate:

$$\mathsf{B} = (2 + 4 + 6 + 8 + \cdots + 4\mathsf{A}) - (1 + 3 + 5 + 7 + 9 + \cdots + (4\mathsf{A} - 1)).$$

Pass on B

Solution: There are 2A terms in each sum, and the difference of each correspondening pair of terms is 1. So total is $\mathsf{B} = 2\mathsf{A}$. With $\mathsf{A} = 24$ get $\mathsf{B} = 48$.

□

I-C. You will receive B.

Three consecutive positive integers sum to B.

Let C be the largest of these three integers. **Pass on C**

Solution: $(x-1) + x + (x+1) = \mathsf{C} \Longrightarrow 3x = \mathsf{C} \Longrightarrow x = \mathsf{C}/3 \Longrightarrow x + 1 = \mathbb{C}/3 + 1$. With $\mathsf{C} = 48$ get $\mathsf{D} = 17$. □

I-D. You will receive C.

The lines $y = 1 + \mathsf{C}x$ and $y = 1 - \mathsf{C}x$ intersect the line $y = \mathsf{C}$ at two points.

Let D be the distance between these points. **Done!**

Solution: The lines intersect at $x = \pm\frac{\mathsf{C}-1}{\mathsf{C}}$ and both have y-coordinate D. Thus the distance between them is $\mathsf{D} = 2|\frac{\mathsf{C}-1}{\mathsf{C}}|$. With $\mathsf{C} = 17$ get $\mathsf{D} = \frac{32}{17}$.

□