CT95

CELEBRATING 50 YEARS OF
CATEGORY THEORY

JULY 9- 15 , 1995
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA

ACCESSIBLE CATEGORIES

- WITH M. MAKKAI – CONTEMPORARY MATH 104

- LARGE CLASS OF CATEGORIES WITH GOOD CLOSURE PROPERTIES

- CONTAINS ALL ALGEBRAIC CATEGORIES

- GIVES RESULTS ABOUT THEM BY GOING OUTSIDE & THEN RETURNING

- GOOD CONTEXT IN WHICH TO STUDY MORE RESTRICTED CLASSES OF CATEGORIES

- PURLY INTRINSIC DEFINITION FREE FROM ANY REF. TO UNDERLYING STRUCTURES

- HAS ITS ROOTS IN
  - MAKKAI-REYES – CAT. LOGIC
  - GABRIEL-ULMER – LOC. PRES CATS
  - EHERESMANN-LAIR – SKETCHES
  - KELLY-STREET – 2-CATS

κ- FILTERED COLIMITS

κ ∞ REG. CARD.

$\underline{\underline{I}}$ **κ- FILTERED** IF EVERY DIAG. $\underline{\underline{D}} \longrightarrow \underline{\underline{I}}$
WITH $\#\underline{\underline{D}} < \kappa$ HAS A COCONE

(0) $\underline{\underline{I}} \neq \emptyset$

(1) $\left.\begin{matrix} I_1 \\ I_2 \\ \vdots \\ I_\alpha \end{matrix}\right\} {<\kappa} \dashrightarrow J$

(2) $I \underset{<\kappa}{\rightrightarrows} J \dashrightarrow K$

FOR POSETS SAY κ- DIRECTED

THM: IN $\underline{\underline{\mathrm{SET}}}$, $\underline{\underline{I}}$ COLIM COMMUTE
WITH $\underline{\underline{J}}$ LIM* FOR ALL $\#\underline{\underline{J}} < \kappa$
IFF $\underline{\underline{I}}$ κ- FILTERED.

* $\Gamma : \underline{\underline{I}} \times \underline{\underline{J}} \to \underline{\underline{\mathrm{SET}}} \rightsquigarrow \varinjlim_{I} \varprojlim_{J} \Gamma(I,J) \cong \varprojlim_{J} \varinjlim_{I} \Gamma(I,J)$

NOTE 1: LIM ALWAYS COMMUTE WITH LIM

NOTE 2: EVERY SET IS A κ- DIRECTED
UNION OF SETS OF CARD $< \kappa$.

NOTE 3: ALSO TRUE IN ALGEBRAIC CATS.

MODELS OF A SKETCH

$\mathcal{S} = (\mathbb{G}, \mathbb{D}, \mathbb{L}, \mathbb{C})$ SMALL

MOD($\mathcal{S}$) = MODELS IN SET. TOO GENERAL? NO!

① MOD($\mathcal{S}$) HAS κ-FILTERED COLIMITS, COMPUTED AS IN SET, FOR ANY REG. CARD κ > CARD OF CONES IN $\mathbb{L}$.

② DOWNWARD LÖWENHEIM-SKOLEM THM SAYS IF A THEORY HAS A MODEL, IT HAS A SMALL ONE. APPLIES IN THIS CASE. IN FACT, FOR ANY "SUBSET" OF A MODEL OF CARD < κ', THERE IS A SUBMODEL OF CARD < κ' CONTAINING THE SUBSET.

SO EVERY MODEL IS A κ'-DIRECTED UNION OF SUBMODELS OF CARD < κ'.

FREE THIS FROM SETS & CARDINALITY.

ACCESSIBLE CATEGORIES

$\underline{A}$ IS κ-ACCESSIBLE IF

① $\underline{A}$ HAS κ-FILT $\varinjlim$

② THERE IS A SMALL SUBCATEGORY $\underline{B} \subseteq \underline{A}$ WHOSE OBJECTS ARE κ-PRESENTABLE SUCH THAT EVERY OBJECT OF $\underline{A}$ IS A κ-FILT COLIM OF OBJECTS OF $\underline{B}$.

A IS κ-PRESENTABLE IFF $\underline{A}(A,-) : \underline{A} \to \underline{\mathrm{SET}}$ PRESERVES κ-FILT COLIM.

$$\begin{tikzcd}
{} \arrow[r] & A_\alpha \arrow[r] & \cdots \arrow[r] & A_\beta \arrow[r] & \cdots \arrow[r] & \varinjlim A_\alpha \\
 & & & A \arrow[u, dotted] \arrow[urr] & &
\end{tikzcd}$$

A FUNCTOR IS κ-ACCESSIBLE IF IT PRESERVES κ-FILT COLIM.

ACCESSIBLE MEANS κ-ACC SOME κ.

$\boxed{\text{SMALLNESS CONDITION}}$

EXAMPLES OF ACCESSIBLE CATEGORIES

- ANY VARIETY GIVES AN $\aleph_0$-ACC. CAT.
- ANY CAT. OF ALG. ON LINTON THY W. RANK
- LOCALLY PRESENTABLE CAT (GABRIEL-ULMER)
- MODELS OF SKETCH
- ANY SMALL CAT W. SPLIT IDEMPOTENTS
- THE CAT OF NON-EMPTY SETS.

EXAMPLES OF ACCESSIBLE FUNCTORS

- MORPH OF SKETCHES $F : \mathcal{J} \to \mathcal{J}'$ INDUCES AN ACC. FUNCT $: \mathrm{Mod}(\mathcal{J}') \to \mathrm{Mod}(\mathcal{J})$
- IN PARTICULAR INTERPRETATIONS OF ONE THY IN ANOTHER
- FORGETFUL FUNCTORS
- FREE FUNCTORS

LAIR'S THEOREM: CATEGORIES OF MODELS OF SMALL SKETCHES ARE EXACTLY THE ACCESSIBLE CATEGORIES.

THE ACCESSIBLE ADJOINT FUNCTOR THEOREM

LET $\Phi : \underline{A} \to \underline{B}$ BE AN ACC FUNCTOR BETWEEN ACC. CATS. IF $\underline{A}$ IS COMPLETE (COCOMP.) THEN Φ HAS A LEFT (RIGHT) ADJOINT IFF IT PRESERVES $\varprojlim$ ($\varinjlim$).

- FOR LEFT ADJ. USE GAFT:

ANY ACC. FUNCT. SATISFIES THE SOLUTION SET CONDITION: $\forall\, B\ \exists\ \underline{\text{SET}}\ \{A_\alpha\}$ S.T.

$$\begin{tikzcd} B \arrow[r, "\forall"] \arrow[dr, dotted] & \Phi A \\ & \Phi A_\alpha \arrow[u, dotted] \end{tikzcd}.$$

- FOR RIGHT ADJ. USE SAFT.

2 RECENT RESULTS

(ROSICKÝ & THOLEN) Φ ACC $\Leftrightarrow$ Φ^2 SAT SSC.

$$\Phi^2 : \underline{A}^2 \longrightarrow \underline{B}^2$$

(HONGDE HU) $\underline{A} \hookrightarrow \underline{B}$, $\underline{B}$ ACC & $\underline{A}$ CLOSED UNDER K-COLIM. THEN, $\underline{A}$ ACC $\Leftrightarrow$ INCLUSION SATISFIES SSC.

THE ADJOINT FUNCTOR THEOREM AT WORK

- $\underline{A}$ ACC, THEN COMPLETE $\Leftrightarrow$ COCOMPLETE.
  THESE ARE LOCALLY PRESENTABLE CATS OF GABRIEL-ULMER.

  LOCALLY FINITLY PRESENTABLE CATS
  $\overset{\text{DEF}}{=}$ $\aleph_0$-ACC + COMPLETE
  $\overset{\text{THM}}{=}$ MODELS OF FINITE $\varprojlim$ SKETCH

  THE OBJECT OF BASE FREE CATEGORICAL UNIVERSAL ALGEBRA.

- DOUBLE GROUPOIDS $\hookrightarrow$ DOUBLE CATS HAS A LEFT ADJOINT.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dashed, "f\cdot g"] \\
C \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\beta^{-1}"] & D \arrow[d, dashed, "g^{-1}"] \\
E \arrow[r] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] \arrow[d, dashed, "f"'] \arrow[dr, phantom, "\gamma"] & G \arrow[d, dashed] \\
F \arrow[r] & H
\end{tikzcd}
$$

- FIN. COPROD $(\underline{B}, \underline{SET}) \hookrightarrow \underline{SET}^{\underline{B}}$ HAS RAJ.

  MODELS OF COLIM SKETCH HAVE COFREE MODELS.

THE LIMIT THEOREM

THE 2-CATEGORY $\underline{\underline{\text{ACC}}}$ OF ACCESSIBLE CATEGORIES HAS ALL SMALL WEIGHTED BILIMITS, WHICH ARE CALCULATED AS IN $\underline{\underline{\text{CAT}}}$.

EXAMPLES: BIPULLBACKS

$$\begin{tikzcd} \underline{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \underline{A} \arrow[d, "F"] \\ \underline{C} \arrow[r, "G"'] & \underline{B} \end{tikzcd}$$

COMMA OBJS

$$\begin{tikzcd} \underline{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \underline{A} \arrow[d, "F"] \\ \underline{C} \arrow[r, "G"'] & \underline{B} \end{tikzcd}$$

FUNCTOR CATS

$$\underline{A}^{\mathbb{C}}$$

ALG. FOR (CO) MONAD

$$\begin{tikzcd} \underline{A}^{\mathbb{T}} \arrow[d, "U"] \\ \underline{A} \arrow[loop right, "T"] \end{tikzcd}$$

EQUIFIERS

$$\begin{tikzcd} \underline{E} \arrow[r, dotted] & \underline{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \underline{B} \end{tikzcd}$$

PROOF COMES FROM

THE UNIFORM SKETCHABILITY THEOREM

ANY 2-DIAGRAM $\underline{D} \xrightarrow{\Gamma} \underline{\underline{\text{ACC}}}$ LIFTS TO $\underline{D}^{op} \xrightarrow{\Phi} \underline{\underline{\text{SKETCH}}}$ S.T. $\Gamma = \text{MOD} \circ \Phi$.

Some Consequences

$\mathcal{J} = (\mathbb{G}, \mathbb{D}, \mathbb{L}, \mathbb{C})$ sketch

$\underline{A}$ acc with all colim of type in $\mathbb{C}$

then $\mathrm{Mod}(\mathcal{J}, \underline{A})$ acc.

<u>Note</u>: Limits are no problem

Colimits are $\rightarrow$ large cardinals!

There is a sketch $\mathcal{E}$ whose models are the epis in $\underline{A}$: $\mathcal{E}$ $\left(\cdot \longrightarrow \cdot \overset{1}{\underset{1}{\rightrightarrows}} \cdot\right)$ $\leftarrow$ P.O.

$$
\begin{tikzcd}
\mathrm{Quo}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathrm{Epi}(\underline{A}) \arrow[d, "\mathrm{dom}"] \\
1 \arrow[r, "\text{“}A\text{”}"'] & \underline{A}
\end{tikzcd}
$$

$\mathrm{Quo}(A)$ poset & acc $\Rightarrow$ small

so $\underline{A}$ cowellpowered. (Needs P.O.)

If $(\underline{V}, \otimes, I, \ldots)$ is acc monoidal cat

$\mathbb{P}$ a prop, then $\mathrm{Mod}(\mathbb{P}, \underline{V})$ acc.

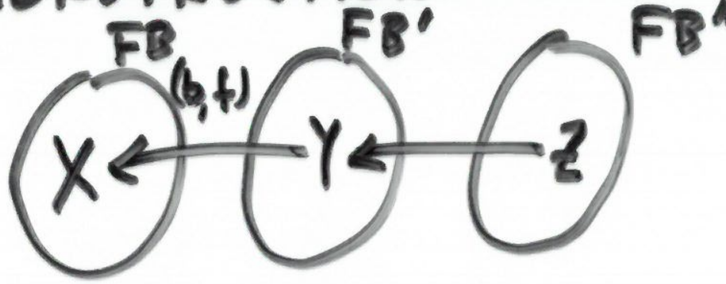
$\underline{R\text{-Coalg}} \longrightarrow \underline{R\text{-Mod}}$ has RAJ.

THE LAX-COLIMIT THEOREM

IF $F : \underline{B}^{op} \longrightarrow \underline{\underline{Acc}}$ THEN THE LAX COLIM EXISTS & IS COMPUTED BY SPLITTING IDEMPOTENTS IN THE LAX COLIM IN $\underline{\underline{CAT}}$.

$$\begin{tikzcd} FB \arrow[d, "Fb"'] \arrow[drr, "\gamma_B"] & & \\ FB' \arrow[d, "Fb'"'] \arrow[rr, "\gamma_{B'}"] & \Uparrow & \underline{C} \\ FB'' \arrow[urr, "\gamma_{B''}"'] & \Uparrow & \end{tikzcd}$$

IN $\underline{\underline{CAT}}$ IT IS GIVEN BY THE GROTHENDIECK CONSTRUCTION



$b : B' \to B$

$f : Y \to F(b)(X)$

WHAT ABOUT FILTERED COLIMITS ?

WITH J. ROSICKÝ : $\to \underline{A}_\alpha \xrightarrow{F_{\alpha\beta}} \underline{A}_\beta \longrightarrow \cdots \to \underline{A}$

① $F_{\alpha\beta}$ FULL & FAITHFUL $\Longrightarrow$ YES.

② $F_{\alpha\beta}$ FAITH & CPT. CARD $\Longrightarrow$ YES

$\neg$ CPT. CARD $\Longrightarrow$?

③ $F_{\alpha\beta}$ ARBITRARY ???

EXAMPLES

- $\underline{\text{SET}} \longrightarrow \underline{\text{SET}}^2 \longrightarrow \underline{\text{SET}}^3 \longrightarrow \underline{\text{SET}}^4 \longrightarrow \cdots \quad \longrightarrow \underline{A}$

  $(A, B) \longmapsto (A, B, B)$ FAITHFUL

  $\underline{A}$: OBJ : SEQ OF SETS $\langle X_n \rangle$ EVENTUALLY CONSTANT

  MORPH = SEQ OF FNS $\langle f_n \rangle$ " "

  ## SKETCHABLE ?

- $\underline{\text{SET}}^{\mathbb{N}} \longrightarrow \underline{\text{SET}}^{\mathbb{N}} \longrightarrow \underline{\text{SET}}^{\mathbb{N}} \longrightarrow \cdots \longrightarrow \underline{B}$

  $\langle X_0, X_1, \ldots \rangle \longmapsto \langle X_1, X_2, \ldots \rangle$ NOT FAITHFUL.

  $\underline{B}$: OBJ : SEQ OF SETS $\langle X_n \rangle$

  MORPH : EQUIV. CLASSES OF SEQ. OF FUNCTIONS $[f_n]$,

  $[f_n] = [g_n] \iff \exists N \, \forall n \geq N \, (f_n = g_n)$.

  ## SKETCHABLE ?