CATEGORY THEORY

AND THE WORK OF

JIM LAMBEK

BETWEEN 1966 & NOW

35 - 49 PAPERS IN CATEGORY TH

PERHAPS MORE!

$$\left.\begin{array}{l}\cdot\ \text{RING THEORY}\\ \cdot\ \text{LOGIC}\\ \cdot\ \text{LINGUISTICS}\end{array}\right\}\leftrightsquigarrow \text{CATEGORY THEORY}$$

$\otimes$, HOM, $[M\otimes N, P] \cong [M, \mathrm{Hom}(N,P)]$

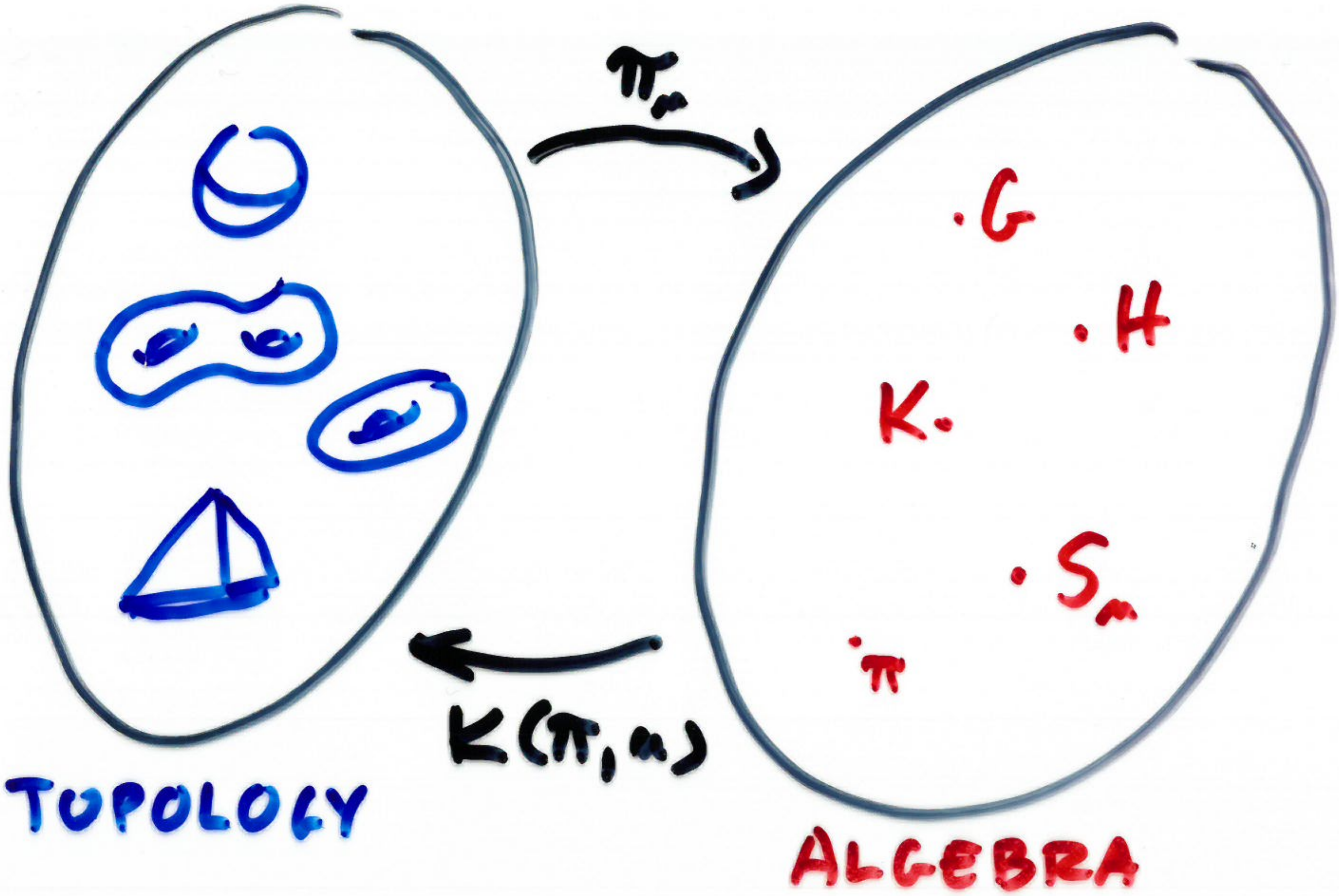
$\oplus$, KER, QUOTIENTS

DEDUCTIVE SYSTEMS

SYNTAX

• 1945 EILENBERG & MAC LANE
"GENERAL THEORY OF NATURAL EQUIVALENCES" (TRANS AMS)

ALGEBRAIC TOPOLOGY



NATURALITY

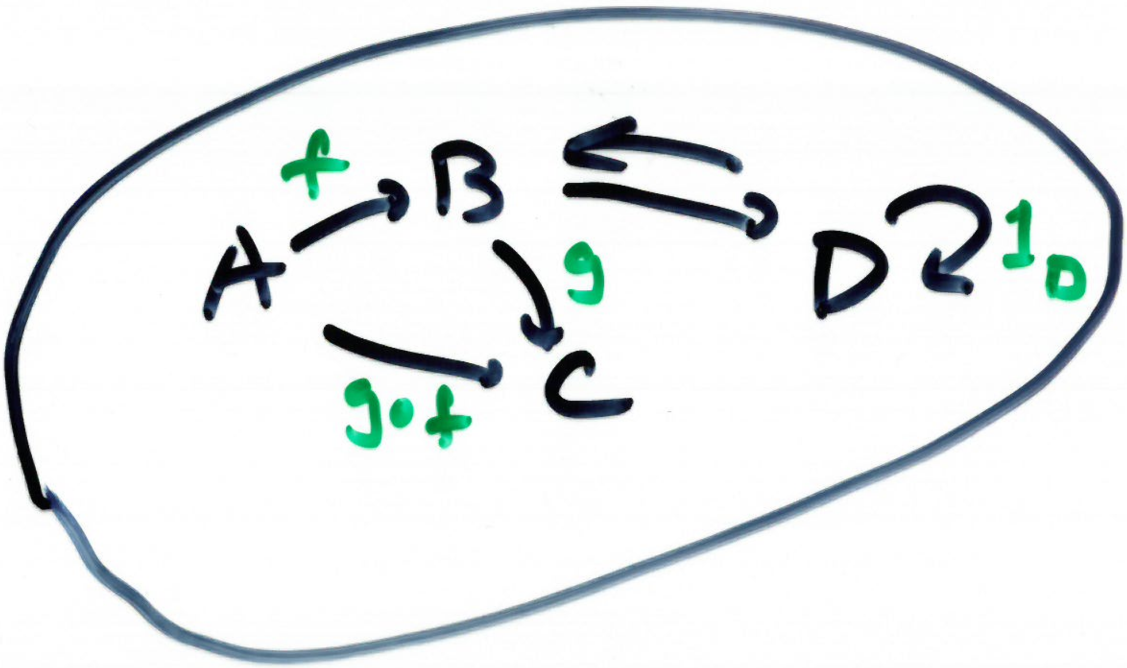
$$V \cong V^* \quad \text{vs} \quad V \xrightarrow{\cong} V^{**}$$

A CATEGORY HAS OBJECTS

$A, B, C, \ldots$

AND ARROWS

$f, g, h, \ldots$



TWO OBJECTS ARE <u>ISOMORPHIC</u>

IF

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

$$f^{-1} \circ f = 1_A \quad \& \quad f \circ f^{-1} = 1_B$$

EILENBERG – MAC LANE

Top – OBJECTS = TOPOLOGICAL SPACES

– ARROWS = CONTINUOUS FUNCTIONS

Gr – OBJECTS = GROUPS

– ARROWS = HOMOMORPHISMS

A CATEGORY CORRESPONDS ROUGHLY TO A BRANCH OF MATEMATICS.

THE NATURAL EVOLUTION OF F. KLEIN'S ERLANGER PROGRAMM

STONE DUALITY (TRANS AMS '37, '38)

<u>BOOLEAN ALGEBRA</u>

B, $\wedge$, $\vee$, $(\,)'$, 0, 1

$a \wedge a = a$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$(a \wedge b)' = a' \vee b'$

<u>Ex</u>: $P(X)$ = SET OF SUBSETS OF X.

STONE: TAKE CLOPEN SETS IN TOP SP.

" BOOLEAN ALGEBRA & TOTALLY DISCONNECTED T_2 SPACES ARE <u>SAME SUBJECT</u> "

Cartesian Products

Topology	$X \times Y$
Groups	$G \times H$
Vector Sp	$V \oplus W$

$$
\begin{tikzcd}
& & A \\
X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, "\langle f, g \rangle"] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
& & B
\end{tikzcd}
$$

$$[X, A \times B] \cong [X, A] \times [X, B]$$

Duality

$$\underline{A} \longleftrightarrow \underline{A}^{op}$$

$$
\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[ur, "f"] \arrow[r, "g \circ f"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& B \arrow[dl, "\bar{f}"'] \\
A & C \arrow[u, "\bar{g}"'] \arrow[l, "\bar{f} \circ \bar{g} = \overline{g \circ f}"]
\end{tikzcd}
$$

EVERY CONCEPT HAS A DUAL
EVERY THEOREM HAS A DUAL

<u>COPRODUCTS</u>

$$\begin{tikzcd}
A \arrow[dr, "i_1"'] \arrow[drr, bend left, "f"] & & \\
& A+B \arrow[r, "{[f,g]}"] & X \\
B \arrow[ur, "i_2"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

TOPOLOGY	$X+Y$
GROUPS	$G*H$
VECTOR SP	$V \oplus W$

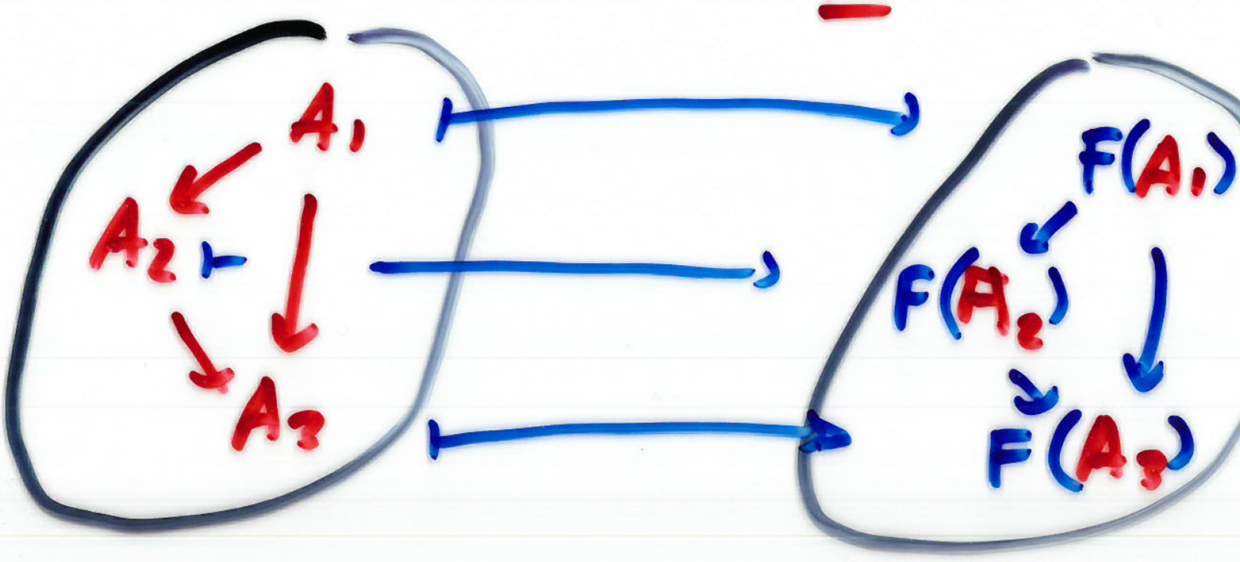
<u>STONE DUALITY</u>

$$\underline{\mathrm{Bool}} \sim \underline{\mathrm{Stone}}^{op}$$

CATEGORY THEORY IN THE 50's

- ABELIAN CATEGORIES
- MITCHELL & FREYD EMBEDDING THEOREMS
- N. YONEDA 1954
  $[X,A] \cong [X,B] \Rightarrow A \cong B$
- D. KAN "ADJOINT FUNCTORS" (TRANS AMS 1958)

FUNCTOR $F : \underline{A} \longrightarrow \underline{B}$



$$\begin{tikzcd} \underline{A} \arrow[r, bend left, "F"] & \underline{B} \arrow[l, bend left, "U"] \end{tikzcd}$$

F IS **LEFT** **ADJOINT** TO U

$$[FA, B] \cong [A, UB]$$

$$\frac{FA \longrightarrow B}{A \longrightarrow UB}$$

$$\begin{tikzcd} & FA \arrow[r, "h"] & B \\ UFA \arrow[r, "Uh"] & B & \\ A \arrow[u, "\eta_A"] \arrow[ur, "g"'] & & \end{tikzcd}$$

YONEDA $\Rightarrow$ ADJOINTS ARE UNIQUE IF THEY EXIST

PROP

$\Sigma = \{p, q, r, s, \ldots\}$ ALPHABET

WFF OF PROPOSITIONAL LOGIC $\wedge, \vee, \Rightarrow, \neg, T, F$

$(p \vee q) \Rightarrow p$, $p \vee \neg p$

OBJ: WFF's

ARROWS: EXACTLY ONE $\varphi \to \psi$ IF $\varphi \vdash \psi$

E.g. $\neg p \vee \neg q \vdash \neg(p \wedge q)$

$$\frac{\theta \vdash \varphi \ \ \& \ \ \theta \vdash \psi}{\theta \vdash \varphi \wedge \psi}$$

$\wedge$ IS PRODUCT

$\vee$ IS COPRODUCT

$$\frac{\varphi \wedge \psi \vdash \theta}{\varphi \vdash \psi \Rightarrow \theta}$$

"DEDUCTION THEOREM"

$$\underline{\mathrm{Prop}} \underset{\psi \Rightarrow (\ )}{\overset{(\ )\wedge\psi}{\rightleftarrows}} \underline{\mathrm{Prop}}$$

$(\ )\wedge\psi$
LEFT ADJ
TO
$\psi \Rightarrow (\ )$

ANY ORDERED SET $(X, \leq)$ CAN BE VIEWED AS A CATEGORY

$$x \longrightarrow y \quad \text{iff} \quad x \leq y$$

PROD IS INF $\quad x \wedge y$
COPROD IS SUP $\quad x \vee y$
MIGHT ALSO HAVE $\quad x \Rightarrow y$
(HEYTING ALGEBRA)

COMPLETIONS OF CATEGORIES

$(X, \leq) \hookrightarrow \hat{X}$

DEDIKIND - MAC NIELLE COMPLETION (CUTS)

$\underline{A} \overset{Y}{\hookrightarrow} \underline{Set}^{\underline{A}^{op}}$

FREE COMPLETION UNDER COLIMITS (SUPS)

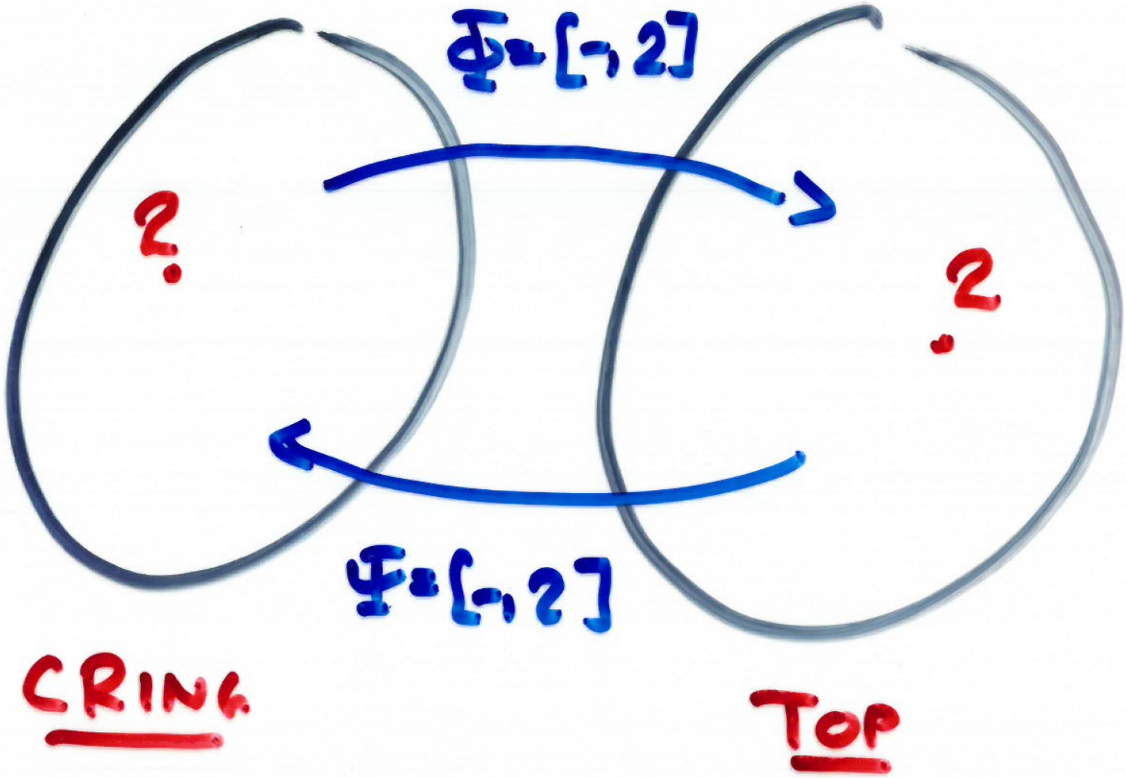
ALSO HAS LIMITS (INFS)

- BETTER PRESERVATION PROPERTIES
- $\underline{A}$ LARGE – PROPER
- JOYAL – FREE BICOMPLETIONS

DUALITY & LOCALIZATION

(w. B. RATTRAY)

· STONE



$$[R, \Psi(X)] \cong [X, \Phi(R)]$$

$$\mathrm{Fix}(\Psi\Phi) \sim \mathrm{Fix}(\Phi\Psi)^{op}$$

$$\left.\begin{array}{l} \alpha_X : X \longrightarrow \Phi\Psi(X) \text{ iso} \\ \quad\Updownarrow \\ \beta_R : R \longrightarrow \Psi\Phi(R) \text{ iso} \end{array}\right\} \Longrightarrow \begin{array}{l} \mathrm{Fix}(\Psi\Phi) \ \& \\ \mathrm{Fix}(\Phi\Psi) \\ \text{REFLECTIVE} \end{array}$$

DEDUCTIVE SYSTEMS & CATEGORIES

- DEDUCTIVE SYSTEMS & CATEGORIES I: SYNTAKTIC CALCULUS & RESIDUATED CATEGORIES (1968)
- DEDUCTIVE SYSTEMS & CATEGORIES II: STANDARD CONSTRUCTIONS & CLOSED CATEGORIES (1969)
- DEDUCTIVE SYSTEMS & CATEGORIES III: CARTESIAN CLOSED CATEGORIES, INTUITIONISTIC PROPOSITIONAL CALCULUS & COMBINATORY LOGIC (1972)
- WITH P. SCOTT - INTRODUCTION TO HIGHER ORDER CATEGORICAL LOGIC (1986)

BUILD FREE CATEGORIES WITH STRUCTURE USING DEDUCTIVE SYSTEMS

RESIDUATED CATEGORIES
(BICLOSED MONOIDAL CATEGORIES)

$(\underline{A}, \otimes, /, \backslash, a, b, c)$

$$\dfrac{\dfrac{A \otimes B \longrightarrow C}{A \longrightarrow B \backslash C}}{B \longrightarrow C/B}$$

$B \Rightarrow C$

$C \Leftarrow B$

<u>Ex</u>: R-R-BIMODULES

SYNTACTIC CALCULUS

WHAT DOES THE FREE RES CAT GENERATED BY A CAT $\underline{A}$ LOOK LIKE?

DEDUCTIVE SYSTEM $\underline{D}(\underline{A})$

<u>TERMS</u> $A, B \in Ob(\underline{A})$

$X \otimes Y$, X/Y, $Y\backslash X$

<u>FORMULAS</u> $X \to Y$

<u>AXIOMS</u> $Ax_f : A \to B$ FOR $f : A \to B$

$Ax_1 : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$

$$\frac{X \otimes Y \to Z}{X \to Z/Y}$$ DEDUCTION

$$\frac{X \otimes Y \to Z}{Y \to X\backslash Z}$$ "

$$\frac{X \to Y \quad Y \to Z}{X \to Z}$$ "

ETC.

FREE RES CAT: OBJS = <u>TERMS</u>

ARROWS = EQUIV. CL. OF <u>PROOFS</u>

INTRODUCE GENTZEN STYLE DEDUCTIVE SYSTEM

PROVE CUT ELIMINATION

GIVES DICISION PROCEDURE FOR EQUALITY OF ARROWS

CAN BE USED TO PROVE COHERENCE THEOREMS

ALSO GET FREE CARTESIAN CLOSED CATEGORIES.

USE DEDUCTIVE SYSTEM FOR POSITIVE INTUITIONISTIC PROPOSITIONAL LOGIC

category is a philosophical term with a narrower meaning than *class*, but under the influence of LOVE OF THE LONG WORD it is used freely as a synonym of the simpler one. For the sake of precision it would be better if *category* were used by no one who was not prepared to state (1) that he does not mean *class*, and (2) that he knows the difference between the two; see WORKING AND STYLISH WORDS, and POPULARIZED TECHNICALITIES.

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