DOUBLE LIMITS

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LIMITS IN 2-CATEGORIES

2-$\varprojlim$, PSEUDO-$\varprojlim$, LAX $\varprojlim$, OP-LAX $\varprojlim$

bi-$\varprojlim$, WEIGHTED $\varprojlim$, FLEXIBLE $\varprojlim$

$\Gamma : \mathbb{I} \longrightarrow \mathbb{A}$ 2-FUNCTOR

$\varprojlim \Gamma$ IS A UNIVERSAL CONE ON Γ.

CONE

$$\begin{tikzcd}
 & \Gamma I \arrow[dd, "\Gamma i"] \\
A \arrow[ur, "\gamma I"] \arrow[dr, "\gamma J"'] \arrow[r, phantom, "\Downarrow \gamma i"] & \\
 & \Gamma J
\end{tikzcd}$$

- γi COMPOSE
- COULD BE IDENTITIES, ISOS, MORPHISMS, OP-MOR.

UNIVERSAL

$$\mathbb{A}(X, A) \longrightarrow \underline{\mathrm{CONE}}(X, \Gamma)$$

- ISO OF CATS
- EQUIV OF CATS

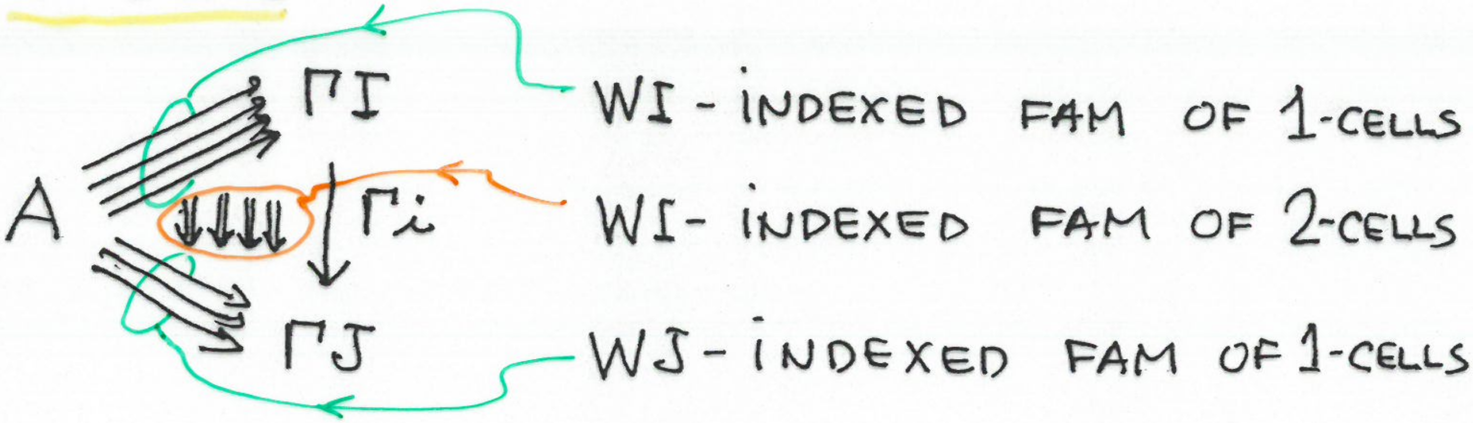
NOT QUITE GENERAL ENOUGH

INTRODUCE WEIGHTED 2-$\varprojlim$

COMES FROM $\underline{V}$-CATEGORIES

WEIGHT: $W : \mathbb{I} \longrightarrow \mathbb{C}\mathrm{AT}$ 2-FUNCT.

W-CONE



i.e. $W \xrightarrow{\gamma} \mathbb{A}(A, \Gamma -)$ (2-NAT, LAX-NAT, ...)

$\varprojlim_W \Gamma$ IS UNIVERSAL W-CONE.

PSEUDO-, LAX-, OP-LAX-$\varprojlim$ ALL CAN BE REDUCED TO WEIGHTED 2-$\varprojlim$

2-$\varprojlim$ ARE TOO GENERAL

INTRODUCE FLEXIBLE $\varprojlim$

IN ORDER TO HANDLE ALL THESE CONCEPTS, NEED SOME THEORY TO GUIDE US.

THE THEORY USED IS THAT OF $\underline{V}$-CATEGORIES FOR $\underline{V} = \underline{\text{CAT}}$

PROPOSAL: USE $\underline{\text{CAT}}$-INDEXED CAT THY

PROGRAM: ORGANIZE & EXTEND THE THEORY OF 2-CATS USING INDEXED CATS

REFINE THE THEORY OF INDEXED CATS BY STUDYING 2-CATEGORIES

CATEGORY OF ELEMENTS

$$F : \underline{A}^{op} \longrightarrow \underline{SET} \quad \rightsquigarrow \quad \begin{array}{c} EL(F) \\ \downarrow P \\ \underline{A} \end{array}$$

- F REPRESENTABLE $\Longleftrightarrow$ $EL(F)$ HAS TERM. OBJ.
- $F = \underline{A}(-,A) \Longrightarrow EL(F) = \underline{A}/A$
- F FLAT $\Longleftrightarrow$ $EL(F)$ IS FILTERED
- $\varinjlim_F \Phi \cong \varinjlim \Phi P$

FOR 2-CATS - MANY CHOICES

$F : \mathbb{A} \longrightarrow \mathbb{C}AT$ 2-FUNCT

$\mathbb{E}L(F)$ OBJ. $(A, X \in FA)$

MORPH. $(A, X) \longrightarrow (B, Y)$

$a : A \longrightarrow B$

$F(a)(Y) = X$

$F(a)(Y) \xrightarrow{\cong} X$

$X \longrightarrow F(a)(Y)$

$F(a)(Y) \longrightarrow X$

2-CELLS ARE
$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$
+ COMPATIBILITY

$F = \mathbb{A}(-, A_0) \rightsquigarrow$

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "X"'] \arrow[rr, phantom, shift right=3, "\xi \Rightarrow"] & & B \arrow[dl, "Y"] \\ & A_0 & \end{tikzcd}$$

ξ EQUALITY, ISO, MORPH, OP-MORPH.

EACH GIVES A USEFUL CONCEPT BUT NONE IS ADEQUATE FOR EVERYTHING, E.G. REPRESENTABILITY THEOREM. OR TRANSFORMING WEIGHTED $\varinjlim$ INTO UNWEIGHTED ONES.

THE THEORY OF INDEXED CATEGORIES TELLS US THAT LIMITS SHOULD BE TAKEN OVER CATEGORY OBJECTS IN THE BASE, I.E. DOUBLE CATEGORIES.

DOUBLE CATEGORIES
(EHRESMANN)

$$\mathbb{D}: \quad \underline{D}_2 \;\overset{p_0}{\underset{p_1}{\overset{\gamma}{\rightrightarrows\!\!\to}}}\; \underline{D}_1 \;\overset{\partial_0}{\underset{\partial_1}{\overset{\mathrm{id}}{\leftrightarrows\!\!\to}}}\; \underline{D}_0$$

CATEGORY OBJ. IN $\underline{\mathrm{CAT}}$

HORIZ. + DOUB.   OBJ. + VERT.

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, dashed, "v'"] \\
\bar{A} \arrow[r, "\bar{h}"'] & \bar{A}'
\end{tikzcd}$$

HORIZ + VERT COMP + IDENTITIES + MIDDLE FOUR INTERCHANGE

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\searrow"] & \cdot \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\searrow"] & \cdot \arrow[d, dashed] \\
\cdot \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\searrow"] & \cdot \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\searrow"] & \cdot \arrow[d, dashed] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Ex 1: $\underline{D}_0$ DISCRETE $\Leftrightarrow$ $\mathbb{D}$ IS A 2-CAT.

Ex 2: 8 DUALITIES $\longleftrightarrow$ SYMMETRIES OF $\square$

GEN BY   op   co   tr

op	co	tr
$\partial_0 \leftrightarrow \partial_1$	pt. wise op	VERT. $\updownarrow$ HORIZ.

Ex 3: $\mathbb{D}$OUB

DOUBLE FUNCTORS $F : \mathbb{D} \longrightarrow \mathbb{E}$

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "v"'] \arrow[dr, phantom, "\searrow\alpha"] & A' \arrow[d, "v'"] \\
\bar{A} \arrow[r, "\bar{h}"'] & \bar{A}'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
FA \arrow[r, "Fh"] \arrow[d, "Fv"'] \arrow[dr, phantom, "\searrow F\alpha"] & FA' \arrow[d, "Fv'"] \\
F\bar{A} \arrow[r, "F\bar{h}"'] & F\bar{A}'
\end{tikzcd}
$$

PRESERVES ALL COMPOSITIONS.

HORIZONTAL NATURAL TRANSFORMATIONS

$t : F \longrightarrow G$

$$
\begin{tikzcd}
A \arrow[d, "v"'] \\
\bar{A}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Fv"'] \arrow[dr, phantom, "\searrow tv"] & GA \arrow[d, "Gv"] \\
F\bar{A} \arrow[r, "t\bar{A}"'] & G\bar{A}
\end{tikzcd}
$$

PRES. VERT COMP & HORIZ. NAT.

VERT. NAT. TRANSF. SIMILAR

DOUBLE NAT. TRANSFS.

$$
\begin{tikzcd}
F \arrow[r, "t"] \arrow[d, "u"'] \arrow[dr, phantom, "\searrow\tau"] & G \arrow[d, "u'"] \\
\bar{F} \arrow[r, "\bar{t}"'] & \bar{G}
\end{tikzcd}
\; : \; A \rightsquigarrow
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "uA"'] \arrow[dr, phantom, "\searrow\tau A"] & GA \arrow[d, "u'A"] \\
\bar{F}A \arrow[r, "\bar{t}A"'] & \bar{G}A
\end{tikzcd}
$$

CONSTRUCTIONS ON DOUBLE CATEGORIES

DOUB. CAT $\rightsquigarrow$ $\underline{Cat}$ INDEXED CAT

$\mathbb{D} \rightsquigarrow \underset{\sim}{D}$: $\underset{\sim}{D}^{\underline{C}}$ OBJ. $\underline{C} \to \underline{D}_0$

MOR. $\underline{C} \longrightarrow \underline{D}_1$

CAN RECOVER $\mathbb{D}$

$\underset{\sim}{D}^{\mathbb{1}} \rightsquigarrow$ HORIZ. (LEADING ASPECT)

$\underset{\sim}{D}^{\mathbb{2}} \rightsquigarrow$ VERT & DOUB

$\underset{\sim}{D}^{\mathbb{3}} \rightsquigarrow$ VERT COMP

USE THIS PROCESS TO TRANSLATE CONCEPTS WELL UNDERSTOOD FOR $\underline{Cat}$ IND. CATS TO DOUBLE CATS.

1. FUNCTOR CATEGORIES

$\mathbb{D}^{\mathbb{E}}$ DOUBLE FUNCTORS, HORIZ. VERT. & DOUBLE NAT. TRANSFS

FOR $\mathbb{D}$ & $\mathbb{E}$ 2-CATEGORIES $\mathbb{D}^{\mathbb{E}}$

IS NOT GENERALLY A 2-CAT

E.G. $\mathbb{D} = \mathbb{C}\mathrm{AT}$, $\mathbb{E} = \mathbb{2}$ (VERT. DISC.)

$\mathbb{C}\mathrm{AT}^{\mathbb{2}}$

OBJ. $\underline{A} \xrightarrow{F} \underline{B}$

HORIZ.

$$\begin{tikzcd} \underline{A} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & \underline{A}' \arrow[d, "F'"] \\ \underline{B} \arrow[r, "K"'] & \underline{B}' \end{tikzcd}$$

VERT.

$$\begin{tikzcd} \underline{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \underline{B} \end{tikzcd}$$

DOUB.

$$\begin{tikzcd} \underline{A} \arrow[rr, "H"] \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\Rightarrow"] \arrow[drr, phantom, "\equiv"] & & \underline{A}' \arrow[d, bend right, "F'"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\Rightarrow"] \\ \underline{B} \arrow[rr, "K"'] & & \underline{B}' \end{tikzcd}$$

2. COMMA CATEGORIES (HORIZ.)

$$\begin{tikzcd} (F,, G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & \mathbb{C} \arrow[d, "F"] \\ \mathbb{D} \arrow[r, "G"'] & \mathbb{E} \end{tikzcd}$$

HORIZ.

$$
\begin{tikzcd}
C \arrow[d, "c"'] & FC \arrow[r, "e"] \arrow[d, "Fc"'] \arrow[dr, phantom, "="] & GD \arrow[d, "Gd"] & D \arrow[d] \\
C' & FC' \arrow[r, "e'"'] & GD' & D'
\end{tikzcd}
$$

(ALL HORIZ.)

VERT.

$$
\begin{tikzcd}
C \arrow[d, dashed] & FC \arrow[r, "e"] \arrow[d, dashed] \arrow[dr, phantom, "\Searrow \varepsilon"] & GD \arrow[d, dashed] & D \arrow[d, dashed] \\
\bar{C} & F\bar{C} \arrow[r, "e'"'] & G\bar{D} & \bar{D}
\end{tikzcd}
$$

DOUB.

$$
\begin{tikzcd}
FC \arrow[rr] \arrow[dr] \arrow[dd, dashed] & & GD \arrow[dr] \arrow[dd, dashed] & \\
& FC' \arrow[rr] \arrow[dd, dashed] & & GD' \arrow[dd, dashed] \\
F\bar{C} \arrow[rr] \arrow[dr] & & G\bar{D} \arrow[dr] & \\
& F\bar{C}' \arrow[rr] & & G\bar{D}'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[r, Rightarrow, "\varepsilon"] \arrow[dr, Rightarrow, "F\gamma"'] \arrow[drr, phantom, "\equiv"] & \cdot \arrow[dr, Rightarrow, "G\delta"] & \\
& \cdot \arrow[r, Rightarrow, "\varepsilon'"'] & \cdot
\end{tikzcd}
$$

3. Double Slice

$\mathbb{A} \| A$

HORIZ.

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "="] & Y \arrow[dl] \\
& A &
\end{tikzcd}
$$

DOUB

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Rightarrow"] & \arrow[d, phantom, "="] & Y \arrow[dl, bend left] \arrow[dl, bend right] \arrow[dl, phantom, "\Rightarrow"] \\
& A &
\end{tikzcd}
$$

VERT.

$$
\begin{tikzcd}
X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A
\end{tikzcd}
$$

4. "Category of Elements"

$F : \mathbb{A}^{op} \longrightarrow \mathbb{C}\mathrm{at}$ 2-FUNCTOR

$\mathbb{E}\mathrm{l}(F)$ OBJ. $(A, X \in FA)$

HORIZ. $$\frac{(A, X \in FA) \longrightarrow (B, Y \in FB)}{A \xrightarrow{a} B \qquad F(a)(Y) = X}$$

VERT. $$\frac{(A, X \in FA) \dashrightarrow (A, X' \in FA)}{x : X \longrightarrow X'}$$

DOUB.
$$\begin{tikzcd}
(A, X) \arrow[r, "{(a,Y)}"] \arrow[d, dashed, "{(A,x)}"'] \arrow[dr, phantom, "{\Searrow (\alpha, y)}"] & (B, Y) \arrow[d, dashed, "{(B,y)}"] \\
(A, X') \arrow[r, "{(b,Y')}"'] & (B, Y')
\end{tikzcd}$$

WHERE
$$\begin{tikzcd}
A \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}$$
AND

$$x = F(\alpha)(Y') \cdot F(a)(y) = F(b)(y) \cdot F(\alpha)(Y)$$

NAT OF $F(\alpha)$

A TERMINAL OBJECT IN A DOUBLE CAT $\mathbb{D}$ IS T SUCH THAT

$\forall D\ \exists!\ D \longrightarrow T$ AND

$$\forall \begin{tikzcd} D \arrow[d, "v"'] \\ \bar{D} \end{tikzcd} \quad \exists! \begin{tikzcd} D \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow"] \arrow[r] & T \arrow[d, equal] \\ \bar{D} \arrow[r] & T \end{tikzcd}$$

THEOREM: $F : \mathbb{A}^{op} \longrightarrow \mathbb{C}\text{AT}$ IS REPRESENTABLE $\iff \mathbb{E}\text{L}(F)$ HAS TERM. OBJ.

IF $F = \mathbb{A}(-, A)$ THEN $\mathbb{E}\text{L}(F) = \mathbb{A} \| A$ AND $1_A : A \longrightarrow A$ IS TERMINAL.

DOUBLE CONES

$\Gamma : \mathbb{I} \longrightarrow \mathbb{D}$ A DOUBLE DIAGRAM.

A DOUBLE CONE IS A HORIZ. NAT TRANSF $\text{CONST}(A) \longrightarrow \Gamma$ FOR A AN OBJ. OF $\mathbb{D}$

I.E.

$$\begin{tikzcd}
 & \Gamma I' \\
A \arrow[ur, "\gamma I'"] \arrow[r, "\gamma I"] \arrow[dr, "\gamma \bar{I}"'] \arrow[r, phantom, shift left=3, "="] \arrow[dr, phantom, shift left=3, "\Downarrow \gamma v"] & \Gamma I \arrow[u, "\Gamma h"'] \arrow[d, dashed, "\Gamma v"] \\
 & \Gamma \bar{I}
\end{tikzcd}$$

γI HORIZ. NAT.

γv COMPOSE VERT.

HORIZ. NAT.

W.R.T. DOUBLE MAPS

THE DOUBLE LIMIT $\underleftarrow{\mathrm{LLIM}}\,\Gamma$

IS A UNIVERSAL SUCH CONE

UNIVERSALITY

ANY CONE SUCH AS γ ABOVE

FACTORS UNIQUELY

$$\begin{tikzcd}
\underleftarrow{\mathrm{LLIM}}\,\Gamma \arrow[rr, "\lambda I"] \arrow[drr, "\lambda \bar{I}"] \arrow[rr, phantom, shift right=3, "\Downarrow \lambda v"] & & \Gamma I \arrow[d, dashed, "\Gamma v"] \\
 & & \Gamma \bar{I} \\
A \arrow[uu, "\exists !"] \arrow[uurr, "\gamma I"] \arrow[urr, "\gamma \bar{I}"'] \arrow[urr, phantom, shift left=2, "\Downarrow \gamma v"] & &
\end{tikzcd}$$

AND...

ANY "DOUBLE CONE"

$$
\begin{tikzcd}
A \arrow[rr, "\gamma I"] \arrow[dd, dashed, "a"'] \arrow[ddrr, "\gamma \bar{I}"' near start] \arrow[r, phantom, shift right=3, "\Downarrow \gamma v"] & & \Gamma I \arrow[dd, dashed, "\Gamma v"] \arrow[l, phantom, shift right=3, "\alpha I \Downarrow"] \\
& & \\
A' \arrow[rr, "\gamma' \bar{I}"'] \arrow[uurr, "\gamma' I" near start] \arrow[r, phantom, shift left=3, "\Downarrow \gamma' v"] & & \Gamma \bar{I} \arrow[l, phantom, shift left=3, "\alpha \bar{I} \Downarrow"]
\end{tikzcd}
$$

FACTORS UNIQUELY AS

$$
\begin{tikzcd}
A \arrow[dd, dashed, "a"'] \arrow[dr] \arrow[dd, phantom, shift left=6, "\exists! \Downarrow"] & & & \Gamma I \arrow[dd, dashed, "\Gamma v"] \\
& \underleftarrow{\mathrm{LLIM}}\, \Gamma \arrow[urr, "\lambda I"] \arrow[drr, "\lambda \bar{I}"'] \arrow[rr, phantom, "\Downarrow \lambda v"] & & \\
A' \arrow[ur] & & & \Gamma \bar{I}
\end{tikzcd}
$$

THIS CONCEPT CAN BE USED IN A 2-CAT. HERE THE Γv WOULD ALL BE IDENTITIES BUT NOT λv.

<u>Ex 0</u>. IF $\Gamma : \emptyset \longrightarrow \mathbb{A}$ IS THE EMPTY DIAGRAM, THEN $\underleftarrow{\mathrm{LLIM}}\, \Gamma$ IS A TERMINAL OBJECT, AS ABOVE.

Ex 1. COMMA OBJECTS

$\mathbb{I}$:

$$\begin{tikzcd} 0 \arrow[r] & 2 \arrow[d, dashed] \\ 1 \arrow[r] & 3 \end{tikzcd}$$

$\Gamma : \mathbb{I} \longrightarrow \mathbb{A}$

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

CONE IS

$$\begin{tikzcd} & A \arrow[dr] \arrow[d, phantom, "="] & \\ X \arrow[ur] \arrow[rr, bend left=15] \arrow[rr, bend right=15] \arrow[dr] \arrow[rr, phantom, "\Downarrow"] & {} \arrow[d, phantom, "="] & C \\ & B \arrow[ur] & \end{tikzcd}$$

i.e.

$$\begin{tikzcd} & A \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur] \arrow[dr] & {} & C \\ & B \arrow[ur] & \end{tikzcd}$$

Ex 2. INSERTERS

$\mathbb{I}$:

$$\begin{tikzcd} & 1 \arrow[d, dashed] \\ 0 \arrow[ur] \arrow[r] & 2 \end{tikzcd}$$

$\Gamma : \mathbb{I} \longrightarrow \mathbb{A}$

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \end{tikzcd}$$

CONE IS

$$\begin{tikzcd} & A \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[ur] \arrow[r, bend left=10] \arrow[r, bend right=10] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Ex 3. Inverters

$$\mathbb{I}: \quad \begin{tikzcd} & 1 \arrow[d, "v"] \arrow[d, phantom, shift right=4ex, "\Downarrow"] \\ 0 \arrow[ur] \arrow[r] & 2 \arrow[u, bend right, dashed, "v^{-1}"'] \end{tikzcd}$$

$$\Gamma: \mathbb{I} \longrightarrow \mathbb{A}$$

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

Cone is

$$\begin{tikzcd} & A \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{t}{\Rightarrow}"] \\ X \arrow[ur, "x"] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "tx \Downarrow\ \Uparrow (tx)^{-1}"] & B \end{tikzcd}$$

i.e.

$$\begin{tikzcd} X \arrow[r, "x"] & A \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow t"] & B \end{tikzcd}$$

s.t. tx inv.

Ex 4. LAX LIMITS

$\Gamma : \underline{I} \longrightarrow \mathbb{A}$ 2-FUNCT ($\underline{I}$ A CAT)

LAX CONE

$$\begin{tikzcd} & \Gamma I \arrow[d, "\Gamma j"] \\ X \arrow[ur, "\gamma I"] \arrow[r, "\gamma J"'] \arrow[r, phantom, shift left=3, "\Downarrow \gamma j"] & \Gamma J \end{tikzcd}$$

THE γj COMPOSE

CONSTRUCT A DOUBLE CAT

$\$\underline{I}$ OBJ

$$\begin{tikzcd} I \arrow[d, "j"] \\ J \end{tikzcd}$$

HORIZ

$$\begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "k"] & \\ J \arrow[rr, "\alpha"'] & & K \end{tikzcd}$$

VERT

$$\begin{tikzcd} I \arrow[rr, "\alpha"] \arrow[dr, "j"'] & & I' \arrow[dl, "j'"] \\ & J & \end{tikzcd}$$

DOUB

$$\begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr] & \\ J \arrow[rr, "\alpha"] \arrow[dr] & & K \arrow[dl, "l"] \\ & L & \end{tikzcd}$$

$$\begin{tikzcd} {[\alpha j]} \arrow[r, "l"] \arrow[d, dashed, "j"'] & {[l\alpha j]} \arrow[d, dashed, "j"] \\ {[\alpha]} \arrow[r, "l"'] & {[l\alpha]} \end{tikzcd}$$

$D_1 : \$\underline{I} \longrightarrow \underline{I}$ TAKE CODOMAIN

THEN $\underleftarrow{\text{LAX}}\ \Gamma \cong \underleftarrow{\text{LLIM}}\ \Gamma D_1$

THEOREM: LET $W : \mathbb{I} \longrightarrow \mathbb{C}\mathrm{AT}$ BE A 2-FUNCTOR AND $P : \mathbb{E}\mathrm{L}(W) \longrightarrow \mathbb{I}$ THE CANONICAL PROJECTION. THEN FOR EVERY $\Gamma : \mathbb{I} \longrightarrow \mathbb{A}$ (2-FUNCTOR)

$$\underleftarrow{\mathrm{LIM}}_W \Gamma \cong \underleftarrow{\mathrm{LLIM}}\, \Gamma P$$

EX: PSEUDO & LAX LIM ARE LLIM's

PROPOSITION: EVERY DOUBLE LIMIT IS AN ITERATED WEIGHTED 2-LIM. IN PARTICULAR 2-COMPLETENESS $\Longrightarrow$ DOUBLE COMPLETENESS.

"PROOF" FOR INDEXED CATS

LIM IS EQUALIZER OF $\prod$.

EQUALIZERS ARE 2-LIM's.

$\prod$ IS $\underleftarrow{\mathrm{LLIM}}$ WHEN $\mathbb{I}$ IS HORIZONTALLY DISCRETE.

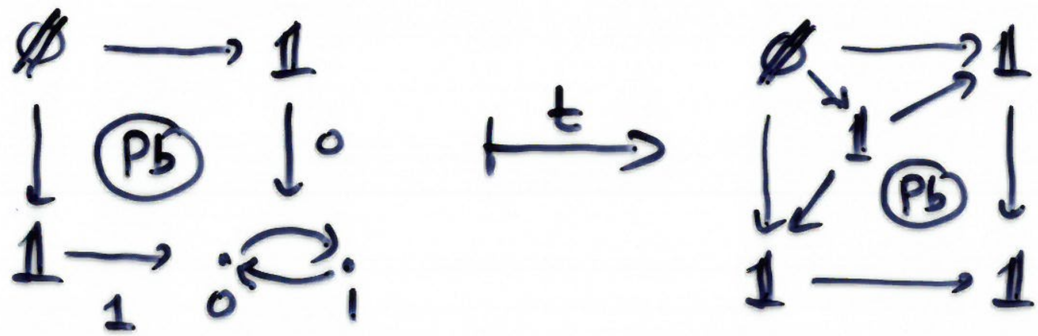
$\Gamma : \mathbb{I} \longrightarrow \mathbb{A}$ (2-CAT) IS CONSTANT ON COMPONENTS.

$$\underleftarrow{\mathrm{LLIM}}\, \Gamma = \prod_{[I] \in \pi_0 \mathbb{I}} \Gamma(I)^{[I]} \qquad \blacksquare$$

PERSISTENT DOUBLE LIMITS

DEF: $\mathbb{I}$ $\underleftarrow{\text{LLIM}}$ ARE PERSISTENT IF FOR ANY TWO DIAGRAMS $\Gamma, \Phi : \mathbb{I} \longrightarrow \mathbb{A}$ AND ANY HORIZ. NAT. TRANSF. $t : \Gamma \longrightarrow \Phi$ SUCH THAT $\forall I$, $tI : \Gamma I \longrightarrow \Phi I$ IS AN EQUIVALENCE, IT FOLLOWS THAT $\underleftarrow{\text{LLIM}}\,\Gamma \longrightarrow \underleftarrow{\text{LLIM}}\,\Phi$ IS AN EQUIVALENCE.

E.G. PULLBACKS ARE NOT PERSISTENT

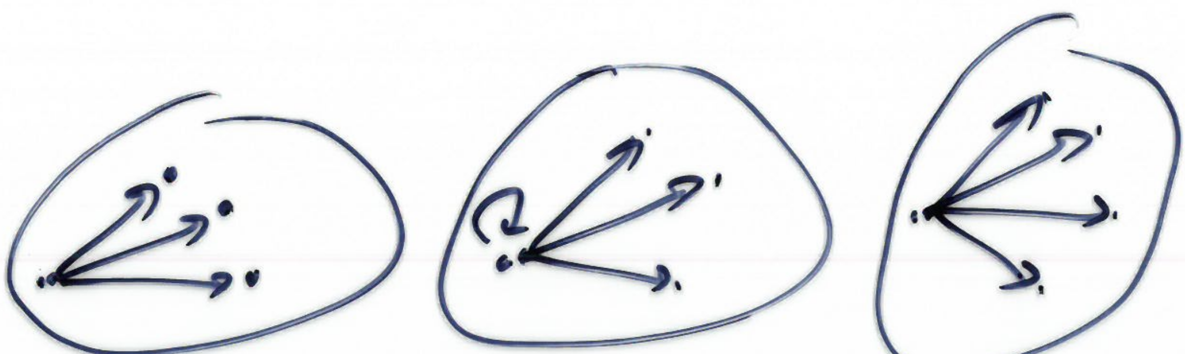


COMMA OBJECTS ARE.

THEOREM: $\mathbb{I}$ $\underleftarrow{\mathrm{LLIM}}$ ARE PERSISTENT IFF THE HORIZONTAL PART OF $\mathbb{I}$ IS COMPONENTWISE CONTRACTIBLE.

THIS MEANS THAT $\exists\ R : \pi_0 \mathbb{I}_h \longrightarrow \mathbb{I}_h$ AND $\exists\ t : RQ \longrightarrow 1_{\mathbb{I}_h}$ SUCH THAT $QR = 1_{\pi_0 \mathbb{I}_h}$, WHERE $Q : \mathbb{I}_h \twoheadrightarrow \pi_0 \mathbb{I}_h$ IS THE QUOTIENT MAP.

MORE CONCRETELY: EACH COMPONENT OF $\mathbb{I}_h$ HAS A NATURAL WEAK INITIAL OBJECT



NOTE: FLEXIBLE $\Longrightarrow$ PERSISTENT