

LAX PRESHEAVES
FOR
DOUBLE CATEGORIES

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DOUBLE CATEGORIES

OBJECTS, TWO KINDS OF ARROWS $\rightarrow, \twoheadrightarrow$

CELLS

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 v \downarrow & \alpha & \downarrow w \\
 C & \xrightarrow{g} & D
 \end{array}$$

HORIZONTAL COMPOSITION OF ARROWS & CELLS

GIVES CATEGORY STRUCTURES,

VERTICAL COMPOSITION GIVES "WEAK CATEGORIES"

HAVE SPECIAL ISOMORPHISMS

$$\begin{array}{ccc}
 A = A & A = A & A = A \\
 \tilde{v} \cdot (\tilde{v} \cdot v) \downarrow & \text{id}_A \downarrow & v \downarrow \\
 \alpha & p & \lambda \\
 \downarrow & \downarrow & \downarrow \\
 C = C & C \rightrightarrows C & C = C
 \end{array}$$

SATISFY SAME CONDITIONS AS BICATEGORIES

BASIC EXAMPLE: SET

OBJECTS ARE SETS, HORIZONTAL ARROWS ARE FUNCTIONS.

VERTICAL ARROWS ARE SPANS $S: X \rightarrow Y$

CELLS ARE COMMUTATIVE DIAGRAMS

$$\begin{array}{ccc}
 & X & \xrightarrow{f} & X' \\
 (c)_0 \nearrow & & \sigma & (c)_0 \nearrow \\
 S & \longrightarrow & S' & \\
 (c)_1 \searrow & & & (c)_1 \searrow \\
 & Y & \xrightarrow{g} & Y'
 \end{array}$$

HORIZONTAL COMPOSITION OBVIOUS

VERTICAL COMPOSITION BY PULLBACK

NOT STRICTLY ASSOCIATIVE

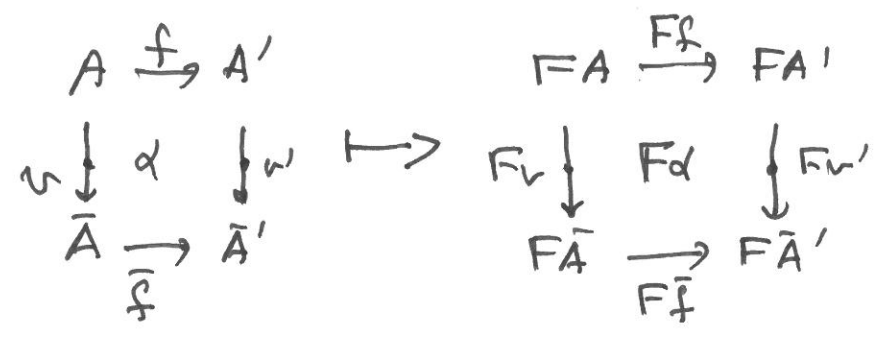
TWO OTHER EXAMPLES

$\underline{\underline{V}} \underline{\underline{A}}$ FOR A BICATEGORY $\underline{\underline{A}}$

$\underline{\underline{H}} \underline{\underline{A}}$ FOR A 2-CATEGORY $\underline{\underline{A}}$

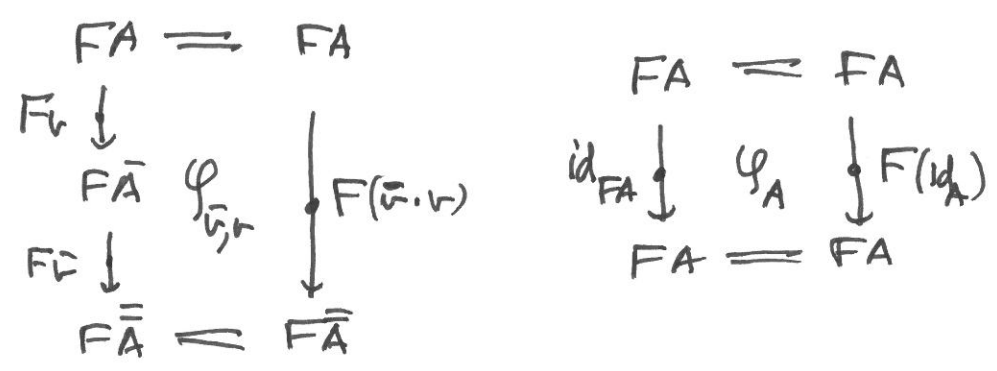
LAX FUNCTORS

A LAX FUNCTOR $F: \mathcal{A} \rightarrow \mathcal{B}$



HORIZONTALLY FUNCTORIAL

VERTICALLY PROVIDES SPECIAL COMPARISONS



SATISFYING ASSOCIATIVITY, UNITY, NATURALITY

EXAMPLE:

$F: \mathcal{V}_{\underline{\mathcal{A}}} \rightarrow \mathcal{V}_{\underline{\mathcal{B}}}$ LAX MORPHISM OF BICATS

$F: \mathcal{H}_{\underline{\mathcal{A}}} \rightarrow \mathcal{H}_{\underline{\mathcal{B}}}$ 2-FUNCTOR OF 2-CATS

HOM FUNCTORS

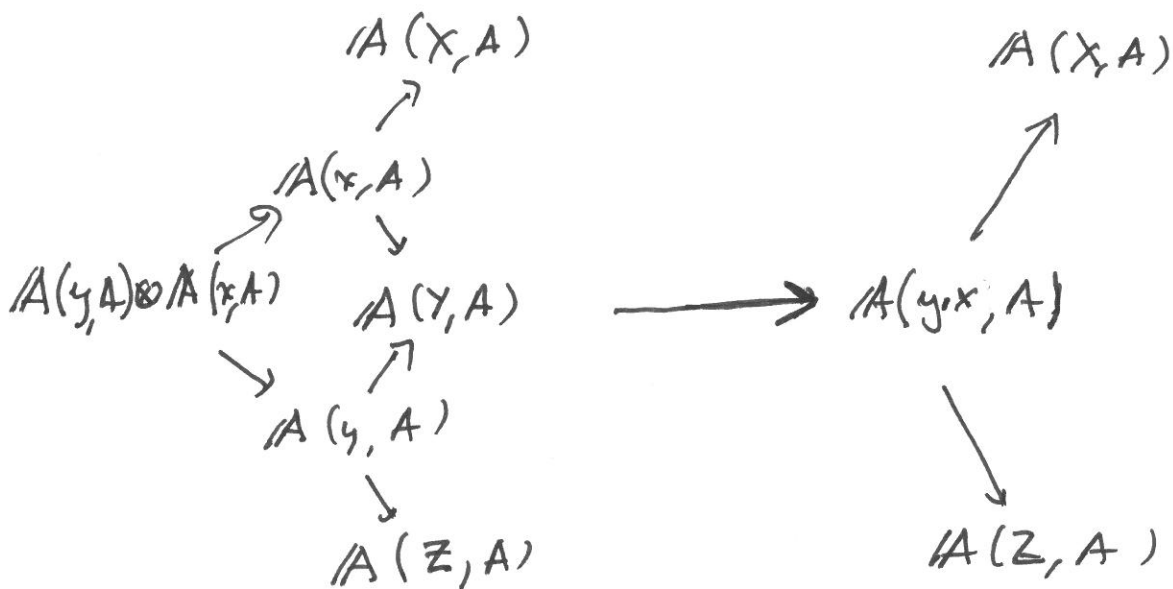
$$A(-, A) : A^{op} \rightarrow SET$$

$$X \mapsto A(X, A) = \{ f : X \rightarrow A \mid f \text{ HORIZONTAL} \}$$

$$\begin{array}{ccc}
 X & & A(X, A) \\
 x \downarrow & \longmapsto & \partial_0 \uparrow \\
 & & A(x, A) \\
 Y & & \partial_1 \downarrow \\
 & & A(Y, A)
 \end{array}
 = \left\{ \begin{array}{ccc} X & \rightarrow & A \\ x \downarrow & \Sigma & \downarrow id_A \\ Y & \rightarrow & A \end{array} \right\}$$

HORIZONTAL FUNCTORIAL BY COMPOSITION

VERTICAL COMPARISONS



$$\begin{array}{ccc}
 X \rightarrow A & & X \rightarrow A \\
 x \downarrow \xi \downarrow & & \downarrow \\
 Y \rightarrow A & \longmapsto & y.x \downarrow \theta.\xi \downarrow \\
 y \downarrow \theta \downarrow & & \downarrow \\
 Z \rightarrow A & & Z \rightarrow A
 \end{array}$$

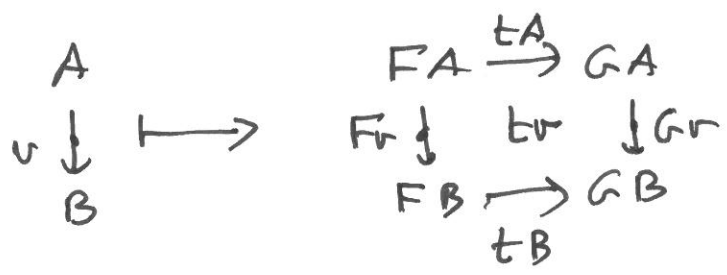
YONEDA EMBEDDING

$$Y: A \rightarrow \text{Lax}(A^{op}, \text{Set})$$

HORIZONTAL MORPHISMS ARE CALLED
NATURAL TRANSFORMATIONS

$$\epsilon: F \rightarrow G$$

$$A \mapsto \epsilon_A: FA \rightarrow GA$$



HORIZONTALLY NATURAL
VERTICALLY FUNCTORIAL

EXAMPLE: $A(-, f): A(-, A) \rightarrow A(-, A')$

EXAMPLE: $F, G: \underline{\underline{V}}_A \rightarrow \underline{\underline{V}}_B$

$\epsilon: F \rightarrow G$ LAX TRANSFORMATION

EXAMPLE: $F, G: \underline{\underline{H}}_A \rightarrow \underline{\underline{H}}_B$

$\epsilon: F \rightarrow G$ 2-NATURAL TRANSFORMATION

MODULES

A VERTICAL MORPHISM $m: F \rightarrow G$ IN $\text{LAX}(A^{\text{op}}, \text{SET})$ IS CALLED A MODULE

FOLLOWING COCKETT, KOSLOWSKI, SEELY, WOOD "MODULES" IN TAC No. 17

- $v: A \rightarrow B \mapsto m v: F A \rightarrow G B$

- $$\begin{array}{ccc}
 A \xrightarrow{f} A' & & F A \xrightarrow{F f} F A' \\
 v \downarrow \alpha \downarrow w & \mapsto & m v \downarrow m \alpha \downarrow m w' \\
 B \xrightarrow{g} B' & & F B \xrightarrow{F g} F B' \\
 s & & F s
 \end{array}$$

- $A \xrightarrow{v} B \xrightarrow{w} C$

$$\begin{array}{ccc}
 F A \rightrightarrows F A & & F A \rightrightarrows F A \\
 F v \downarrow & P & \downarrow m(w \cdot v) \\
 F B & & G B \quad \lambda \\
 m v \downarrow & & \downarrow m(w \cdot v) \\
 G C \rightrightarrows G C & & G w \downarrow \\
 & & G C \rightrightarrows G C
 \end{array}$$

- HORIZONTALLY FUNCTORIAL
- NATURALITY OF P, λ
- ASSOCIATIVITY & UNITY OF P, λ

EXAMPLE $\mathbb{A} = \mathbb{1}$

LAX $F, G : \mathbb{1} \rightarrow \text{SET} \iff \text{CATS } \underline{B}, \underline{C}$

MODULE $m : F \rightarrow G \iff \text{PROFUNCTOR } \underline{B} \rightarrow \underline{C}$

I.E. $P : \underline{B}^{\text{op}} \times \underline{C} \rightarrow \underline{\text{SET}}$

EXAMPLE $\mathbb{A} = \underline{\forall \underline{A}}$ \underline{A} CAT

LAX $F, G : \underline{\forall \underline{A}} \rightarrow \underline{\text{SET}} \iff \text{CATS } \underline{A}, \begin{array}{ccc} \underline{B} & & \underline{C} \\ & \searrow & \swarrow \\ & \underline{A} & \end{array}$

MODULE $m : F \rightarrow G \iff \text{PROFUNCTOR OVER } \underline{A}$

$$\begin{array}{ccc} \underline{B} & \xrightarrow{P} & \underline{C} \\ \downarrow & \Downarrow \pi & \downarrow \\ \underline{A} & \xrightarrow{\text{Id}_A} & \underline{A} \end{array}$$

EXAMPLE $\nu : \underline{A} \rightarrow \bar{\underline{A}}$ GIVES A MODULE

$$\underline{A}(-, \nu) : \underline{A}(-, \underline{A}) \rightarrow \underline{A}(-, \bar{\underline{A}}).$$

PROP: LAX FUNCTOR $(\underline{A} \times \underline{\forall \underline{A}})^{\text{op}} \rightarrow \underline{\text{SET}}$

IS EQUIV TO A MODULE $m : F \rightarrow G$

MODULATIONS (LOC. CIT.)

$$\begin{array}{ccc}
 F & \xrightarrow{t} & F' \\
 m \downarrow & & \downarrow m' \\
 G & \xrightarrow{s} & G'
 \end{array}$$

$$\begin{array}{ccc}
 A & & FA \xrightarrow{tA} F'A \\
 v \downarrow & \longrightarrow & mv \downarrow \quad \mu v \quad \downarrow m'v \\
 \bar{A} & & G\bar{A} \xrightarrow{sA} G'\bar{A}
 \end{array}$$

- HORIZONTAL NATURALITY
- EQUIVARIANCE

EXAMPLE FOR $A \xrightarrow{f} B$ IN \mathcal{A}

$$\begin{array}{ccc}
 v \downarrow & \alpha & \downarrow w \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{A}(-, B) & \xrightarrow{\mathcal{A}(-, f)} & \mathcal{A}(-, A) \\
 \downarrow & & \downarrow \\
 \mathcal{A}(-, \bar{B}) & \xrightarrow{\mathcal{A}(-, \bar{f})} & \mathcal{A}(-, \bar{A})
 \end{array}$$

MODULATION

Lax (A^{op}, Set)

- 2-DIMENSIONAL VERSION OF SET ^{A^{op}}
- BASIS OF CATEGORICAL UNIVERSAL ALGEBRA
- CATEGORICAL LOGIC
- SHEAF THEORY
- WHAT ARE ITS PROPERTIES?
 - COMPLETENESS ?
 - CARTESIAN CLOSED ?
 - COMPOSITION OF MODULES ?

NOTATION

Lax₀ (A^{op}, Set) CATEGORY OF LAX FUNCTORS
 $A^{op} \rightarrow \text{Set}$ WITH NATURAL TRANSFORMATIONS.

Lax₁ (A^{op}, Set) CATEGORY OF MODULES AND
 MODULATIONS.

NOTE: Lax₀ ($\forall A^{op}, \text{Set}$) \cong CAT/A NOT CC !

SPAN \underline{B}

B CATEGORY WITH PULLBACKS

SPAN \underline{B} DOUBLE CATEGORY

- OBJECTS THOSE OF B
- HORIZONTAL ARROWS ARE MORPHISMS OF B
- VERTICAL ARROWS ARE SPANS IN B
- CELLS ARE COMMUTATIVE DIAGRAMS

$$\begin{array}{ccc}
 B & \longrightarrow & C \\
 \uparrow & & \uparrow \\
 S & \xrightarrow{\sigma} & T \\
 \downarrow & & \downarrow \\
 \underline{B} & \longrightarrow & \underline{C}
 \end{array}$$

- VERTICAL COMPOSITION BY PULLBACK

EXAMPLE $\underline{\text{SET}} = \text{SPAN}(\underline{\text{SET}})$

EXAMPLE / PROPOSITION

$$\text{Lax}(\underline{\text{KHA}}^{\text{OP}}, \underline{\text{SET}}) \simeq \text{SPAN}(\underline{\text{SET}}^{\text{AOP}})$$

THEOREM LET \mathcal{A} BE A DOUBLE CATEGORY
 AND \underline{B} A CATEGORY WITH PULLBACKS.
 THEN THE CATEGORY $\underline{\text{LAX}}_0(\mathcal{A}^{\text{op}}, \text{SPAN } \underline{B})$ OF
 LAX FUNCTORS $\mathcal{A}^{\text{op}} \rightarrow \text{SPAN } \underline{B}$ WITH NATURAL
 TRANSFORMATIONS IS EQUIVALENT TO THE
 CATEGORY $\underline{\text{MOD}}_{\underline{B}}(\mathcal{Jd} \mathcal{A})$ OF MODELS IN \underline{B}
 OF A PULLBACK SKETCH $\mathcal{Jd} \mathcal{A}$.

DEFINITION A PULLBACK SKETCH IS A
 CATEGORY WITH SOME DESIGNATED PULLBACKS.
 A MODEL IS A FUNCTOR PRESERVING THOSE
 PULLBACKS. A MORPHISM IS A NATURAL
 TRANSFORMATION.

COROLLARY $\underline{\text{LAX}}_0(\mathcal{A}^{\text{op}}, \text{SET})$ AND $\underline{\text{LAX}}_1(\mathcal{A}^{\text{op}}, \text{SET})$
 $\simeq \underline{\text{LAX}}_0((\mathcal{A} \times \mathbb{2})^{\text{op}}, \text{SET})$ ARE LOCALLY FINITELY
 PRESENTABLE (LFP) CATEGORIES.

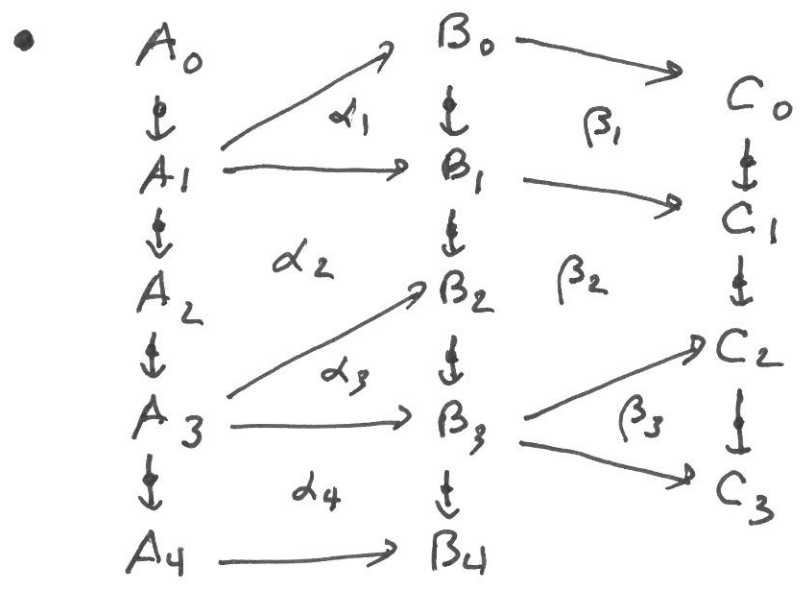
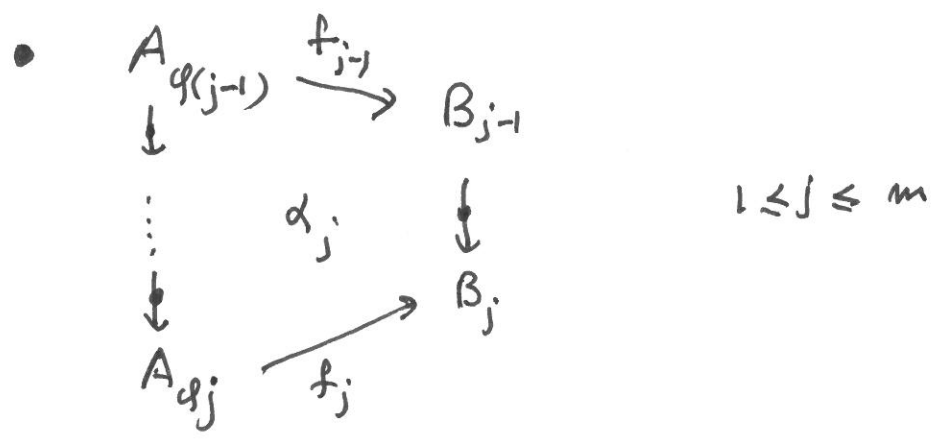
BARYCENTRIC SUBDIVISION SdA

OBJECT $(m, A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_m} A_m) = (m, A_0, v_*)$

MORPHISM $(\varphi, f, \alpha) : (m, A_0, v_*) \rightarrow (m, B_0, w_*)$

• ORDER PRES $\varphi : [m] \rightarrow [m] = \{0 < 1 < 2 < \dots < m\}$

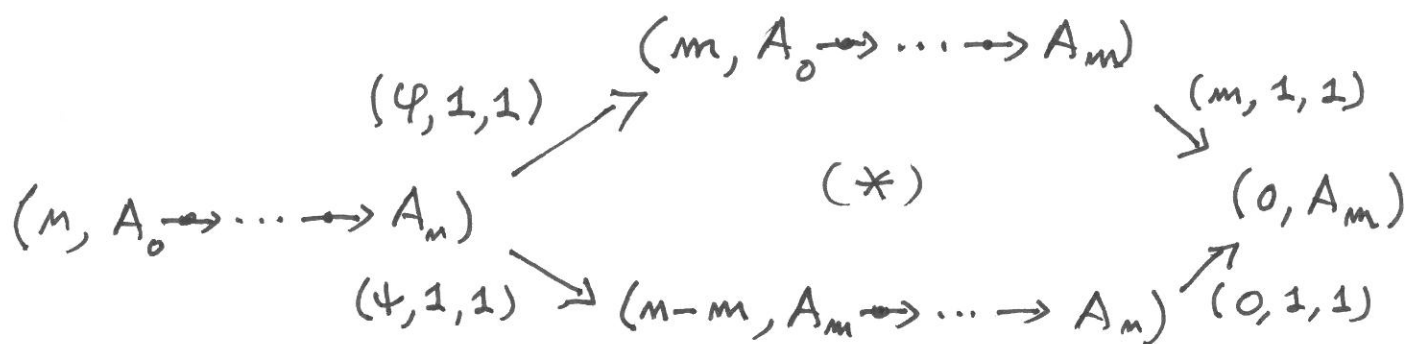
• $f_j : A_{\varphi(j)} \rightarrow B_j \quad 0 \leq j \leq m$



PROPOSITION SdA IS A CATEGORY.

DESIGNATED PULLBACKS

FOR $0 < m < n$ AND $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_n} A_n$



PROPOSITION ALL DIAGRAMS (*) ARE PULLBACKS.

THIS DETERMINES THE PULLBACK SKETCH $\mathcal{J}dA$

FUNCTORIALITY

A LAX FUNCTOR $F: A \rightarrow B$ INDUCES
A SKETCH MORPHISM

$$\mathcal{J}dF: \mathcal{J}dA \longrightarrow \mathcal{J}dB$$

COROLLARY F INDUCES A MORPHISM OF LFP

CATEGORIES $F^* : \underline{\text{Lax}}_0(\mathcal{B}^{op}, \text{Set}) \longrightarrow \underline{\text{Lax}}_0(\mathcal{A}^{op}, \text{Set})$

I.E. F^* HAS FINITE RANK (PRESERVES FILTERED COLIMITS) AND HAS A LEFT ADJOINT $F_!$. F^* ALSO PRESERVES COPRODUCTS.

COROLLARY $\partial_0, \partial_1 : \underline{\text{Lax}}_1(\mathcal{A}^{op}, \text{Set}) \longrightarrow \underline{\text{Lax}}_0(\mathcal{A}^{op}, \text{Set})$

AND $\text{id} : \underline{\text{Lax}}_0(\mathcal{A}^{op}, \text{Set}) \longrightarrow \underline{\text{Lax}}_1(\mathcal{A}^{op}, \text{Set})$

ARE LFP MORPHISMS THAT PRESERVE Σ .

THUS $\underline{\text{Lax}}(\mathcal{A}^{op}, \text{Set})$ HAS ALL LIMITS AND FILTERED COLIMITS AND COPRODUCTS WITH THE APPROPRIATE DOUBLE UNIVERSAL PROPERTY. AN LFP DOUBLE CATEGORY?

COMPOSITION OF MODULES

THE CATEGORY $\underline{\text{Lax}}_3(A^{\text{op}}, \text{Set}) = \underline{\text{Lax}}_0((A \times V\mathcal{B})^{\text{op}}, \text{Set})$

HAS OBJECTS 3 MODULES AND A MODULATION

$$\begin{array}{ccc}
 F = F & & \\
 m \downarrow & & \downarrow \\
 G \quad M & & P \\
 m \downarrow & & \downarrow \\
 H = H & &
 \end{array}$$

IS ALSO LOCALLY FINITELY PRESENTABLE.

WE CAN ALSO SHOW THAT THE CATEGORY

$\underline{\text{Lax}}_2(A^{\text{op}}, \text{Set})$ WHOSE OBJECTS ARE

COMPOSABLE PAIRS OF MODULES $F \xrightarrow{m} G \xrightarrow{m} H$

IS LFP AND THAT THE FORGETFUL FUNCTOR

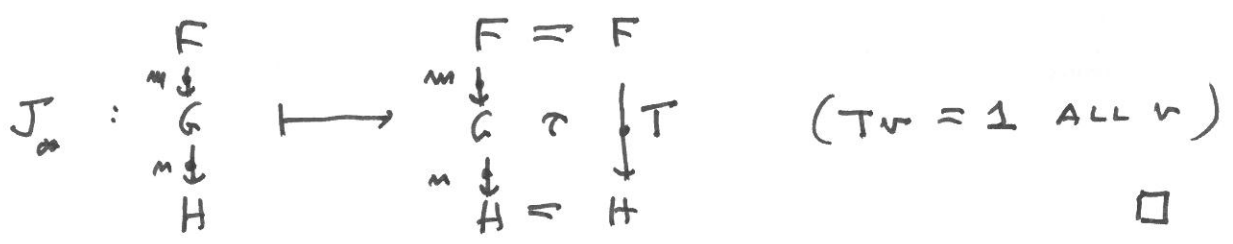
$$\underline{\text{Lax}}_3(A^{\text{op}}, \text{Set}) \longrightarrow \underline{\text{Lax}}_2(A^{\text{op}}, \text{Set})$$

IS AN LFP MORPHISM J^* .

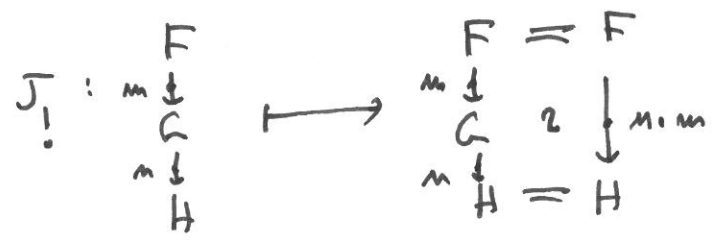
THE LEFT ADJOINT $J_! \dashv J^*$ WILL GIVE COMPOSITION, BUT WE NEED

LEMMA J^* HAS A RIGHT ADJOINT J_* AND $J^* J_* \cong 1$.

PROOF



FOLLOWS $J^* J_! \cong 1$ I.E.



m.m IS THE COMPOSITE, I.E. SATISFIES THE UNIVERSAL PROPERTY

DON'T KNOW IF IT IS ASSOCIATIVE - QUITE SURE IT'S NOT ALWAYS BUT IS IN GOOD CASES

SAME TECHNIQUE GIVES k-FOLD COMPOSITION