Polya's Paragon

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For this month's installment we are going to take a little detour from the standard realm of mathematics and examine a very interesting problem in logic. This problem involves a group of 10 pirates who have come across a treasure consisting of 10 equally sized bars of gold. They struggle to come up with some equitable way to divide the bounty and eventually conclude that a democratic method would be best. So, the pirates number themselves by rank with the captain being 1, and the lackey being 10. Pirate 10 begins by suggesting a way to divide the gold. Then, a vote is taken among all the pirates (including the one who gave the proposal) on whether they like this particular suggestion. If at least 50% are in favour, the gold is divided up according to the proposal and they go on their way. If more than 50% refuse, the pirate giving the proposal is killed and they repeat the procedure with the pirate with the next lower number.

There are a couple of assumptions that we need to make about the pirates in question. First of all, they are perfectly logical and they know that every other pirate is as well. Secondly, they are infinitely greedy - that is to say they will always reject a proposal if they know they can get more later by doing so. Thirdly, they are bloodthirsty. That is to say, if they can get the same amount of gold from a later proposal, they will refuse the current proposal just because they would rather kill another pirate than not (Hey, they are pirates after all!).

So, the problem is as follows: If you are the 10^{th} pirate, what do you propose to get the most gold bars?

A slight variation of this problem appeared in the May, 1999 issue of Scientific American. Some of the assumptions I've used are a little different and lead to some very interesting results.

STOP READING NOW until you have tried the problem yourself.

First of all, it may not be entirely clear that there is anything you can suggest that will not end in your death! In problems like this one, we can only decide what the 10^{th} pirate will do if we know with certainty what the 9^{th} pirate will do if you die. Therefore, it seems like we should work backwards towards a solution.

If there happened to be only 1 pirate, he would simply suggest taking all the gold and get a unanimous vote! Unfortunately, this situation will never occur. If there were two pirates, number 2 would simply suggest that he would take all the gold and give the captain nothing. He will clearly get at least 50% of the votes since he will vote for himself!

We now run into something a little more unexpected. If there were 3 pirates left, what does number 3 do? He only needs one other vote besides his own to make his proposal pass. Since he knows the captain gets nothing if he dies, he can suggest 9 for himself and 1 for the captain. Clearly this is better for both of them than if he dies, so the captain will agree and pirate number 2 goes away empty handed.

Continuing along these lines, number 4's proposal should be 9 bars for himself and 1 for pirate number 2 (who would otherwise get nothing if number 4 is killed). Number 5 should then "bribe" both 1 and 3 with 1 gold bar each and keep 8 for himself to get a majority vote of 3/5. Clearly a pattern is emerging. We can easily verify that pirate number 10 can suggest keeping 6 for himself and giving 1 bar to each of 2, 4, 6 and 8. He gets exactly five of the ten votes and comes away quite well off!

The following table lists the proposals each pirate makes if it becomes their turn. The entry in the n^{th} row and m^{th} column represents how much pirate m will receive with pirate n's proposal.

-	1	2	3	4	5	6	7	8	9	10
1	10	-	-	-	-	-	-	-	-	-
2	0	10	-	-	-	-	-	-	-	-
3	1	0	9	-	-	-	-	-	-	-
4	0	1	0	9	-	-	-	-	-	-
5	1	0	1	0	8	-	-	-	-	-
6	0	1	0	1	0	8	-	-	-	-
7	1	0	1	0	1	0	7	-	-	-
8	0	1	0	1	0	1	0	7	-	-
9	1	0	1	0	1	0	1	0	6	-
10	0	1	0	1	0	1	0	1	0	6

This leads to other interesting questions about pirates such as these. Consider a variant to the original problem where a proposal requires $> 50\%$ of the votes to be accepted. We can repeat process as above and discover some interesting results. Again, I suggest that you try working on this new problem for a while before continuing.

When there are only 2 pirates left, the captain will never accept the proposal because he would always rather kill number 2 and take all the gold. Since 50% is no longer good enough, pirate 2 cannot save his own life. So given this, what does pirate 3 propose if it becomes his turn? Clearly we can use the logic from above and say that he can keep 9 bars and give 1 bar to number 2. What would happen if he proposes to keep all the gold himself? In particular, how does pirate 2 vote? If he votes for the proposal he gets no gold but if he votes against it he dies because he cannot make any good proposal next! We did specify, however, that pirates are bloodthirsty so if he is going to get 0 gold he would rather vote no despite the fact that it would mean he dies next.

Now we get to number 4. He needs 2 of the other 3 to vote yes to get over 50% approval so he must “bribe” the first two pirates with 1 and 2 gold respectively - keeping 7 for himself. We can continue in this line of reasoning to get the following table:

-	1	2	3	4	5	6	7
1	10	-	-	-	-	-	-
2	-	-	-	-	-	-	-
3	0	1	9	-	-	-	-
4	1	2	0	7	-	-	-
5	2	0	1	0	7	-	-
6	0	1	2	1	0	6	-
7	1	2	0	0	1	0	6

At this point I should mention that this is not the only proposal that pirate 7 can make and still be accepted. For instance he could just as easily have proposed:

-	1	2	3	4	5	6	7
6	0	1	2	1	0	6	-
7	1	0	0	2	1	0	6

This gets him the same amount of gold so there is no reason for him to pick one of these proposals over the other. Now the really interesting consequence of this is that pirate 8 must account for the fact that it is unknown which of these proposals pirate 7 will make. So, if he tries to get pirate 2's vote by giving him 1 gold bar, he will not succeed because pirate 2 may get more by refusing. So, to avoid the use of probabilities we can say that pirate 8 wants to ensure that his proposal will always pass the vote. So to get a particular pirate's vote he must offer more than the maximum that pirate could get in the next round. I will use *'s to indicate values that can be swapped within a particular proposal. Now, the 6th and later rows look like this:

-	1	2	3	4	5	6	7	8	9	10
6	0	1	2	1	0	6	-	-	-	-
7	1	0*	0	2*	1	0	6	-	-	-
8	2	0	1	0	2	1	0	4	-	-
9	0	1	2*	1	0	0*	1	0	5	-
10	1	2*	0	2*	1	0	0*	1	0	3

It is interesting to notice that the 9^{th} pirate can actually do better than the 8^{th} if he ever gets to make a proposal. Also, the 10^{th} pirate has 3 different ways to make a proposal that will certainly pass the vote. It is a worthwhile exercise to see how far this table can be extended. Eventually we will reach a pirate that can not make a proposal that is guaranteed to pass. The first such pirate where this is true is number 14. I found it truly amazing, however, to discover that the 15^{th} pirate actually has a proposal that will be accepted!

Naturally, there are many more questions we can ask about such interesting pirates:

1. For both versions, given n pirates and m bars of gold, how many bars can the n^{th} pirate get?

2. For both versions, which values of n and m is there no good proposal for the n^{th} pirate?

3. In the second version, what happens if we introduce probabilities and say that pirates will vote yes if their "expected number of gold bars" is greater than if they vote no?

4. For both versions, what if the pirates decide that after a refused proposal ALL pirates that voted yes are killed? (This problem begins to infringe on classical game theory and is similar is some respects to a famous problem called the prisoner's dilemma).