LAX PRESHEAVES FOR DOUBLE CATEGORIES

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JUNE 2011

DOUBLE CATEGORIES

OBJECTS, TWO KINDS OF ARROWS $\rightarrow$, $\overset{\bullet}{\rightarrow}$

CELLS

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

HORIZONTAL COMPOSITION OF ARROWS & CELLS

GIVES CATEGORY STRUCTURES,

VERTICAL COMPOSITION GIVES "WEAK CATEGORIES"

HAVE <u>SPECIAL</u> ISOMORPHISMS

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\tilde{v}\cdot(\bar{v}\cdot v)"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "(\tilde{v}\cdot\bar{v})\cdot v"] \\
C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "id_A"'] \arrow[ddr, phantom, "\rho"] & A \arrow[dd, "v"] \\
A \arrow[d, "v"'] & \\
C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"'] \arrow[ddr, phantom, "\lambda"] & A \arrow[dd, "v"] \\
C \arrow[d, "id_C"'] & \\
C \arrow[r, equal] & C
\end{tikzcd}$$

SATISFY SAME CONDITIONS AS BICATEGORIES

BASIC EXAMPLE: $\mathbb{S}\mathrm{ET}$

OBJECTS ARE SETS, HORIZONTAL ARROWS ARE FUNCTIONS.

VERTICAL ARROWS ARE SPANS $S : X \to\!\!\!\!\bullet\; Y$

CELLS ARE COMMUTATIVE DIAGRAMS

$$\begin{tikzcd}
& X \arrow[rr, "f"] & & X' \\
S \arrow[ur, "(\,)_0"] \arrow[dr, "(\,)_1"'] \arrow[rr, "\sigma"] & & S' \arrow[ur, "(\,)_0"] \arrow[dr, "(\,)_1"] & \\
& Y \arrow[rr, "g"'] & & Y'
\end{tikzcd}$$

HORIZONTAL COMPOSITION OBVIOUS

VERTICAL COMPOSITION BY PULLBACK

NOT STRICTLY ASSOCIATIVE

TWO OTHER EXAMPLES

$\mathbb{V}\underline{\underline{A}}$ FOR A BICATEGORY $\underline{\underline{A}}$

$\mathbb{H}\underline{\underline{A}}$ FOR A 2-CATEGORY $\underline{\underline{A}}$

LAX FUNCTORS

A LAX FUNCTOR $F : \mathbb{A} \longrightarrow \mathbb{B}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\ \bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fv"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'"] \\ F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}' \end{tikzcd}$$

HORIZONTALLY FUNCTORIAL

VERTICALLY PROVIDES SPECIAL COMPARISONS

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "F(\bar{v}\cdot v)"] \\ F\bar{A} \arrow[d, "F\bar{v}"'] \arrow[r, phantom, "\varphi_{\bar{v},v}"] & \\ F\bar{\bar{A}} \arrow[r, equal] & F\bar{\bar{A}} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "id_{FA}"'] \arrow[dr, phantom, "\varphi_A"] & FA \arrow[d, "F(id_A)"] \\ FA \arrow[r, equal] & FA \end{tikzcd}$$

SATISFYING ASSOCIATIVITY, UNITY, NATURALITY

EXAMPLE:

$F : \mathbb{V}\underline{\underline{A}} \longrightarrow \mathbb{V}\underline{\underline{B}}$ LAX MORPHISM OF BICATS

$F : \mathbb{H}\underline{\underline{A}} \longrightarrow \mathbb{H}\underline{\underline{B}}$ 2-FUNCTOR OF 2-CATS

HOM FUNCTORS

$$\mathbb{A}(-,A) : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{ET}$$

$$X \longmapsto \mathbb{A}(X,A) = \{ f : X \to A \mid f \text{ HORIZONTAL} \}$$

$$\begin{tikzcd} X \arrow[d, "x"'] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} \mathbb{A}(X,A) \\ \mathbb{A}(x,A) \arrow[u, "\partial_0"] \arrow[d, "\partial_1"'] \\ \mathbb{A}(Y,A) \end{tikzcd} = \left\{ \begin{tikzcd} X \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\xi"] & A \arrow[d, "id_A"] \\ Y \arrow[r] & A \end{tikzcd} \right\}$$

HORIZONTALLY FUNCTORIAL BY COMPOSITION

VERTICAL COMPARISONS

$$\begin{tikzcd}[column sep=small] & & \mathbb{A}(X,A) & & & \mathbb{A}(X,A) \\ & \mathbb{A}(x,A) \arrow[ur] \arrow[dr] & & & & \\ \mathbb{A}(y,A)\otimes\mathbb{A}(x,A) \arrow[ur] \arrow[dr] & & \mathbb{A}(Y,A) \arrow[rr] & & \mathbb{A}(y\cdot x,A) \arrow[uur] \arrow[ddr] & \\ & \mathbb{A}(y,A) \arrow[ur] \arrow[dr] & & & & \\ & & \mathbb{A}(Z,A) & & & \mathbb{A}(Z,A) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\xi"] & A \arrow[d] \\ Y \arrow[r] \arrow[d, "y"'] \arrow[dr, phantom, "\theta"] & A \arrow[d] \\ Z \arrow[r] & A \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[r] \arrow[d, "y\cdot x"'] \arrow[dr, phantom, "\theta\cdot\xi"] & A \arrow[d] \\ Z \arrow[r] & A \end{tikzcd}$$

YONEDA EMBEDDING

$$Y : \mathbb{A} \longrightarrow \mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{ET})$$

HORIZONTAL MORPHISMS ARE CALLED

NATURAL TRANSFORMATIONS

$$t : F \longrightarrow G$$

$$A \longmapsto tA : FA \longrightarrow GA$$

$$\begin{tikzcd} A \arrow[d, "v"'] \\ B \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "Fv"'] \arrow[dr, phantom, "tv"] & GA \arrow[d, "Gv"] \\ FB \arrow[r, "tB"'] & GB \end{tikzcd}$$

HORIZONTALLY NATURAL

VERTICALLY FUNCTORIAL

EXAMPLE: $\mathbb{A}(-, f) : \mathbb{A}(-, A) \longrightarrow \mathbb{A}(-, A')$

EXAMPLE: $F, G : \mathbb{V}\underline{A} \longrightarrow \mathbb{V}\underline{\underline{B}}$

$t : F \to G$ LAX TRANSFORMATION

EXAMPLE: $F, G : \mathbb{H}\underline{A} \longrightarrow \mathbb{H}\underline{\underline{B}}$

$t : F \to G$ 2-NATURAL TRANSFORMATION

MODULES

A VERTICAL MORPHISM $m : F \to G$ IN $\mathbb{L}ax(\mathbb{A}^{op}, \mathbb{S}et)$ IS CALLED A MODULE

FOLLOWING COCKETT, KOSLOWSKI, SEELY, WOOD "MODULES" IN TAC No. 17

- $v : A \to B \quad \longmapsto \quad mv : FA \to GB$

- $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "w", "\bullet" marking] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "mv"', "\bullet" marking] \arrow[dr, phantom, "m\alpha"] & FA' \arrow[d, "mv'", "\bullet" marking] \\ FB \arrow[r, "Fg"'] & FB' \end{tikzcd}$$

- $A \xrightarrow{v} B \xrightarrow{w} C$

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"', "\bullet" marking] & FA \arrow[dd, "m(w \cdot v)", "\bullet" marking] \\ FB \arrow[d, "mw"', "\bullet" marking] \arrow[r, phantom, "\rho"] & \\ GC \arrow[r, equal] & GC \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "mv"', "\bullet" marking] & FA \arrow[dd, "m(w \cdot v)", "\bullet" marking] \\ GB \arrow[d, "Gw"', "\bullet" marking] \arrow[r, phantom, "\lambda"] & \\ GC \arrow[r, equal] & GC \end{tikzcd}$$

- HORIZONTALLY FUNCTORIAL
- NATURALITY OF ρ, λ
- ASSOCIATIVITY & UNITY OF ρ, λ

$\underline{\text{EXAMPLE}}$ $\mathbb{A} = \mathbb{1}$

LAX $F, G : \mathbb{1} \to \mathbb{S}\text{ET} \iff$ CATS $\underline{B}, \underline{C}$

MODULE $m : F \to G \iff$ PROFUNCTOR $\underline{B} \to \underline{C}$

I.E. $P : \underline{B}^{op} \times \underline{C} \to \underline{\text{SET}}$

$\underline{\text{EXAMPLE}}$ $\mathbb{A} = \mathbb{V}\underline{A}$ $\quad \underline{A}$ CAT

LAX $F, G : \mathbb{V}\underline{A} \to \mathbb{S}\text{ET} \iff \text{CATS}/\underline{A}$,
$$\begin{tikzcd} \underline{B} \arrow[dr] & & \underline{C} \arrow[dl] \\ & \underline{A} & \end{tikzcd}$$

MODULE $m : F \to G \iff$ PROFUNCTOR OVER $\underline{A}$

$$\begin{tikzcd} \underline{B} \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\Downarrow \pi"] & \underline{C} \arrow[d] \\ \underline{A} \arrow[r, "\mathrm{Id}_{\underline{A}}"'] & \underline{A} \end{tikzcd}$$

$\underline{\text{EXAMPLE}}$ $v : A \to \bar{A}$ GIVES A MODULE
$$\mathbb{A}(-, v) : \mathbb{A}(-, A) \longrightarrow \mathbb{A}(-, \bar{A}).$$

$\underline{\text{PROP}}$: LAX FUNCTOR $(\mathbb{A} \times \mathbb{V}2)^{op} \longrightarrow \mathbb{S}\text{ET}$

IS EQUIV TO A MODULE $m : F \to G$

MODULATIONS (LOC. CIT.)

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[d, "m"'] \arrow[dr, phantom, "\mu"] & F' \arrow[d, "m'"] \\ G \arrow[r, "s"'] & G' \end{tikzcd}$$

- $$\begin{tikzcd} A \arrow[d, "v"'] \\ \bar{A} \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "mv"'] \arrow[dr, phantom, "\mu v"] & F'A \arrow[d, "m'v"] \\ G\bar{A} \arrow[r, "sA"'] & G'\bar{A} \end{tikzcd}$$

- HORIZONTAL NATURALITY
- EQUIVARIANCE

EXAMPLE FOR $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ \bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \end{tikzcd}$ IN $\mathbb{A}$

$$\begin{tikzcd} \mathbb{A}(-,B) \arrow[r, "\mathbb{A}(-,f)"] \arrow[d, "\mathbb{A}(-,w)"'] \arrow[dr, phantom, "\mathbb{A}(-,\alpha)"] & \mathbb{A}(-,A) \arrow[d, "\mathbb{A}(-,v)"] \\ \mathbb{A}(-,\bar{B}) \arrow[r, "\mathbb{A}(-,\bar{f})"'] & \mathbb{A}(-,\bar{A}) \end{tikzcd}$$

MODULATION

$\mathbb{L}ax(\mathbb{A}^{op}, \mathbb{S}et)$

- 2-DIMENSIONAL VERSION OF $\underline{Set}^{\underline{A}^{op}}$
- BASIS OF CATEGORICAL UNIVERSAL ALGEBRA
- CATEGORICAL LOGIC
- SHEAF THEORY

- WHAT ARE ITS PROPERTIES?
  - COMPLETENESS ?
  - CARTESIAN CLOSED ?
  - COMPOSITION OF MODULES ?

NOTATION

$\underline{Lax}_0(\mathbb{A}^{op}, \mathbb{S}et)$ CATEGORY OF LAX FUNCTORS $\mathbb{A}^{op} \to \mathbb{S}et$ WITH NATURAL TRANSFORMATIONS.

$\underline{Lax}_1(\mathbb{A}^{op}, \mathbb{S}et)$ CATEGORY OF MODULES AND MODULATIONS.

NOTE: $\underline{Lax}_0(\mathbb{V}\underline{A}^{op}, \mathbb{S}et) \simeq \underline{Cat}/\underline{A}$ NOT CC !

$\mathbb{S}\mathrm{PAN}\ \underline{B}$

$\underline{B}$ CATEGORY WITH PULLBACKS

$\mathbb{S}\mathrm{PAN}\ \underline{B}$ DOUBLE CATEGORY

- OBJECTS THOSE OF $\underline{B}$
- HORIZONTAL ARROWS ARE MORPHISMS OF $\underline{B}$
- VERTICAL ARROWS ARE SPANS IN $\underline{B}$
- CELLS ARE COMMUTATIVE DIAGRAMS

$$\begin{tikzcd}
B \arrow[r] & C \\
S \arrow[u] \arrow[d] \arrow[r, "\sigma"] & T \arrow[u] \arrow[d] \\
\bar{B} \arrow[r] & \bar{C}
\end{tikzcd}$$

- VERTICAL COMPOSITION BY PULLBACK

<u>EXAMPLE</u> $\mathbb{S}\mathrm{ET} = \mathbb{S}\mathrm{PAN}(\underline{\mathrm{SET}})$

<u>EXAMPLE/PROPOSITION</u>

$$\mathbb{L}\mathrm{AX}(\mathbb{H}\underline{A}^{op}, \mathbb{S}\mathrm{ET}) \simeq \mathbb{S}\mathrm{PAN}\left(\underline{\mathrm{SET}}^{\underline{A}^{op}}\right)$$

<u>THEOREM</u> LET $\mathbb{A}$ BE A DOUBLE CATEGORY AND $\underline{B}$ A CATEGORY WITH PULLBACKS. THEN THE CATEGORY $\underline{\mathrm{LAX}}_0(\mathbb{A}^{op}, \mathbb{S}\mathrm{PAN}\,\underline{B})$ OF LAX FUNCTORS $\mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{PAN}\,\underline{B}$ WITH NATURAL TRANSFORMATIONS IS EQUIVALENT TO THE CATEGORY $\underline{\mathrm{MOD}}_{\underline{B}}(\mathcal{J}d\,\mathbb{A})$ OF MODELS IN $\underline{B}$ OF A PULLBACK SKETCH $\mathcal{J}d\,\mathbb{A}$.

<u>DEFINITION</u> A <u>PULLBACK SKETCH</u> IS A CATEGORY WITH SOME DESIGNATED PULLBACKS. A <u>MODEL</u> IS A FUNCTOR PRESERVING THOSE PULLBACKS. A <u>MORPHISM</u> IS A NATURAL TRANSFORMATION.

<u>COROLLARY</u> $\underline{\mathrm{LAX}}_0(\mathbb{A}^{op}, \mathbb{S}\mathrm{ET})$ AND $\underline{\mathrm{LAX}}_1(\mathbb{A}^{op}, \mathbb{S}\mathrm{ET}) \simeq \underline{\mathrm{LAX}}_0((\mathbb{A}\times\mathbb{V}2)^{op}, \mathbb{S}\mathrm{ET})$ ARE LOCALLY FINITELY PRESENTABLE (LFP) CATEGORIES.

BARYCENTRIC SUBDIVISION $Sd\mathcal{A}$

OBJECT $\quad (m, A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \cdots \xrightarrow{v_m} A_m) = (m, A_\bullet, v_\bullet)$

MORPHISM $\quad (\varphi, f_\bullet, \alpha_\bullet) : (m, A_\bullet, v_\bullet) \longrightarrow (n, B_\bullet, w_\bullet)$

- ORDER PRES $\varphi : [n] \longrightarrow [m] = \{0 < 1 < 2 < \cdots < m\}$
- $f_j : A_{\varphi(j)} \longrightarrow B_j \qquad 0 \leq j \leq n$
- $$\begin{tikzcd}
A_{\varphi(j-1)} \arrow[r, "f_{j-1}"] \arrow[d] & B_{j-1} \arrow[dd] \\
\vdots \arrow[d] \arrow[r, phantom, "\alpha_j"] & \\
A_{\varphi j} \arrow[r, "f_j"] & B_j
\end{tikzcd} \qquad 1 \leq j \leq n$$
- $$\begin{tikzcd}
A_0 \arrow[d] \arrow[dr, phantom, "\alpha_1"] & B_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1"] & C_0 \arrow[d] \\
A_1 \arrow[d] \arrow[ur] \arrow[r] \arrow[dr, phantom, "\alpha_2"] & B_1 \arrow[d] \arrow[r] \arrow[dr, phantom, "\beta_2"] & C_1 \arrow[d] \\
A_2 \arrow[d] \arrow[dr, phantom, "\alpha_3"] & B_2 \arrow[d] \arrow[dr, phantom, "\beta_3"] & C_2 \arrow[d] \\
A_3 \arrow[d] \arrow[ur] \arrow[r] \arrow[dr, phantom, "\alpha_4"] & B_3 \arrow[d] \arrow[ur] \arrow[r] & C_3 \\
A_4 \arrow[r] & B_4 &
\end{tikzcd}$$

PROPOSITION $\quad Sd\mathcal{A}$ IS A CATEGORY.

DESIGNATED PULLBACKS

FOR $0 < m < n$ AND $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \cdots \xrightarrow{v_n} A_n$

$$
\begin{tikzcd}
 & (m, A_0 \to \cdots \to A_m) \arrow[dr, "{(m,1,1)}"] & \\
(n, A_0 \to \cdots \to A_n) \arrow[ur, "{(\varphi,1,1)}"] \arrow[dr, "{(\psi,1,1)}"'] & (*) & (0, A_m) \\
 & (n-m, A_m \to \cdots \to A_n) \arrow[ur, "{(0,1,1)}"'] &
\end{tikzcd}
$$

PROPOSITION ALL DIAGRAMS $(*)$ ARE PULLBACKS.

THIS DETERMINES THE PULLBACK SKETCH $\mathcal{S}d\,\mathbb{A}$

FUNCTORIALITY

A LAX FUNCTOR $F : \mathbb{A} \to \mathbb{B}$ INDUCES A SKETCH MORPHISM

$$\mathcal{S}d\,F : \mathcal{S}d\,\mathbb{A} \longrightarrow \mathcal{S}d\,\mathbb{B}$$

<u>COROLLARY</u> F INDUCES A MORPHISM OF LFP CATEGORIES $F^*: \underrightarrow{\text{LAX}}_0(\mathbb{B}^{op}, \text{SET}) \longrightarrow \underrightarrow{\text{LAX}}_0(\mathbb{A}^{op}, \text{SET})$ I.E. F^* HAS FINITE RANK (PRESERVES FILTERED COLIMITS) AND HAS A LEFT ADJOINT $F_!$. F^* ALSO PRESERVES COPRODUCTS.

<u>COROLLARY</u> $\partial_0, \partial_1 : \underrightarrow{\text{LAX}}_1(\mathbb{A}^{op}, \text{SET}) \to \underrightarrow{\text{LAX}}_0(\mathbb{A}^{op}, \text{SET})$ AND $\text{id} : \underrightarrow{\text{LAX}}_0(\mathbb{A}^{op}, \text{SET}) \longrightarrow \underrightarrow{\text{LAX}}_1(\mathbb{A}^{op}, \text{SET})$ ARE LFP MORPHISMS THAT PRESERVE Σ.

THUS $\mathbb{L}\text{AX}(\mathbb{A}^{op}, \text{SET})$ HAS ALL LIMITS AND FILTERED COLIMITS AND COPRODUCTS WITH THE APPROPRIATE DOUBLE UNIVERSAL PROPERTY. AN LFP DOUBLE CATEGORY?

COMPOSITION OF MODULES

THE CATEGORY $\underline{\mathrm{LAX}}_3(\mathbb{A}^{op}, \mathbf{Set}) = \underline{\mathrm{LAX}}_0((\mathbb{A}\times\mathbb{B})^{op}, \mathbf{Set})$

HAS OBJECTS 3 MODULES AND A MODULATION

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, "m"'] & F \arrow[dd, "p"] \\ G \arrow[r, phantom, "\mu"] \arrow[d, "n"'] & \\ H \arrow[r, equal] & H \end{tikzcd}$$

IS ALSO LOCALLY FINITELY PRESENTABLE.

WE CAN ALSO SHOW THAT THE CATEGORY

$\underline{\mathrm{LAX}}_2(\mathbb{A}^{op}, \mathbf{Set})$ WHOSE OBJECTS ARE

COMPOSABLE PAIRS OF MODULES $F \xrightarrow{m} G \xrightarrow{n} H$

IS LFP AND THAT THE FORGETFUL FUNCTOR

$$\underline{\mathrm{LAX}}_3(\mathbb{A}^{op}, \mathbf{Set}) \longrightarrow \underline{\mathrm{LAX}}_2(\mathbb{A}^{op}, \mathbf{Set})$$

IS AN LFP MORPHISM J^*.

THE LEFT ADJOINT $J_! \dashv J^*$ WILL GIVE COMPOSITION, BUT WE NEED

<u>LEMMA</u> J^* HAS A RIGHT ADJOINT J_∞ AND $J^* J_\infty \cong 1$.

<u>PROOF</u>

$$
J_\infty : \begin{tikzcd} F \arrow[d, "m"'] \\ G \arrow[d, "m"'] \\ H \end{tikzcd}
\longmapsto
\begin{tikzcd} F \arrow[r, equal] \arrow[d, "m"'] & F \arrow[dd, "T"] \\ G \arrow[d, "m"'] \arrow[r, phantom, "\tau"] & \\ H \arrow[r, equal] & H \end{tikzcd}
\qquad (T\tau = 1 \text{ ALL } \tau)
$$

$\square$

FOLLOWS $J^* J_! \cong 1$ I.E.

$$
J_! : \begin{tikzcd} F \arrow[d, "m"'] \\ G \arrow[d, "m"'] \\ H \end{tikzcd}
\longmapsto
\begin{tikzcd} F \arrow[r, equal] \arrow[d, "m"'] & F \arrow[dd, "m \circ m"] \\ G \arrow[d, "m"'] \arrow[r, phantom, "2"] & \\ H \arrow[r, equal] & H \end{tikzcd}
$$

$m \circ m$ IS THE COMPOSITE, I.E. SATISFIES THE UNIVERSAL PROPERTY

DON'T KNOW IF IT IS ASSOCIATIVE — QUITE SURE IT'S NOT ALWAYS BUT IS IN GOOD CASES

SAME TECHNIQUE GIVES k-FOLD COMPOSITION