Yoneda Theory for Double Categories

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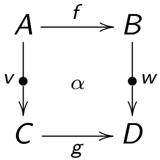
Introduction

- Basic tenet: The right framework for two-dimensional category theory is that of double categories
- The traditional approach was via:
  - 2-categories (like $\mathcal{C}at$)
  - bicategories (like $\mathcal{P}rof$)
- (Weak) double categories are a lot like bicategories
  - good because much of the well-developed theory of bicategories can be easily adapted to double categories
  - interesting when the theories differ
  - the Yoneda lemma is an instance of this
- The Yoneda lemma is the cornerstone of category theory
  - categorical universal algebra
  - categorical logic
  - sheaf theory
  - representability and adjointness
- The further development of double category theory depends on understanding the Yoneda lemma in this context

- Not *a priori* clear what representables should be
    - Where do they take their values?
    - What kind of "functor" are they?
- These questions lead to the double category $\mathbb{S}$et
    - sets, functions and spans
    - this is the double category version of the category of sets
    - the most basic double category
- Representables are lax functors into $\mathbb{S}$et
    - hence the double category $\mathbb{L}\text{ax}(\mathbb{A}^{op}, \mathbb{S}\text{et})$
    - a presheaf double category
    - understand its properties
    - horizontal and vertical arrows
    - composition (not trivial)
    - completeness
    - etc.

(Weak) Double Categories

- Objects
- Two kinds of arrows (horizontal and vertical)
- Cells that tie them together



(also denoted $\alpha : v \longrightarrow w$)

- Horizontal composition of arrows and cells give category structures
- Vertical composition gives "weak categories", i.e. composition is associative and unitary up to coherent *special* isomorphism (like for bicategories)
- Vertical composition of cells is as associative and unitary as the structural isomorphisms allow
- Interchange holds

The Basic Example: $\mathbb{S}\mathrm{et}$

- Objects are sets
- Horizontal arrows are functions
- Vertical arrows are spans
- Cells are commutative diagrams

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \uparrow \\
S & \longrightarrow & T \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

- Vertical composition uses pullback – it is associative and unitary up to coherent special isomorphism

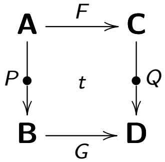
A Related Example: $\mathbf{V}$-$\mathbb{S}$et

- $\mathbf{V}$ a $\otimes$-category with coproducts
- Objects are sets
- Horizontal arrows are functions
- Vertical arrows $A \multimapdotbothA B$ are $A \times B$ matrices of objects of $\mathbf{V}$
- Cells are matrices of $\mathbf{V}$ morphisms
- Vertical composition is matrix multiplication

When $\mathbf{V} = \mathbf{Set}$, $\times$, we get $\mathbb{S}$et
When $\mathbf{V} = \mathbf{2}$, $\wedge$, we get sets with relations as vertical arrows

An Important Example: $\mathbb{C}\mathrm{at}$

- Objects are small categories
- Horizontal arrows are functors
- Vertical arrows $\mathbf{A} \to\!\!\!\!\bullet\;\, \mathbf{B}$ are profunctors, i.e. $\mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$
- Cells

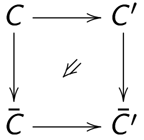


are natural transformations

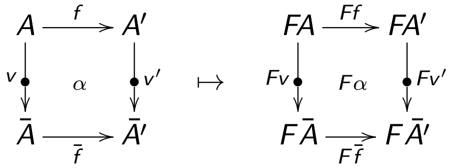
$$t : P(-,-) \longrightarrow Q(F-, G-)$$

Three General Constructions: $\mathbb{V}, \mathbb{H}, \mathbb{Q}$

- For $\mathcal{B}$ a bicategory, $\mathbb{V}\mathcal{B}$ is $\mathcal{B}$ made into a double category vertically, i.e. horizontal arrows are identities
- For $\mathcal{C}$ a 2-category, $\mathbb{H}\mathcal{C}$ is $\mathcal{C}$ made into a double category horizontally, i.e. vertical arrows are identities
- For $\mathcal{C}$ a 2-category, $\mathbb{Q}\mathcal{C}$ is Ehresmann's double category of "quintets". A general cell is



Lax Functors $F : \mathbb{A} \longrightarrow \mathbb{B}$



- Preserve horizontal compositions and identities
- Provide comparison special cells for vertical composition and identities

$$\phi(\bar{v}, v) : F\bar{v} \cdot Fv \Rightarrow F(\bar{v} \cdot v), \quad \phi(A) : \mathrm{id}_{FA} \Rightarrow F(\mathrm{id}_A)$$

- Satisfy naturality and coherence conditions, like for lax morphisms of bicategories

There are also oplax, normal, strong, and strict double functors

Examples

- $F : \mathbb{V}\mathcal{B} \longrightarrow \mathbb{V}\mathcal{B}'$ is a (lax) morphism of bicategories
- $F : \mathbb{H}\mathcal{C} \longrightarrow \mathbb{H}\mathcal{C}'$ is a 2-functor
- $\mathbb{C}\text{at} \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathbb{S}\text{et}$
  - $\text{Ob} : \mathbb{C}\text{at} \longrightarrow \mathbb{S}\text{et}$ is lax
  - $\text{Disc} : \mathbb{S}\text{et} \longrightarrow \mathbb{C}\text{at}$ is strong
  - $\pi_0 : \mathbb{C}\text{at} \longrightarrow \mathbb{S}\text{et}$ is oplax normal
- $\mathbf{1} \longrightarrow \mathbb{A}$ is a vertical monad
  - $\mathbf{1} \longrightarrow \mathbb{S}\text{et}$ is a small category
  - $\mathbf{1} \longrightarrow \mathbf{V}\text{-}\mathbb{S}\text{et}$ is a small $\mathbf{V}$-category

The Main Example: $\mathbb{A}(-, A) : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{et}$

$$\mathbb{A}(X, A) = \{f : X \longrightarrow A\}$$

$$\mathbb{A}(v, A) = \left\{ \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle v}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_A} \\ Y & \xrightarrow[g]{} & A \end{array} \right\}$$

The span projections are domain and codomain

- Horizontal functoriality is by composition
- Vertical comparisons

$$h(w, v) : \mathbb{A}(w, A) \otimes \mathbb{A}(v, A) \Rightarrow \mathbb{A}(w \cdot v, A)$$

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle v}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_A} \\ Y & \xrightarrow{g} & A \\ {\scriptstyle w}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_A} \\ Z & \xrightarrow[h]{} & A \end{array} \quad \mapsto \quad \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle w\cdot v}\big\downarrow\!\bullet & \beta\cdot\alpha & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_A} \\ Z & \xrightarrow[h]{} & A \end{array}$$

Natural Transformations of Lax Functors

$$t : F \longrightarrow G$$

- For every A, $tA : FA \longrightarrow GA$ (horizontal)
- For every $v : A \,-\!\!\bullet\!\!\!\rightarrow\, \bar{A}$,

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Fv}\,\downarrow\!\bullet & tv & \bullet\!\downarrow\,{\scriptstyle Gv} \\ F\bar{A} & \xrightarrow[t\bar{A}]{} & G\bar{A} \end{array}$$

- Horizontally natural
- Vertically functorial

Examples

- For lax $\mathbf{1} \longrightarrow \mathbb{S}\mathrm{et}$, we get functors
- For lax $\mathbf{1} \longrightarrow \mathbf{V}\text{-}\mathbb{S}\mathrm{et}$, we get $\mathbf{V}$-functors
- For $\mathbb{V}\mathcal{B} \longrightarrow \mathbb{V}\mathcal{B}'$, we get lax transformations which are identities on objects
- For $\mathbb{H}\mathcal{C} \longrightarrow \mathbb{H}\mathcal{C}'$, we get 2-natural transformations
- Every horizontal $f : A \longrightarrow A'$ gives a natural transformation

$$\mathbb{A}(-, f) : \mathbb{A}(-, A) \longrightarrow \mathbb{A}(-, A')$$

$$X \xrightarrow{x} A \qquad \mapsto \qquad X \xrightarrow{x} A \xrightarrow{f} A'$$

$$\begin{array}{ccc} X & \xrightarrow{x} & A \\ {\scriptstyle v}\,\big\downarrow\!\bullet & \xi & \bullet\!\big\downarrow\,{\scriptstyle \mathrm{id}_A} \\ Y & \xrightarrow[y]{} & A \end{array} \qquad \mapsto \qquad \begin{array}{ccccc} X & \xrightarrow{x} & A & \xrightarrow{f} & A' \\ {\scriptstyle v}\,\big\downarrow\!\bullet & \xi & \bullet\!\big\downarrow & \mathrm{id}_f & \bullet\!\big\downarrow\,{\scriptstyle \mathrm{id}_{A'}} \\ Y & \xrightarrow[y]{} & A & \xrightarrow[f]{} & A' \end{array}$$

The Yoneda Lemma

Theorem

For a lax functor $F : \mathbb{A}^{op} \longrightarrow \mathbb{S}\text{et}$ *and an object* A *of* $\mathbb{A}$*, there is a bijection between natural transformations* $t : \mathbb{A}(-, A) \longrightarrow F$ *and elements* $x \in FA$ *given by* $x = t(A)(1_A)$*.*

Corollary

Every natural transformation $t : \mathbb{A}(-, A) \longrightarrow \mathbb{A}(-, A')$ *is of the form* $\mathbb{A}(-, f)$ *for a unique* $f : A \longrightarrow A'$*.*

“Application”

The theory of adjoints for double categories was set out in [Grandis-Paré, *Adjoint for Double Categories*, Cahiers (2004)]. The left adjoint is typically oplax and the right adjoint lax. It is expressed in terms of conjoints in a strict double category $\mathbb{D}\mathrm{oub}$.

Example:

$$\pi_0 \dashv \mathrm{Disc} \dashv \mathrm{Ob}$$

Theorem

For $F : \mathbb{A} \longrightarrow \mathbb{B}$ oplax and $U : \mathbb{B} \longrightarrow \mathbb{A}$ lax, there is a bijection between adjunctions $F \dashv U$ and natural isomorphisms

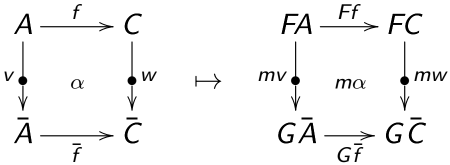
$$\mathbb{B}(F-,-) \longrightarrow \mathbb{A}(-,U-)$$

of lax functors $\mathbb{A}^{op} \times \mathbb{B} \longrightarrow \mathbb{S}\mathrm{et}$.

Vertical Structure of $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{B})$

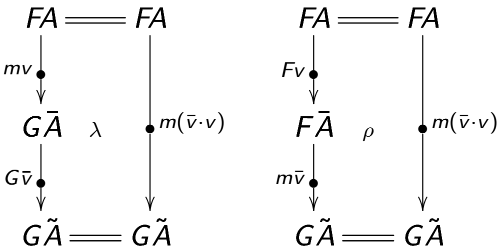
For F and G lax functors, $\mathbb{A} \longrightarrow \mathbb{B}$, a *module* [Cockett, Koslowski, Seely, Wood – *Modules*, TAC 2003] $m : F \multimap G$ is given by the following data.

- For every vertical arrow $v : A \multimap \bar{A}$ in $\mathbb{A}$ a vertical arrow $mv : FA \multimap G\bar{A}$
- For every cell α a cell $m\alpha$



Modules (continued)

- For every pair of vertical arrows $v : A \multimap \bar{A}$ and $\bar{v} : \bar{A} \multimap \tilde{A}$, left and right actions



satisfying

- Horizontal functoriality
- Naturality of λ and ρ
- Left and right unit laws
- Left, right and middle associativity laws

Examples

- For $\mathbf{1} \longrightarrow \mathbb{S}\mathrm{et}$, modules are profunctors
- For $F : \mathbb{A} \longrightarrow \mathbb{B}$ lax, $\mathrm{id}_F : F \,\bullet\!\!\!\rightarrow\, F$ is given by

$$\mathrm{id}_F(v) = \begin{array}{c} FA \\ {\scriptstyle Fv}\big\downarrow\bullet \\ F\bar{A} \end{array}$$

- (Main Example) For $v : A \,\bullet\!\!\!\rightarrow\, \bar{A}$ in $\mathbb{A}$

$$\mathbb{A}(-, v) : \mathbb{A}(-, A) \,\bullet\!\!\!\rightarrow\, \mathbb{A}(-, \bar{A})$$

$$\mathbb{A}(z, v) = \left\{ \begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle z}\big\downarrow\bullet & \xi & \bullet\big\downarrow{\scriptstyle v} \\ Y & \longrightarrow & \bar{A} \end{array} \right\}$$

Modulations

The cells of $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{B})$ are called *modulations* following [CKSW]

$$\begin{array}{ccc} F & \xrightarrow{t} & F' \\ {\scriptstyle m}\downarrow\bullet & \mu & \bullet\downarrow{\scriptstyle m'} \\ G & \xrightarrow[s]{} & G' \end{array}$$

- For every vertical $v : A \to\!\!\!\bullet\; \bar{A}$ we are given

$$\begin{array}{ccc} FA & \xrightarrow{tA} & F'A \\ {\scriptstyle mv}\downarrow\bullet & \mu v & \bullet\downarrow{\scriptstyle m'v} \\ G\bar{A} & \xrightarrow[s\bar{A}]{} & G'\bar{A} \end{array}$$

satisfying

- Horizontal naturality
- Equivariance

Example: A cell α of $\mathbb{A}$ produces a modulation $\mathbb{A}(-, \alpha)$

The Yoneda Lemma II

Theorem

Let $m : F \,-\!\!\!\bullet\!\!\!\to\, G$ be a module in $\mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{et})$ and $v : A \,-\!\!\!\bullet\!\!\!\to\, \bar{A}$ a vertical arrow of $\mathbb{A}$. Then there is a bijection between modulations

$$\begin{array}{ccc} \mathbb{A}(-, A) & \longrightarrow & F \\ {\scriptstyle \mathbb{A}(-,v)}\,\big\downarrow\!\bullet & \mu & \bullet\!\big\downarrow\,{\scriptstyle m} \\ \mathbb{A}(-, \bar{A}) & \longrightarrow & G \end{array}$$

and elements $r \in m(v)$ given by $r = \mu(v)(1_r)$.

Corollary

For $F : \mathbb{A}^{op} \longrightarrow \mathbb{S}\text{et}$ lax, an element $r \in F(v)$ is uniquely determined by a modulation

$$\begin{array}{ccc} \mathbb{A}(-,A) & \longrightarrow & F \\ {\scriptstyle \mathbb{A}(-,v)}\downarrow\bullet & \mu & \bullet\downarrow{\scriptstyle \mathrm{id}_F} \\ \mathbb{A}(-,\bar{A}) & \longrightarrow & F \end{array}$$

Corollary

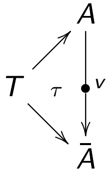
For $v : A \dashrightarrow\!\!\!\!\bullet\ \bar{A}$ and $v' : A' \dashrightarrow\!\!\!\!\bullet\ \bar{A}'$ in $\mathbb{A}$, every modulation

$$\begin{array}{ccc} \mathbb{A}(-,A) & \longrightarrow & \mathbb{A}(-,A') \\ {\scriptstyle \mathbb{A}(-,v)}\downarrow\bullet & \mu & \bullet\downarrow{\scriptstyle \mathbb{A}(-,v')} \\ \mathbb{A}(-,\bar{A}) & \longrightarrow & \mathbb{A}(-,\bar{A})' \end{array}$$

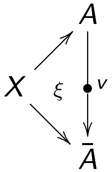
is of the form $\mathbb{A}(-,\alpha)$ for a unique cell $\alpha : v \longrightarrow v'$.

Application: Tabulators

A *tabulator* for a vertical arrow $v : A \to \bar{A}$ in a double category is an object T and a cell



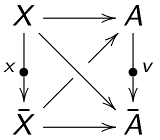
with universal properties:
(T1) For every cell



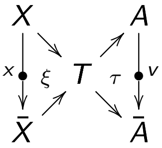
there is a unique horizontal arrow $x : X \longrightarrow T$ such that $\tau x = \xi$

Tabulators: 2-Dimensional Property

(T2) For every commutative tetrahedron of cells



there is a unique cell ξ such that



gives the tetrahedron in the "obvious" way.

Tabulators in $\mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{et})$

Let $m : F \xrightarrow{\bullet} G$ be a module. If it has a tabulator, T, we can use Yoneda to discover what it is. By Yoneda, elements of TA are in bijection with natural transformations $t : \mathbb{A}(-, A) \longrightarrow T$ which by T1 are in bijection with modulations

$$\begin{array}{ccc} \mathbb{A}(-,A) & \longrightarrow & F \\ {\scriptstyle \mathrm{Id}_{\mathbb{A}(-,A)}}\ \big\downarrow\bullet & \mu & \bullet\big\downarrow\ {\scriptstyle m} \\ \mathbb{A}(-,A) & \longrightarrow & G \end{array}$$

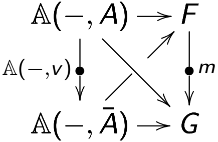
and as $\mathrm{Id}_{\mathbb{A}(-,A)} = \mathbb{A}(-, \mathrm{id}_A)$, such μ are in bijection with elements $r \in m(\mathrm{id}_A)$ by Yoneda II. So we define

$$T(A) = m(\mathrm{id}_A)$$

By Yoneda II, elements of $T(v)$ are in bijection with modulations

$$\begin{array}{ccc} \mathbb{A}(-,A) & \longrightarrow & T \\ {\scriptstyle \mathbb{A}(-,v)}\downarrow\!\bullet & \mu & \bullet\!\downarrow{\scriptstyle \mathrm{id}_T} \\ \mathbb{A}(-,\bar{A}) & \longrightarrow & T \end{array}$$

which correspond to commutative tetrahedra



and this tells us that we must have

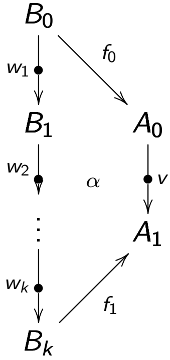
$$T(v) = (Gv \otimes m(\mathrm{id}_A)) \times_{m(v)} (m(\mathrm{id}_{\bar{A}}) \otimes Fv)$$

It is now straightforward to check that with these definitions, T is indeed the tabulator.

Lax Double Categories

Also called **fc**-multicategories, virtual double categories, multicategories with several objects

Like double categories, except vertical arrows don't compose. Instead multicells are given



denoted

$$\alpha : w_k, \ldots, w_2, w_1 \longrightarrow v$$

Lax Double Categories (continued)

There are identities $1_v : v \longrightarrow v$ and multicomposition: for compatible

$$\beta_i : x_{i1}, \ldots, x_{il_i} \longrightarrow w_i$$

we are given

$$\alpha(\beta_k, \ldots, \beta_1) : x_{11}, \ldots, x_{kl_k} \longrightarrow v$$

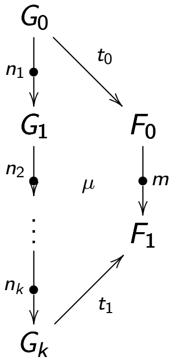
which is associative and unitary in the appropriate sense.
The composite $w_k, \ldots, w_1$ *exists* (or is *representable*) if there is a vertical arrow w and a special multicell

$$\iota : w_k, \ldots, w_1 \Rightarrow w$$

such that for every multicell α as above there exists a unique $\bar{\alpha} : w \longrightarrow v$ such that $\bar{\alpha}\iota = \alpha$.
The composite $w_k, \cdot \ldots \cdot w_1$ is *strongly representable* if ι has a stronger universal property for α's whose domain is a string containing the w's as a substring.

Multimodulations

A *multimodulation*



- For each path $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_k} A_k$, we are given

$$\mu(v_k, \dots, v_1) : n_k v_k \cdot \dots \cdot n_1 v_1 \longrightarrow m(v_k \cdot \dots \cdot v_1)$$

satisfying

- Horizontal naturality
- Left, right, inner equivariance ($k - 1$ conditions)

The Multivariate Yoneda Lemma

Theorem

For $m : F \nrightarrow G$ in $\mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{et})$ and $v_k, \ldots, v_1$ a path in $\mathbb{A}$, we have a bijection between multimodulations

$$\mathbb{A}(-, v_k), \ldots, \mathbb{A}(-, v_1) \longrightarrow m$$

and elements

$$r \in m(v_k \cdot \ldots \cdot v_1)$$

Corollary

The composite $\mathbb{A}(-, v_k) \cdot \ldots \cdot \mathbb{A}(-, v_1)$ is represented by $\mathbb{A}(-, v_k \cdot \ldots \cdot v_1)$.

Theorem

All composites (k-fold) are representable in $\mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{et})$.

Remark: Don't know if they are strongly representable. Don't think so, but we conjecture that they are if $\mathbb{A}$ satisfies a certain factorization of cells condition.

The Yoneda Embedding

$$Y : \mathbb{A} \longrightarrow \mathbb{L}\mathsf{ax}(\mathbb{A}^{op}, \mathbb{S}\mathsf{et})$$

$$Y(A) = \mathbb{A}(-, A)$$

$$Y(v) = \mathbb{A}(-, v)$$

- Y is a morphism of lax double categories
- It preserves identities and composition (up to iso)
- It is full on horizontal arrows
- It is full on multicells
- It is dense

Density

For $F : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{et}$ construct the double category of elements of F $\mathbb{E}\mathrm{l}(F)$

- Objects are (A, x) with $x \in FA$
- Horizontal arrows $f : (A, x) \longrightarrow (A', x')$ are $f : A \longrightarrow A'$ such that $F(f)(x') = x$
- Vertical arrows $(v, r) : (A, x) \bullet\!\!\longrightarrow (\bar{A}, \bar{x})$ are $v : A \bullet\!\!\longrightarrow \bar{A}$ and $r \in F(v)$ such that $r_0 = x$ and $r_1 = \bar{x}$
- Cells are cells α of $\mathbb{A}$ such that $F(\alpha)(r') = r$

There is a strict double functor $P : \mathbb{E}\mathrm{l}(F) \longrightarrow \mathbb{A}$

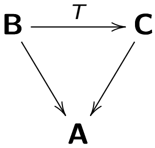
Theorem

$$F \cong \lim_{\Rightarrow} YP$$

Example: $\mathbb{A}$ Horizontally Discrete

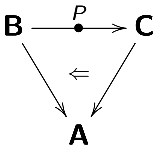
Let $\mathbb{A} = \mathbb{V}\mathbf{A}$ for a category $\mathbf{A}$.

- For a lax functor $F : \mathbb{V}\mathbf{A}^{op} \longrightarrow \mathbb{S}\mathrm{et}$, $\mathbb{E}\mathrm{l}(F)$ is also horizontally discrete, i.e. $\mathbb{V}\mathbf{B}$. Thus F corresponds to an arbitrary category over $\mathbf{A}$, $\mathbf{B} \longrightarrow \mathbf{A}$ (Bénabou)
- The representable $\mathbb{A}(-, A)$ corresponds to $A : \mathbf{1} \longrightarrow \mathbf{A}$
- A natural transformation $t : F \longrightarrow G$ corresponds to a functor over $\mathbf{A}$,

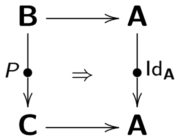


Example (continued)

- A module $m : F \nrightarrow G$ corresponds to a "profunctor over $\mathbf{A}$"

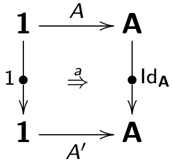


i.e.



a cell in $\mathbb{C}\mathrm{at}$

- The representable $\mathbb{A}(-, a)$ corresponds to



- Modulations are commutative prisms
- Thus $\mathbb{L}\mathrm{ax}(\mathbb{V}\mathbf{A}^{op}, \mathbb{S}\mathrm{et}) \simeq \mathbb{C}\mathrm{at} /\!\!/ \mathbf{A}$