Nova Scotia Math League

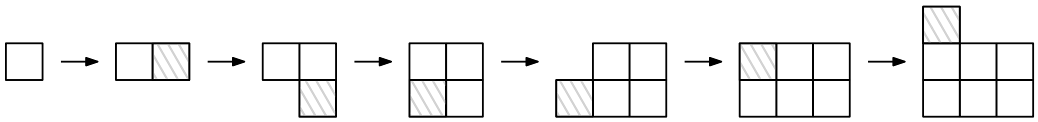
2018–2019

**Game Three**

# Problems and Solutions

# Team Questions

1. A sequence of figures is obtained by successively adjoining 1 cm $\times$ 1 cm squares in a spiral pattern, as shown below. (The striped square is the most recently added.)



What is the perimeter of the figure whose area is 750 cm$^2$?

**Solution:** The largest square less than 750 is $27^2 = 729$. Therefore the figure is a $27 \times 27$ square with an additional partial side of length $750 - 729 = 21$. Its perimeter is the same as that of a $28 \times 27$ rectangle, namely $2(27 + 28) = 110$. $\square$

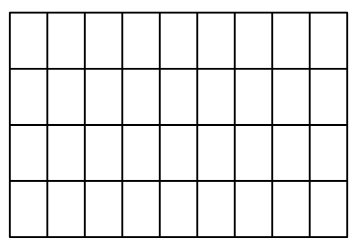
2. Three numbers sum to 2019. If one of the numbers is doubled, the sum triples. And if one of the numbers is quadrupled, the sum doubles.

Find the numbers.

**Solution:** Say $a + b + c = 2019$. If doubling $a$ triples the sum, then $a = 2 \cdot 2019 = 4038$. And if quadrupling $b$ doubles the sum then $3b = 2019$, or $b = 673$ Finally, $a + b + c = 2019$ gives $c = -2692$.

Alternatively, solve the system $\{a + b + c = S, 2a + b + c = 3S, a + 4b + c = 2S\}$ to get $(a, b, c) = (2S, \frac{1}{3}S, -\frac{4}{3}S)$. $\square$

3. Each small rectangle in the figure below is similar to the large rectangle. If the perimeter of the large rectangle is 16, what is the perimeter of each small rectangle?



**Solution:** Say the sides of each small rectangle are $k$ times those of the large. If $A$ is the area of the large rectangle then the area of the small rectangle is $k^2 A$. The diagram indicates that 36 small rectangles fill the big rectangle, so $A = 36k^2 A$, hence $k = \frac{1}{6}$. The perimeter of the small rectangle is therefore $\frac{1}{6}(16) = \frac{8}{3}$.

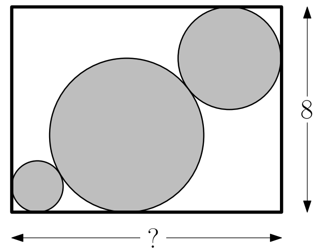
Alternatively, let the small rectangle have sides $x$ and $y$ and from similarity deduce that $\frac{x}{y} = \frac{4y}{9x} \implies 9x^2 = 4y^2 \implies 3x = 2y$. The perimeter of the large rectangle is $16 = 2(9x + 4y) = 2(9x + 6x) = 30x$, so $x = \frac{8}{15}$. The perimeter of the small rectangle is then $2x + 2y = 5x = \frac{8}{3}$. $\square$

4. How many integers between 1 and 100 (inclusive) **cannot** be written as a sum of 10 consecutive integers?

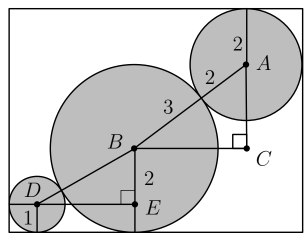
*Note:* The summands can be negative.

**Solution:** Since the sum of 10 consecutive integers $x - 4, x - 3, \ldots, x, \ldots, x + 4, x + 5$ is $10x + 5$, we want the number of integers between 1 and 100 that do not have 5 as their units digit. There are $100 - 10 - 90$ such numbers. $\square$

5. Three circular buttons are placed tightly in a rectangular box as shown below. The radii of the buttons are 1 cm, 2 cm, and 3 cm and the box has a width of 8 cm, as indicated. Find the length of the box.




**Solution:** Label the diagram as below.




Then $|AB| = 3 + 2 = 5$ and $|AC| = 8 - (2 + 3) = 3$, so Pythagorean theorem gives $|BC| = 4$. Similarly $|BD| = 3 + 1 = 4$ and $|BE| = 3 - 1 = 2$ so $|DE| = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. Thus the desired length is $1 + 2\sqrt{3} + 4 + 2 = 7 + 2\sqrt{3}$. $\square$

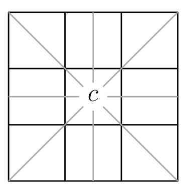
6. The empty cells of the square below are to be filled with positive real numbers such that the product of the three entries along each row, column, and diagonal is 1. Find $x$.

| | | |
|---|---|---|
| | $9$ | |
| | | $4$ |
| | | $x$ |

**Solution:** Enforcing unit products along each row, column and diagonal leads to the configuration below, where cells A, B, C and D were filled in that order. Now consider the lower left cell. Taking products along the first column and along the northeast diagonal force this cell to have the common value $\frac{81}{4x^3} = \frac{16x^3}{9}$, giving $x = \frac{3}{2}$.

| | | |
|---|---|---|
| $\frac{4x}{9}$ B | $9$ | $\frac{1}{4x}$ A |
| $\frac{x^2}{9}$ D | $\frac{9}{4x^2}$ C | $4$ |
| | | $x$ |

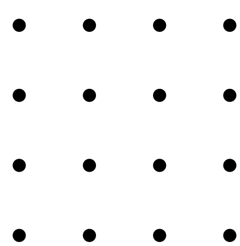
*Note:* Suppose each row/column/diagonal were to have product $P$. Let $c$ be the value of the central cell. Then the product of all 9 entries of the square is clearly $P^3$. However this must also equal $P^4/c^3$, since each entry is contained in exactly one of the four lines passing through $c$, aside from $c$ itself which is contained in all four (see diagram below). Thus $c = \sqrt[3]{P}$. In particular, if $P = 1$ then the central cell must take value 1, independent of the values of the remaining cells. This simplifies the anlaysis above and leads to the general solution shown below right.



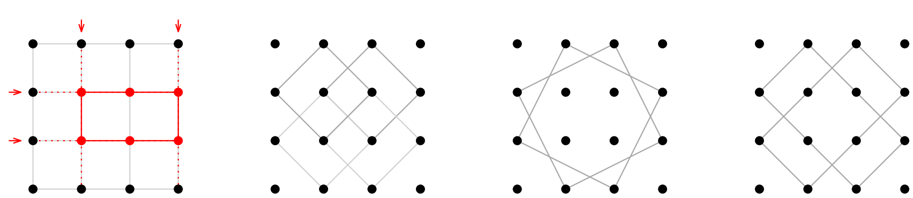

| | | |
|---|---|---|
| $\sqrt{\frac{b}{a}}$ | $a$ | $\frac{1}{\sqrt{ab}}$ |
| $\frac{1}{b}$ | $1$ | $b$ |
| $\sqrt{ab}$ | $\frac{1}{a}$ | $\sqrt{\frac{a}{b}}$ |

Identical arguments apply to *additive* magic squares. If each row/column/diagonal sums to $S$, then the central square is $S/3$. In fact we can pass between multiplicative and additive magic squares via the logarithm/exponential. ☐

7. How many rectangles can be formed by joining four dots in the $4 \times 4$ square lattice?



**Solution:** Each rectangle with sides parallel to the lines of the lattice is uniquely determined by choosing two points in the first row and first column, as illustrated below (left). Thus there are $\binom{4}{2}\binom{4}{2} = 36$ such rectangles.



One then finds four oblique squares with sides $\sqrt{2}$, two with side $\sqrt{5}$, and two $\sqrt{2} \times 2\sqrt{2}$ rectangles. This is a total of $36 + 4 + 2 + 2 = 44$ rectangles. □

8. The sequence $a_0, a_1, a_2, \ldots$ satisfies $a_n = a_{n-1} + a_{n+1}$ for $n \geq 1$.

   If $a_1 = 2019$ and $a_{2019} = 1$, find $a_{2000}$.

**Solution:** We have $a_{n+1} = a_n - a_{n-1}$ for $n \geq 1$. Applying this rule with $a_1 = 2019$ and $a_2 = x$ gives $(a_1, a_2, a_3, \ldots) = (2019, x, x - 2019, -2019, -x, 2019 - x, 2019, x, \ldots)$. We see that the sequence repeats itself with a period of length 6. Since $2000 \equiv 2 \pmod{6}$, we have $a_{2000} = a_2 = x$. But $2019 \equiv 3 \pmod{6}$ gives $1 = a_{2019} = a_3 = x - 2019$, so $x = 2020$.

Alternatively, get $a_n = a_{n-1} - a_{n-2} = (a_{n-2} - a_{n-3}) - an - 2 = -a_{n-3}$ for $n \geq 3$, so that $a_n = -a_{n-3} = -(-a_{n-6}) = a_{n-6}$ for $n \geq 6$. Thus the sequence is periodic with period 6. As above, it follows that $a_3 = a_{2019} = 1$, and then $a_2 = a_1 + a_3 = 2020$.

*Note:* The periodicity of this sequence is related to the fact that setting $a_n = \alpha^n$ converts $a_n = a_{n+1} + a_{n-1}$ into $\alpha^n = \alpha^{n+1} - \alpha^{n-1}$. Dividing by $\alpha^{n-1}$ gives $\alpha = \alpha^2 + 1$. The roots of this equation satisfy $\alpha^3 = -1$, since $\alpha^3 + 1 = (\alpha + 1)(\alpha^2 - \alpha + 1)$, and it turns out that this accounts for the fact that $a_n = -a_{n-3}$. □

# Pairs Relay

P-A. My brother is 5 years older than me, and I am 2 years older than my sister. In seven years our average age will be twice what it is now.

Let $\mathsf{A}$ be my current age. $\boxed{\text{Pass on } \mathsf{A}}$

**Solution:** Today the average age is $(\mathsf{A}+\mathsf{A}+5+\mathsf{A}-2)/3 = \mathsf{A}+1$. In 7 years the average age will be $\mathsf{A}+8$. So $2(\mathsf{A}+1) = \mathsf{A}+8$ gives $\mathsf{A} = 6$. □

P-B. You will receive $\mathsf{A}$.

Let $n!$ denote the product of all positive integers less than or equal to $n$. For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Calculate:

$$\mathsf{B} = \frac{(\mathsf{A}+1)! - \mathsf{A}!}{(\mathsf{A}-1)!}$$

$\boxed{\text{Pass on } \mathsf{B}}$

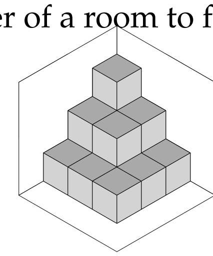
**Solution:** Common factor and cancel to get

$$\mathsf{B} = \frac{\mathsf{A}!(\mathsf{A}+1-1)}{(\mathsf{A}-1)!} = \frac{\mathsf{A}(\mathsf{A}-1)! \cdot \mathsf{A}}{(\mathsf{A}-1)!} = \mathsf{A}^2.$$

With $\mathsf{A} = 6$ get $\mathsf{B} = 36$. □

P-C. You will receive $\mathsf{B}$.

Unit cubes are stacked in the corner of a room to form a pyramid, as shown below.



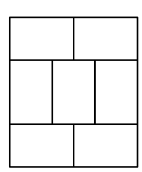
The stacking continues in this manner until the bottom layer contains $\mathsf{B}$ cubes. Let $\mathsf{C}$ be the exposed surface area of the resulting pyramid (*i.e.* the area that can be seen from inside the room).

$\boxed{\text{Pass on } \mathsf{C}}$

**Solution:** We must have $\mathsf{C} = n^2$ for some $n$. Then the surface area is $\mathsf{D} = n^2 + 2(1+2+\cdots+n) = n^2 + n(n+1) = n(2n+1)$. With $\mathsf{C} = 36$ get $n = 6$ and thus $\mathsf{D} = 6(13) = 78$. □

P-D. You will receive $\mathsf{C}$

A rectangle with perimeter $\mathsf{C}$ has been tiled with congruent rectangles in the manner illustrated below.



Let $\mathsf{D}$ be the perimeter of each small rectangle. $\boxed{\text{Done!}}$

**Solution:** Let $x, y$ be the width and height of each rectangle. Clearly $2x = 3y$. The perimeter of the large rectangle is $\mathsf{C} = 2(2x+2y+x) = 6x+4y = 13y$, and the perimeter we seek is $\mathsf{D} = 2x+2y = 5y$. Thus $\mathsf{D} = 5\mathsf{C}/13$. With $\mathsf{C} = 78$ get $\mathsf{D} = 30$. □

# Individual Relay

I-A. Let $\mathsf{A}$ be the number of points with integer coordinates that lie on the line segment joining $P = (-8, -13)$ and $Q = (10, 11)$.

**Note:** Include $P$ and $Q$ in your count! $\boxed{\text{Pass on A}}$

**Solution:** The slope of the segment is $\frac{11-(-13)}{10-(-8)} = \frac{24}{18} = \frac{4}{3}$, so we get one integer point for every three $x$-units. Thus $\mathsf{A} = \frac{18}{3} + 1 = 7$. $\square$

I-B. You will receive $\mathsf{A}$.

A collection of identical unit cubes are fastened together to form a large $\mathsf{A} \times \mathsf{A} \times \mathsf{A}$ cube. The big cube is painted on all sides and then separated again into the original cubes. Let $\mathsf{B}$ be the number of these small cubes which remain completely unpainted.

$\boxed{\text{Pass on B}}$

**Solution:** The unpainted small cubes form a cube of side $\mathsf{A} - 2$ within the large cube. Thus $\mathsf{B} = (\mathsf{A} - 2)^3$. With $\mathsf{A} = 5$ get $\mathsf{B} = 125$. $\square$

I-C. You will receive $\mathsf{B}$.

Let $\mathsf{C}$ be the number of ways the digits $2, 8$ and $\sqrt[3]{\mathsf{B}}$ can be rearranged to form a three-digit number divisible by 3.

$\boxed{\text{Pass on C}}$

**Solution:** A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So either none or all of the arrangements will be divisible by 3, depending on whether $2 + 8 + \sqrt[3]{\mathsf{B}}$ is divisible by 3. When $\mathsf{B} = 125$ this sum is 15, which is divisible by 3. Hence $\mathsf{C} = 3! = 6$. $\square$

I-D. You will receive $\mathsf{C}$.

Two sides of an isosceles triangle have lengths $\mathsf{C}$ and $\mathsf{C}/3$.

Let $\mathsf{D}$ be the area of the triangle. $\boxed{\text{Done!}}$

**Solution:** Since $\mathsf{C} > 2(\frac{\mathsf{C}}{3})$, the sides are $\mathsf{C}, \mathsf{C}$ and $\frac{1}{3}\mathsf{C}$. The altitude to the side of length $\frac{1}{3}\mathsf{C}$ is of length $\mathsf{C}\sqrt{1 - (\frac{1}{6})^2} = \frac{1}{6}\mathsf{C}\sqrt{35}$. The area is therefore $\mathsf{D} = \frac{1}{2}(\frac{1}{3}\mathsf{C})(\frac{1}{6}\mathsf{C}\sqrt{35}) = \frac{1}{36}\mathsf{C}^2\sqrt{35}$. With $\mathsf{C} = 6$ get $\mathsf{D} = \sqrt{35}$. $\square$

# Team Questions Answer Key

1. 110
2. $\{4038, 673, -2692\}$.
3. $\frac{8}{3}$
4. 90
5. $7 + 2\sqrt{3}$
6. $\frac{3}{2}$
7. 44
8. 2020

# Pairs Relay Answer Key

A. 6

B. 36

C. 78

D. 30

# Individual Relay Answer Key

A. 7

B. 125

C. 6

D. $\sqrt{35}$