The “Triple Category” of Bicategories

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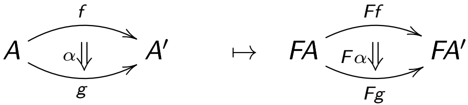
Introduction

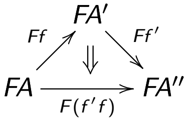
- Common wisdom: The study of 2-dimensional structures leads inexorably to 3-dimensional ones.
- Consider the theory of bicategories: What 3-dimensional structure do they present?
- No matter how it's set up, something doesn't work!
- Grandis and I looked at weak triple categories. We found no full-fledged natural examples!
- Now think this is a basic fact about triple categories. (Like the symmetry isomorphisms for monoidal categories.)

Bicategories: What's there (Bénabou)

- 0-cells - Bicategories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$
- 1-cells
  - Lax functors $F : \mathcal{A} \longrightarrow \mathcal{B}$



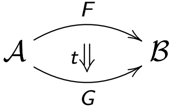
  - Functorial on the α's (vertical composition)
  - Horizontally lax

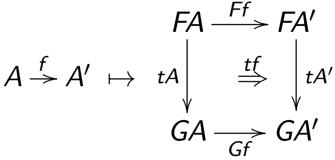


  - Satisfy coherence conditions
  - Horizontal composition of 2-cells is preserved as much as it can be
- Occur in practice
- Compose, strictly associative and unitary

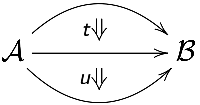
What's there: 2-cells

Lax transformations





- t respects composition of 1-cells
- t is natural with respect to 2-cells
- Occur in practice
- Compose vertically



associative and unitary up to "canonical" isomorphisms (3-cells)

What's there: Other 2-cells

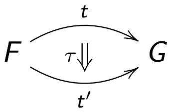
- Oplax transformations

$$A \xrightarrow{f} A' \quad \mapsto \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle tA}\downarrow & \Downarrow tf & \downarrow{\scriptstyle tA'} \\ GA & \xrightarrow[Gf]{} & GA' \end{array}$$

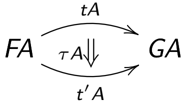
- Also occur in practice
- No hint as to which should be called lax or oplax
- There are also strong transformations

What's there: 3-cells

- Modifications



  - For every A

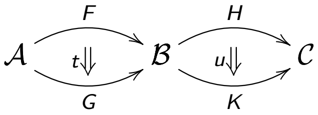


  - For every $f : A \longrightarrow A'$

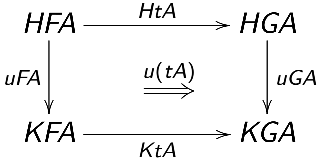
$$\begin{CD} Gf \cdot tA @>{tf}>> tA' \cdot Ff \\ @V{Gf \cdot \tau A}VV @VV{\tau A' \cdot Ff}V \\ Gf \cdot t'A @>>{t'f}> t'A' \cdot Ff \end{CD}$$

- Lax functors, lax transformations, modifications yield a bicategory $\mathbb{L}\mathrm{ax}(\mathcal{A}, \mathcal{B})$

What's not there: Horizontal composition



- For A there are two choices for $(ut)A : HFA \longrightarrow KGA$



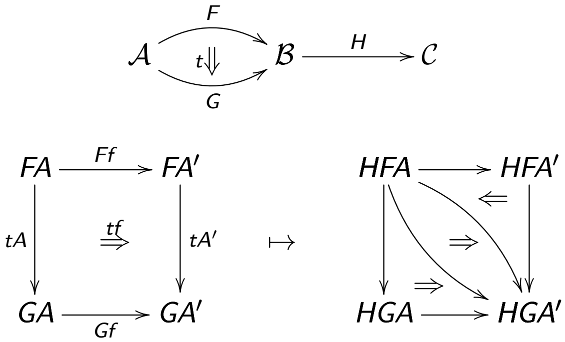
- Neither choice is associative: e.g. if $v : L \longrightarrow M$ is another transformation, and we use the top composite

$$(v(ut))A = vKGA \cdot L(uGA \cdot HtA)$$

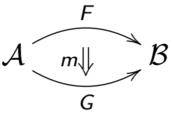
$$((vu)t)A = vKGA \cdot LuGA \cdot LHtA$$

Horizontal composition - continued

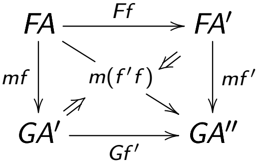
- Worse - neither choice will produce a laxity cell $ut(f)$
- Even whiskering doesn't



Modules [CKSW]



- $m : A \xrightarrow{f} A' \quad \mapsto \quad mf : FA \longrightarrow GA'$
- $A \xrightarrow{f} A' \xrightarrow{f'} A''$



  - $\rho_m : mf' \cdot Ff \longrightarrow m(f'f)$
  - $\lambda_m : Gf' \cdot mf \longrightarrow m(f'f)$
- m is functorial on 2-cells
- ρ_m and λ_m are natural
- ρ_m and λ_m are associative (3 equations)
- ρ_m, λ_m are unitary (2 equations)

Modules: Examples

- $\mathrm{Lax} \quad \mathbf{1} \longrightarrow \mathbb{S}\mathrm{pan}$ is a category
- Module is a profunctor
- Functors are *not* lax transformations
  They are oplax transformations for which tA is a *function*
- More generally $\mathrm{Lax} \quad \mathbf{1} \longrightarrow \mathbf{V}\text{-}\mathcal{M}\mathrm{at}$ is a **V**-category
  Module is a **V**-profunctor

Modules: Lax transformations

- A lax transformation $t : F \longrightarrow G$ gives a module $\overline{t} : F \longrightarrow G$

  For $f : A \longrightarrow A'$, $\quad \overline{t}f = FA \xrightarrow{Ff} FA' \xrightarrow{tA'} FA'$

$$\rho_{\overline{t}} : \overline{t}f' \cdot Ff \longrightarrow \overline{t}(f'f)$$

$$tA'' \cdot Ff' \cdot Ff \xrightarrow{tA'' \cdot \phi} tA'' \cdot F(f'f)$$

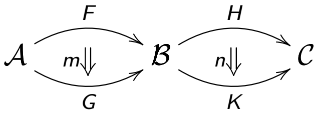
$$\lambda_{\overline{t}} : Gf' \cdot \overline{t}f \longrightarrow \overline{t}(f'f)$$

$$Gf' \cdot tA' \cdot Ff \xrightarrow{tf' \cdot Ff} tA'' \cdot Ff' \cdot Ff \xrightarrow{tA'' \cdot \phi} tA'' \cdot F(f'f)$$

- An oplax transformation $t : F \longrightarrow G$ also produces a module $\tilde{t} : F \longrightarrow G$

  $\tilde{t}f = FA \xrightarrow{tA} GA \xrightarrow{Gf} GA'$

Modules: Horizontal composition



$$(A \xrightarrow{f} A') \quad \xmapsto{m} \quad (FA \xrightarrow{mf} GA') \quad \xmapsto{n} \quad (HFA \xrightarrow{n(mf)} KGA')$$

$$(nm)(f) = n(mf)$$

$$\rho_{nm} : (nm)f' \cdot HFf \longrightarrow (nm)(f'f)$$

$$n(mf') \cdot HFf \xrightarrow{\rho_n} n(mf' \cdot Ff) \xrightarrow{n(\rho_m)} n(m(f'f))$$

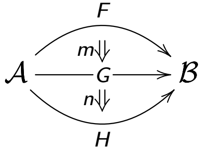
$$\lambda_{nm} : KGf' \cdot (nm)f \longrightarrow (nm)(f'f)$$

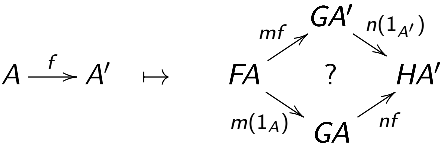
$$KGf' \cdot n(mf) \xrightarrow{\lambda_n} n(Gf' \cdot mf) \xrightarrow{n(\lambda_m)} n(m(f'f))$$

- Strictly associative and unitary

Modules: Vertical composition

Does not usually exist!





Theorem (CKSW)

If $\mathcal{B}$ is locally cocomplete and $\mathcal{A}$ small, then the composite $n \cdot m$ exists, and is associative and unitary up to coherent isomorphism.

Proof.

- $(n \cdot m)(f) = \varinjlim_{\alpha}(nf_2 \cdot mf_1) \qquad \alpha : f_2 f_1 \longrightarrow f.$
- There is an unbiased triple composite

$$(p \cdot n \cdot m)(f) = \varinjlim_{\alpha}(pf_3 \cdot nf_2 \cdot mf_1)$$

  taken over all $\alpha : f_3 f_2 f_1 \longrightarrow f$

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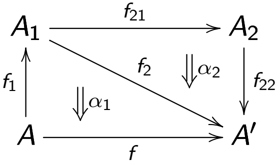
$$((p \cdot n) \cdot m)(f) \simeq \varinjlim (p \cdot n)(f_2) \cdot mf_1$$

$$\simeq \varinjlim (\varinjlim \ f_{22} \cdot nf_{21}) \cdot mf_1$$

$$\simeq \varinjlim (pf_{22} \cdot nf_{21} \cdot mf_1)$$

  where the last colimit is taken over all pairs $\alpha_1 : f_2 f_1 \longrightarrow f$ and

  $\alpha_2 : f_{22} f_{21} \longrightarrow f_2$



□

Theorem (CKSW)

If n is of the form $\bar{t}$ for a lax transformation t, then $n \cdot m$ exists. Similarly if m is of the form $\tilde{t}$ for an oplax transformation then $n \cdot m$ exists.

- $(\bar{t} \cdot m)(f) = (FA \xrightarrow{mf} GA' \xrightarrow{tA'} HA')$

  $(n \cdot \tilde{t})(f) = (FA \xrightarrow{tA} GA \xrightarrow{nf} HA')$
- $\bar{t'} \cdot \bar{t} = \overline{t' \cdot t}$ and $\tilde{t'} \cdot \tilde{t} = \widetilde{t' \cdot t}$

Three-dimensional structure: Modulations [CKSW]

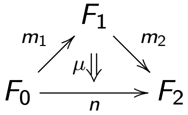
There are morphisms of modules, 3-cells for $\mathcal{B}icat$, called *modulations*

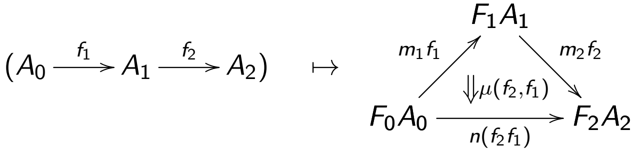
$$\mu : m \longrightarrow n : F \longrightarrow G$$

- $$f : A \longrightarrow A' \quad \mapsto \quad FA \underset{nf}{\overset{mf}{\underset{\mu f \Downarrow}{\rightrightarrows}}} GA'$$
- Natural in f
- Equivariant

Three-dimensional structure: Multimodulations

Bimodulation



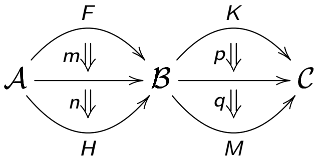
- 

- μ is natural in f_1 and f_2
- "Equivariant"

They determine the composite of m_1 and m_2

- Composition becomes a question of representability

Interchange



$$(qn)\cdot(pm)(f) = \varinjlim_{\alpha} q(nf_2)\cdot p(mf_1); \qquad (\alpha : f_2 f_1 \longrightarrow f)$$

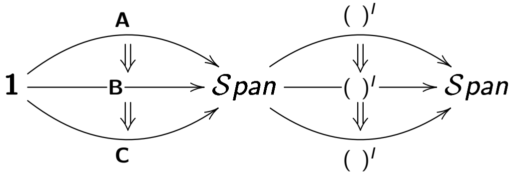
$$(q\cdot p)(n\cdot m)(f) = \varinjlim_{\alpha} q((n\cdot m)f)\cdot p((n\cdot m)f)$$

$$= \varinjlim_{\alpha} q(\varinjlim_{\alpha_2}(nf_{22}\cdot mf_{21}))\cdot p(\varinjlim_{\alpha_1}(nf_{12}\cdot mf_{11}))$$

$$(\alpha_1 : f_{12}f_{11} \longrightarrow f_1 \quad \alpha_2 : f_{22}f_{21} \longrightarrow f_2)$$

- No reason for p or q to preserve $\varinjlim$
- p and q don't preserve composition

Example



$P : \mathbf{A} \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow \mathbf{B}$, $Q : \mathbf{B} \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow \mathbf{C}$ profunctors

$(\ )^I \longrightarrow (\ )^I$ identity modules

$$P^I : \mathbf{A}^I \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow \mathbf{B}^I$$

$$P^I(\langle A_i \rangle, \langle B_i \rangle) = \prod_i P(A_i, B_i)$$

An element is a family of elements $A_i \xrightarrow{x_i} B_i$

$$Q^I \otimes P^I \longrightarrow (Q \otimes P)^I$$

An element on LHS is an equivalence class of pairs of families

$$[\langle A_i \overset{x_i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} B_i \rangle_i, \langle B_i \overset{y_i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} C_i \rangle_i]_{\langle B_i \rangle}$$

An element on RHS is a family of equivalence classes

$$\langle [A_i \overset{x_i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} B_i \overset{y_i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} C_i] \rangle_i$$

In the first, the equivalence involves a path of families whereas in the second, it's a family of paths which can have unbound length if I is infinite.

Conclusion

- [CKSW] talk about a smooth three-dimensional structure of bicategories, lax functors, modules and modulations.
- What is this?
- Perhaps we can't have vertical and horizontal composition simultaneously.
- Interchange should be relaxed.