SPANS FOR 2-CATEGORIES

(JOINT WITH R. DAWSON & D. PRONK)

WHITE POINT, NS
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SPAN

$\underline{A}$ CAT W. PB $\rightsquigarrow$ $\underline{\text{SPAN}}(\underline{A})$

$$A \xleftarrow{s} S \xrightarrow{s'} A'$$

- BICATEGORY
- $\underline{A} \xrightarrow{(\ )_*} \underline{\text{SPAN}}(\underline{A})$ STRONG MORPH
- f_* HAS RT ADJOINT f^*
- BECK CONDITION

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & \text{PB} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} P & \xrightarrow{p_*} & A \\ {\scriptstyle q^*}\uparrow & = & \uparrow{\scriptstyle f^*} \\ B & \xrightarrow[g_*]{} & C \end{array}$$

THEOREM (DPP)

(PB) $\underline{A} \xrightarrow{(\ )_*} \underline{\text{SPAN}}(\underline{A})$

$\forall F$ (SINISTER): $\underline{A} \to \underline{B}$ (BICAT), $\exists!\ \Phi : \underline{\text{SPAN}}(\underline{A}) \to \underline{B}$

- OPLAX $\begin{cases} \Phi(ST) \to \Phi(S)\Phi(T) \\ \Phi(I_A) \to 1_{\Phi A} \end{cases}$
- NORMAL
- PRES f_*S & Tf^*

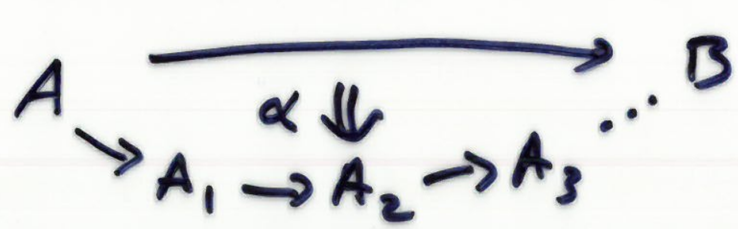
(PARANORMAL)

- UNIV PROP DOES NOT REFER TO PB
  → DROP IT !
- BUT NEED IT FOR COMPOSITION IN SPAN
  → DROP IT TOO!!
  (Φ DOESN'T PRESERVE IT.)
- NEED SOME TRACE OF COMPOSITION TO EXPRESS OPLAX, PARANORMAL

SEVERAL OBJECT MULTICATEGORIES

(HERMIDA, LEINSTER, LAMBEK)

CELLS:



CAN DEFINE IDENTITIES & COMPOSITES BY UNIVERSAL PROPERTIES

CAN'T EXPRESS ADJOINTS !

$\mathbb{S}\mathrm{PAN}(\underline{A})$

FOR $\underline{A}$ W. PB, THE (WEAK) DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}(\underline{A})$ HAS CELLS

$$\begin{array}{ccccc} A & \leftarrow & S & \rightarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ B & \leftarrow & T & \rightarrow & B' \end{array}$$

- BETTER THAN $\underline{\mathrm{SPAN}}(\underline{A})$ BECAUSE IT REMEMBERS THE ARROWS OF $\underline{A}$ (cf SHULMAN)

- IN A DOUBLE CATEGORY THE NOTION OF ADJOINTNESS FACTORS INTO TWO SIMPLER & DUAL NOTIONS:

COMPANION
+
CONJOINT

COMPANIONS

(BROWN - CONNECTIONS, GP)

IN A DOUBLE CAT $\mathbb{D}$ $f : A \to B$ (HORIZ)

AND $g : A \to B$ (VERT) ARE **COMPANIONS**

IF THERE ARE CELLS

$$\begin{array}{ccc} A & = & A \\ \| & \eta & \downarrow g \\ A & \xrightarrow[f]{} & B \end{array} \quad \& \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \varepsilon & \| \\ B & = & B \end{array}$$

S.T. $\varepsilon \eta = id_f$ & $\varepsilon \cdot \eta = 1_g$

CONJOINTS (DUAL)

$$\begin{array}{ccc} B & \xrightarrow{u} & A \\ \| & \beta & \downarrow g \\ B & = & B \end{array} \quad \& \quad \begin{array}{ccc} A & = & A \\ g \downarrow & \alpha & \| \\ B & \xrightarrow[u]{} & A \end{array}$$

S.T $\alpha\beta = id_u$ & $\beta \cdot \alpha = id_g$

ADJOINTS

ADJOINTNESS IS THE USUAL IN THE BICATEGORY $\underline{\underline{\mathrm{Hor}}}(\mathbb{D})$ OF HORIZONTAL ARROWS AND **SPECIAL CELLS**

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \| & \alpha & \| \\ A & \xrightarrow[k]{} & B \end{array}$$

__PROP__: ① COMPANIONS (CONJOINTS) ARE UNIQUE UP TO SPECIAL ISO

② COMPANIONS (CONJOINTS) COMPOSE

③ f COMPANION TO g & u CONJOINT TO g $\Rightarrow$ f LEFT ADJOINT TO u.

__EX__: QUINTETS $\mathbb{Q}\underline{\underline{A}}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

Ex: In $\mathbb{S}\mathrm{pan}(\underline{A})$

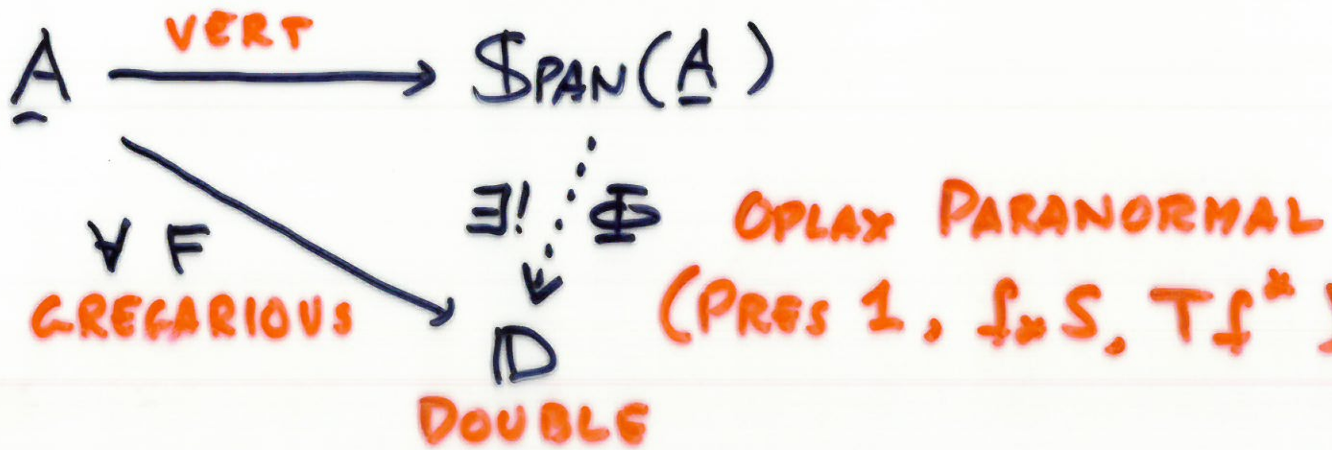
$$\begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{1} & A \\ \| & & \| & & \downarrow f \\ A & \xleftarrow[1]{} & A & \xrightarrow[f]{} & B \end{array} \quad \& \quad \begin{array}{ccccc} A & \xleftarrow{1} & A & \xrightarrow{f} & B \\ f\downarrow & & f\downarrow & & \| \\ B & \xleftarrow[1]{} & B & \xrightarrow[1]{} & B \end{array}$$

MAKE f_* COMPANION TO f

SIMILARLY f^* CONJOINT TO f

$\Rightarrow f_* \dashv f^*$

UNIVERSAL PROPERTY



OPLAX DOUBLE CATEGORIES

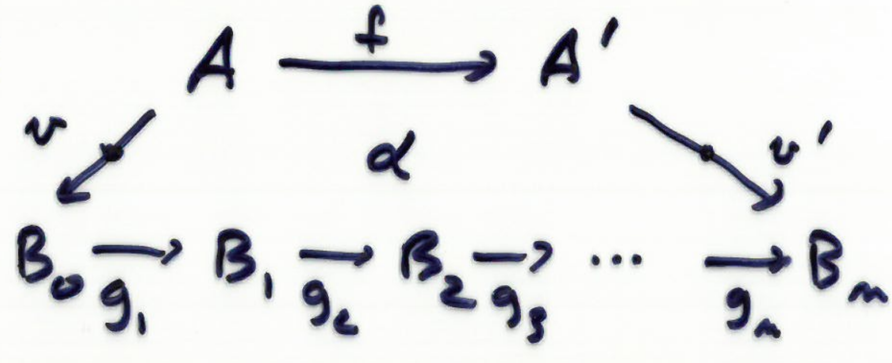
(LEINSTER'S $\underline{fc}$-MULTICATEGORIES)

WHEN $\underline{A}$ DOESN'T HAVE PB SPANS DON'T COMPOSE — IT IS NATURAL TO CONSIDER OPLAX DOUBLE CATEGORIES:

HAVE: OBJECTS, HORIZONTAL ARROWS

VERTICAL ARROWS – FORM A CAT

CELLS



CELLS COMPOSE VERTICALLY IN THE OBVIOUS WAY ($(\beta_n, \ldots, \beta_2, \beta_1) \cdot \alpha$)

ASSOCIATIVE + UNITARY

HORIZONTAL COMPOSITES

SAY THAT THE COMPOSITE OF $g_1, \dots, g_m$ EXISTS (OR IS STRONGLY REPRESENTABLE)

IF THERE IS A CELL

$$\begin{array}{ccc} B_0 & \xrightarrow{\;g_m \cdots g_2 g_1\;} & B_m \\ \| & \iota & \| \\ B_0 \xrightarrow[g_1]{} B_1 \xrightarrow[g_2]{} B_2 \xrightarrow[g_3]{} & \cdots & \xrightarrow[g_m]{} B_m \end{array}$$

IF EVERY CELL

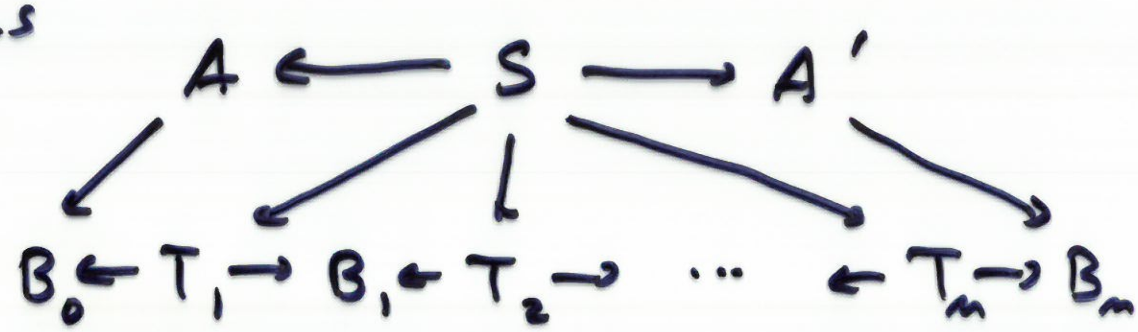
$$\begin{array}{ccc} A & \longrightarrow & A' \\ \swarrow & \alpha & \searrow \\ X_k \to \cdots \to X_1 \to B_0 \to B_1 \to \cdots & \to B_m \to & Y_1 \to \cdots \to Y_\ell \end{array}$$

FACTORS UNIQUELY AS

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \swarrow & \bar{\alpha} & \searrow \\ X_k \to \cdots \to X_1 \to B_0 & \longrightarrow & B_m \to Y_1 \to \cdots \to Y_\ell \\ \| \quad\quad \| \;\; \mathrm{id} \;\; \| & \iota & \| \;\; \mathrm{id} \;\; \| \quad\quad \| \\ X_k \to \cdots \to X_1 \to B_0 \to B_1 \to & \cdots & \to B_m \to Y_1 \to \cdots \to Y_\ell \end{array}$$

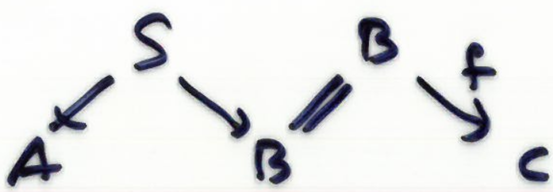
SPAN(A)

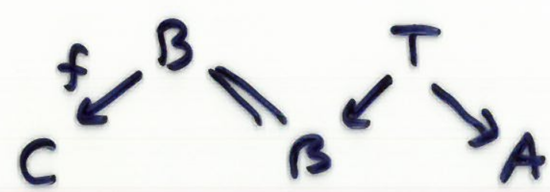
CELLS



- THE COMPOSITE OF $T_1 \cdots T_n$ EXISTS IFF THE GENERALIZED PB (= LIM) EXISTS.

- IDENTITIES EXIST $\quad A \xleftarrow{1} A \xrightarrow{1} A$
- ALSO $f_* S$ AND $T f^*$ EXIST





PARANORMAL

AN OPLAX DOUBLE CATEGORY IS **NORMAL** IF ALL IDENTITIES EXIST

COMPANIONS & CONJOINTS MAKE SENSE — NORMAL MORPHISMS PRESERVE THEM

IT IS **PARANORMAL** IF

(1) IT IS NORMAL

(2) EVERY VERTICAL v HAS A COMPANION v_* AND A CONJOINT v^*

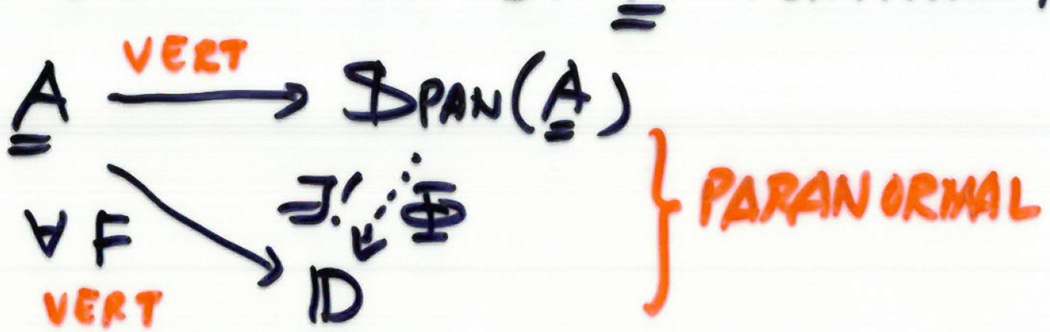
(3) ALL COMPOSITES $v_* h$ & $k v^*$ EXIST.

THEOREM $\mathbb{S}\text{PAN} : \underline{\underline{\text{CAT}}} \longrightarrow \underline{\underline{\text{PARA}}}$ IS 2-LAJ TO $\text{VERT} : \underline{\underline{\text{PARA}}} \longrightarrow \underline{\underline{\text{CAT}}}$.

2-CATEGORIES

$\underline{\underline{A}}$ 2-CATEGORY

WANT THE FREE PARANORMAL DOUBLE CATEGORY GENERATED BY $\underline{\underline{A}}$ VERTICALLY

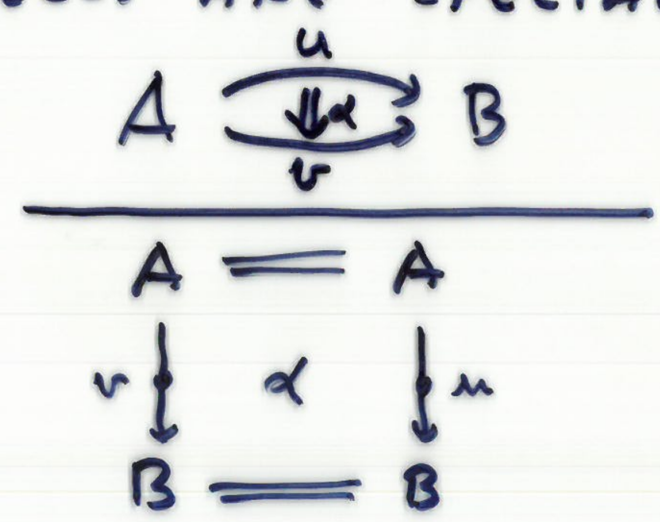


THE UNDERLYING 2-CATEGORY OF A PDC

$\underline{\text{VERT}}(\mathbb{D})$ HAS SAME OBJECTS AS $\mathbb{D}$,

THE ARROWS ARE THE VERTICAL ARROWS OF $\mathbb{D}$,

THE 2-CELLS ARE SPECIAL CELLS



PRECELLS

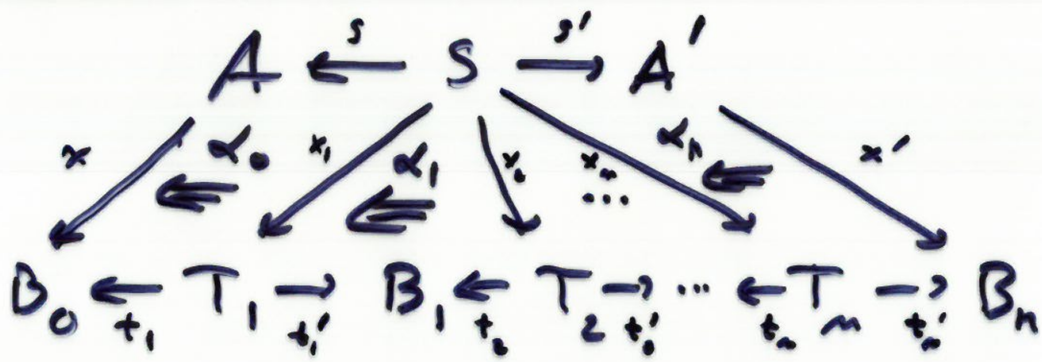
CONSTRUCT $\mathbb{S}\text{pan}'(\underline{\underline{A}})$

OBJECTS – THOSE OF $\underline{\underline{A}}$

VERTICAL ARROWS – ARROWS OF $\underline{\underline{A}}$

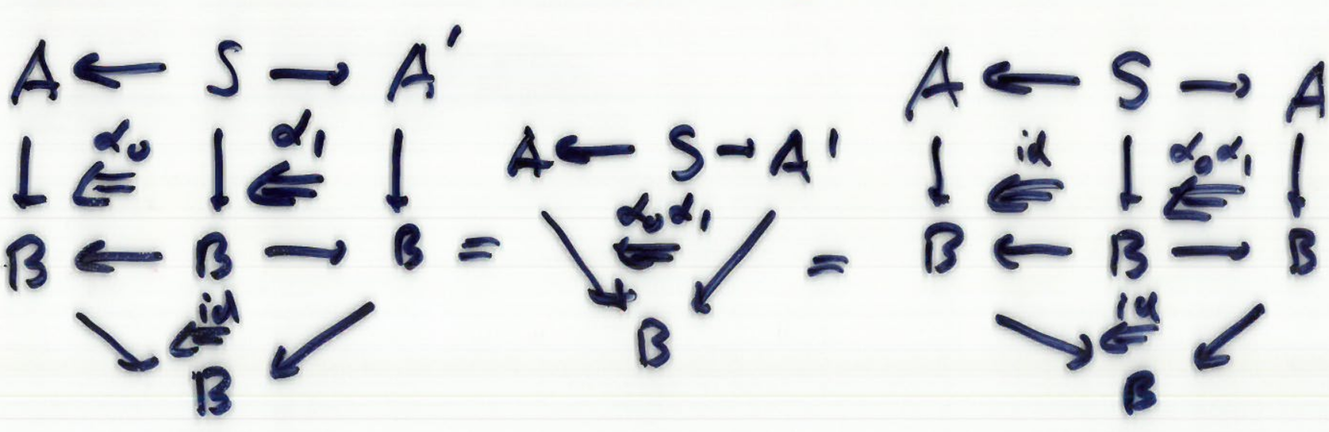
HORIZONTAL ARROWS – SPANS IN $\underline{\underline{A}}$

PRECELLS

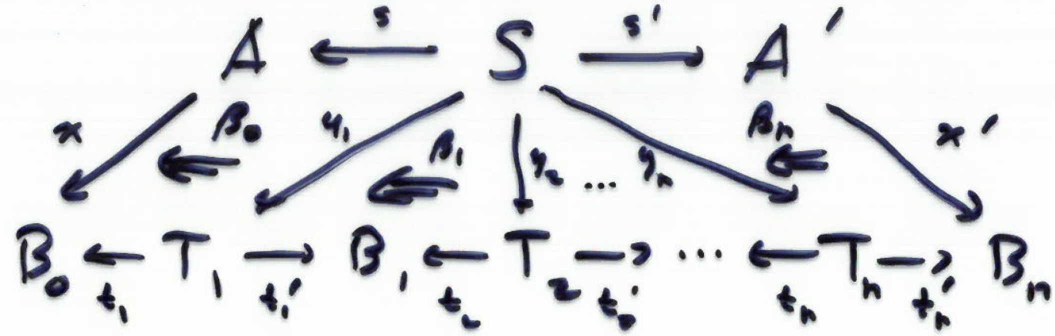


WITH OBVIOUS COMPOSITION OF PRECELLS $\mathbb{S}\text{pan}'(\underline{\underline{A}})$ IS AN OPLAX DOUB CAT.

BUT NOT NORMAL !



EQUIVALENCE RELATION

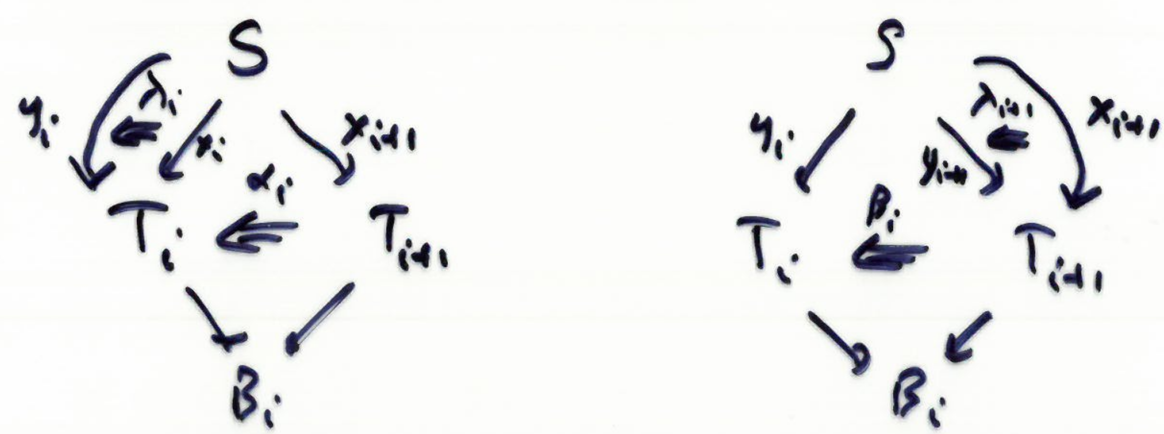


ANOTHER PRECELL WITH SAME BOUNDARY

IDENTIFY IT WITH PREVIOUS ONE IF

THERE ARE $\lambda_i : x_i \to y_i \quad i = 1, \dots, n$

S.T.



(END PTS FIXED – $\lambda_0 = id$, $\lambda_{n+1} = id$)

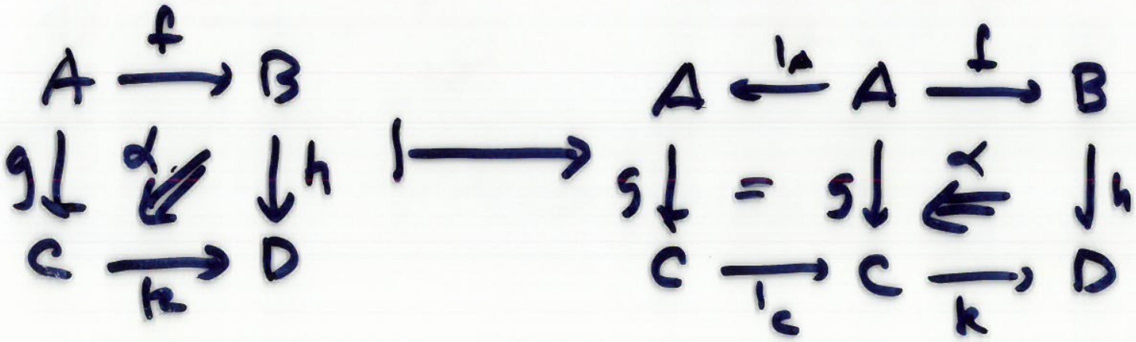
TAKE EQUIVALENCE RELATION GENERATED

DENOTE EQUIV CLASS BY $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n$

$\mathbb{S}\text{PAN}(\underline{\underline{A}})$

THEOREM TAKING CELLS TO BE EQUIVALENCE CLASSES OF DIAGRAMS AS ABOVE GIVES US AN OPLAX DOUBLE CATEGORY $\mathbb{S}\text{PAN}(\underline{\underline{A}})$

- $\mathbb{S}\text{PAN}(\underline{\underline{A}})$ IS PARANORMAL
- $\mathbb{S}\text{PAN} : \underline{\underline{2\text{-CAT}}} \longrightarrow \underline{\underline{\text{PARA}}}$ IS LEFT 2-ADJOINT TO $\text{VERT} : \underline{\underline{\text{PARA}}} \longrightarrow \underline{\underline{2\text{-CAT}}}$
- $\mathbb{Q}(\underline{\underline{A}}) \xrightarrow{(\ )_*} \mathbb{S}\text{PAN}(\underline{\underline{A}})$



IS LOCALLY FULL & FAITHFUL ∎

PROPERTIES

- $$\begin{array}{ccccc} A & \xleftarrow{s} & S & \xrightarrow{s'} & A' \\ {\scriptstyle x}\downarrow & \overset{\alpha_0}{\Leftarrow} & {\scriptstyle x_1}\downarrow & \overset{\alpha_1}{\Leftarrow} & \downarrow{\scriptstyle x'} \\ B & \xleftarrow[1]{} & B & \xrightarrow[f]{} & B' \end{array} \quad = \quad \begin{array}{ccccc} A & \xleftarrow{s} & S & \xrightarrow{s'} & A' \\ {\scriptstyle x}\downarrow & = & {\scriptstyle xs}\downarrow & \overset{\alpha_0 \cdot \alpha_1}{\Leftarrow} & \downarrow{\scriptstyle x'} \\ B & \xleftarrow[1]{} & B & \xrightarrow[f]{} & B' \end{array}$$

- **LEMMA**

$$\begin{array}{ccccccc} A & \leftarrow & S & & S & \longrightarrow & A' \\ \| & = & \downarrow & l \dashv r & \uparrow & = & \| \\ A & \leftarrow & T & & T & \longrightarrow & A' \end{array}$$

$$\Rightarrow (A \leftarrow S \rightarrow A') \cong (A \leftarrow T \rightarrow A') \quad \blacksquare$$

- **COROLLARY** $f \dashv u$

$$\Rightarrow (A \xleftarrow{f} S \xrightarrow{g} B) \cong (A \xleftarrow{1_A} A \xrightarrow{gu} B) \quad \blacksquare$$

COMMA OBJECTS

THEOREM IF $\underline{\underline{A}}$ HAS COMMA OBJECTS THEN COMPOSITES EXIST IN $\mathbf{Span}(\underline{\underline{A}})$. $\mathbf{Span}(\underline{\underline{A}})$ SATISFIES THE BECK CONDITION:

$$\begin{array}{ccc} K & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \swarrow & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \quad \text{COMMA} \quad \Rightarrow \quad \begin{array}{ccc} K & \xrightarrow{q_*} & B \\ {\scriptstyle p^*}\uparrow & \cong & \uparrow{\scriptstyle g^*} \\ A & \xrightarrow[f_*]{} & C \end{array}$$

∎

REMARK 1: COMPOSING SPANS WITH COMMA OBJECTS IS ASSOCIATIVE WITHOUT EQUIVALENCE RELATION **BUT** NOT UNITARY

REMARK 2: COMPOSITES IN $\mathbf{Span}(\underline{\underline{A}})$ $\not\Rightarrow$ COMMA OBJECTS IN $\underline{\underline{A}}$

PROPOSITION IF $B \leftarrow T \rightarrow B'$ IS A BIFIBRATION, EVERY CELL

$$\begin{array}{ccccc} A & \leftarrow & S & \rightarrow & A' \\ \downarrow & \overset{\alpha_0}{\Leftarrow} & \downarrow & \overset{\alpha_1}{\Leftarrow} & \downarrow \\ B & \leftarrow & T & \rightarrow & B' \end{array}$$

IS EQUAL TO ONE (NOT UNIQUE) IN WHICH α_0, α_1 ARE IDENTITIES. ∎

PROPOSITION IF $\underline{\underline{A}}$ HAS COMMA OBJECTS

(1) EVERY SPAN IS ISOMORPHIC TO A BIFIBRATION

(2) $\left(A \xleftarrow{d_1} A^2 \xrightarrow{d_0} A\right) \cong \left(A \xleftarrow{1} A \xrightarrow{1} A\right)$

"PROOF" (1) $S \cong (I_B S) I_A$

(2) $I_A I_A \cong I_A$ ∎

PROPOSITION CONSIDER SPANS

$$A \xleftarrow{s} S \xrightarrow{s'} B \qquad B \xleftarrow{t} T \xrightarrow{t'} C$$

AND ASSUME THE 2-PULLBACK

$$\begin{array}{ccc} P & \xrightarrow{p'} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ S & \xrightarrow[s']{} & B \end{array}$$

EXISTS. IF t IS AN OPFIBRATION OR s' A FIBRATION, THEN THE COMPOSITE OF S & T EXISTS AND IS ISOMORPHIC TO $A \xleftarrow{sp} P \xrightarrow{t'p'} C$ ■

SO IN PRESENCE OF COMMA OBJECTS AND 2-PULLBACKS WE CAN RESTRICT TO BIFIBRATIONS, COMMUTING DIAGRAMS AS CELLS & PB AS COMPOSITION. BUT THE EQUIVALENCE REL NOT TRIVIAL!