LIMITS IN MULTIPLE CATEGORIES
(ON WEAK AND LAX MULTIPLE CATEGORIES, II)

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Abstract. Continuing our first paper in this series, we study multiple limits in infinite-dimensional multiple categories. The general setting is *chiral multiple categories* - a weak, partially lax form with directed interchanges.

After defining multiple limits we prove that all of them can be constructed from (multiple) *products*, *equalisers* and *tabulators* - all of them assumed to be respected by faces and degeneracies. Tabulators appear thus to be the basic higher limits, as was already the case for double categories.

Intercategories, a laxer form of multiple category already studied in two previous papers, are also considered. In this more general setting the basic multiple limits mentioned above can still be defined, but their general theory is not developed here.

Introduction

Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories; their links with strict ω-categories are made clear in the article [ABS].

The present series studies various 'forms' of weak or lax multiple categories, of finite or infinite dimension. They extend weak double categories [GP1] - [GP4] and weak cubical categories [G1, G2, GP5]. More information on literature on higher dimensional category theory can be found in the Introduction of the first paper [GP8], here referred to as Part I.

Our main framework, a *chiral multiple category*, is briefly reviewed here, in Section 1; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the *transversal* direction), weak compositions $x +_i y$ in all positive (or *geometric*) directions i and directed interchanges for the i- and j-compositions (for $0 < i < j$)

$$\chi_{ij} \colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) \qquad \textit{(ij-interchanger).} \qquad (1)$$

Part I also considers a laxer form already studied in two previous papers [GP6, GP7] for the 3-dimensional case, under the name of 'intercategory', that is particularly powerful: it covers duoidal categories, Gray categories, Verity double bicategories, monoidal double categories, etc. In this framework, extended in Part I to infinite dimension and recalled

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here in 1.9, there are also *lower interchangers* $(\tau_{ij}, \mu_{ij}, \delta_{ij})$ where positive degeneracies (i.e. weak identities) intervene; in particular degeneracies are *no longer assumed to commute*, but have a directed interchange for $0 < i < j$

$$\tau_{ij} \colon e_j e_i(x) \to_0 e_i e_j(x) \qquad\qquad (ij\text{-}interchanger\ for\ identities). \qquad (2)$$

Here we study multiple limits in the setting of *chiral multiple categories*. Part of the theory is briefly extended to intercategories, *with the problems discussed below*.

Our general definition of multiple limits (in 4.4) requires their preservation by faces and degeneracies (as in the cubical case [G2]). We prove that all of them can be constructed from (multiple) *products*, *equalisers* and *tabulators*. The latter appear thus to be the basic higher form of a limit, as was already the case for double and cubical categories. In particular this holds in a 2-category, where tabulators (of vertical identities) reduce to cotensors by the ordinal **2**; the previous result agrees thus with Theorem 10 of R. Street [St1], according to which all weighted limits in a 2-category can be constructed from such cotensors and ordinary limits.

More analytically, Section 1 contains a review of the basic notions of strict, weak and chiral multiple categories. We also introduce the 'lift functors' that will play a relevant role below.

Then, in Section 2, we begin our study of limits with the simple case of **i**-*level limits*, for a *positive* multi-index $\mathbf{i} = \{i_1, ..., i_n\}$. In a chiral multiple category A, **i**-level limits are ordinary limits in the transversal category $\mathrm{tv}_\mathbf{i}(\mathsf{A})$. When all these exist, *and are preserved by faces and degeneracies* between transversal categories, we say that A has *level multiple limits*. Of course, multiple products and multiple equalisers generate all of them.

Non-level limits, where the diagram and the limit object are not confined to a transversal category, are studied in the next two sections. The main theorems on the construction and preservation of multiple limits are stated in 3.6 and 4.5, and proved in Section 5.

The main example treated here is the chiral triple category $\mathsf{SC}(\mathbf{C})$ of *spans and cospans* over a category $\mathbf{C}$ with pushouts and pullbacks (see 1.8, 2.1, 2.2, 3.7 and 4.6). One can similarly study multiple limits (and colimits) in other weak or chiral multiple categories of finite or infinite dimension, listed at the beginning of Section 2.

The relationship with the double limits of [GP1] are discussed in 2.6 and 4.7. In the case of level limits (see 2.6) there are only slight differences in terminology. For non-level limits a real difference appears, which is already evident in the basic case of tabulators (see 4.7). We think that the present terminology is preferable.

The general theory of multiple colimits is dual to that of multiple limits and is not written down explicitly. Showing this requires some technical expedient because - as we have seen in Part I - transversal duality turns a (right) chiral multiple category into a *left-hand version* where all interchangers have the opposite direction. Thus, a multiple colimit in the chiral multiple category A is a multiple limit in a *left* chiral multiple category A^{tr}; but it can also be viewed as a *multiple limit in a right chiral multiple category* $(\mathsf{A}^{\mathrm{tr}})^-$ indexed by the negative integers (reversing indices).

An extension of the general theory of multiple limits from the *chiral* case to *intercategories* presents serious problems, linked to the crucial fact that *degeneracies no longer commute*. Yet, the basic limits can be easily extended.

To begin with, *level limits* can be defined as here, in 2.2; one should nevertheless be aware that they do not behave so well as in the chiral case: see the end of Proposition 2.3. Tabulators can also be extended *and even acquire richer forms*: for instance, the total tabulator of a 12-cube gives now rise to two distinct notions, the e_1e_2-*tabulator* and the e_2e_1-*tabulator*, as already shown in Part I, Section 6. However, a general definition of limit seems to fail: in a situation where degeneracies do not commute, even defining the diagonal functor becomes complicated (see 3.1).

Notation. We follow the notation of Part I; the reference I.2.3 points to its Subsection 2.3. The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, often written as $\{-, +\}$ in superscripts. The symbol $\subset$ denotes weak inclusion.

1. Multiple categories

After a review of the basic notions of strict multiple categories, taken from Part I, we introduce the 'lift functors' that will play a relevant role in the study of multiple limits. As it will be made clear later (in 4.8) they are a surrogate for the path endofunctor of symmetric cubical categories. These notions are then extended to *chiral* multiple categories, a weak and partially lax version introduced in Part I.

1.1. Multiple sets. A *multi-index* $\mathbf{i}$ is a finite set of natural numbers, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it will be understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ we always mean that $\mathbf{i}$ has n *distinct* elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer n is called the *dimension* of $\mathbf{i}$.

We use the following symbols

$$\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} \qquad (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \qquad\qquad \mathbf{i}|j = \mathbf{i} \setminus \{j\} \qquad (\text{for } j \in \mathbf{i}). \tag{3}$$

A *multiple set* is system of sets and mappings $X = ((X_\mathbf{i}), (\partial_i^\alpha), (e_i))$ under the following two assumptions.

(mls.1) For every multi-index $\mathbf{i} = \{i_1, ..., i_n\}$, $X_\mathbf{i}$ is a set whose elements are called $\mathbf{i}$-*cells* of X and said to be of *dimension* n. We write X_*, X_i, X_{ij},... instead of $X_\varnothing$, $X_{\{i\}}$, $X_{\{i,j\}}$,...; thus X_* is of dimension 0 while X_0, X_1,... are of dimension 1.

(mls.2) For $j \in \mathbf{i}$ and $\alpha = 0, 1$ we have mappings, called *faces* and *degeneracies* of $X_\mathbf{i}$

$$\partial_j^\alpha \colon X_\mathbf{i} \to X_{\mathbf{i}|j}, \qquad\qquad e_j \colon X_{\mathbf{i}|j} \to X_\mathbf{i}, \tag{4}$$

satisfying the *multiple relations*

$$\begin{aligned} &\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha \quad (i \neq j), && e_i.e_j = e_j.e_i \quad (i \neq j), \\ &\partial_i^\alpha.e_j = e_j.\partial_i^\alpha \quad (i \neq j), && \partial_i^\alpha.e_i = \text{id}. \end{aligned} \tag{5}$$

Faces commute and degeneracies commute, but ∂_i^α and e_i do not. These relations look similar to the cubical ones but much simpler, because here an index i stands for a particular sort, instead of a mere position, and is never 'renamed'. Note also that ∂_i^α acts on $X_{\mathbf{i}}$ if i belongs to the multi-index $\mathbf{i}$ (and cancels it), while e_i acts on $X_{\mathbf{i}}$ if i does not belong to $\mathbf{i}$ (and inserts it); therefore $\partial_i^\alpha.\partial_i^\beta$ and $e_i.e_i$ make no sense, here: one cannot cancel or insert twice the same index.

If $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ is a disjoint union and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)$ is a mapping $\mathbf{k} = \{k_1, ..., k_r\} \to 2$, we have an *iterated face* and an *iterated degeneracy* (independent of the order of composition)

$$\partial_{\mathbf{k}}^{\boldsymbol{\alpha}} = \partial_{k_1}^{\alpha_1} ... \partial_{k_r}^{\alpha_r} \colon X_{\mathbf{i}} \to X_{\mathbf{j}}, \qquad\qquad e_{\mathbf{k}} = e_{k_1} ... e_{k_r} \colon X_{\mathbf{j}} \to X_{\mathbf{i}}. \tag{6}$$

In particular, the *total* $\mathbf{i}$-*degeneracy* is the mapping

$$e_{\mathbf{i}} = e_{i_1} ... e_{i_n} \colon X_* \to X_{\mathbf{i}}. \tag{7}$$

1.2. Multiple categories. We recall the definition, from Part I.

(mlc.1) A *multiple category* A is, first of all, a multiple set of components $A_{\mathbf{i}}$, whose elements are called $\mathbf{i}$-*cells*. As above, $\mathbf{i}$ is any multi-index, i.e. any finite subset of $\mathbb{N}$, and we write A_*, A_i, A_{ij}... for $A_\varnothing$, $A_{\{i\}}$, $A_{\{i,j\}}$,...

(mlc.2) Given two $\mathbf{i}$-cells x, y which are i-*consecutive* (i.e. $\partial_i^+(x) = \partial_i^-(y)$, with $i \in \mathbf{i}$), the i-*composition* $x +_i y$ is defined and satisfies the following interactions with faces and degeneracies

$$\begin{aligned} &\partial_i^-(x +_i y) = \partial_i^-(x), && \partial_i^+(x +_i y) = \partial_i^+(y), \\ &\partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_i \partial_j^\alpha(y), && e_j(x +_i y) = e_j(x) +_i e_j(y) \qquad (j \neq i). \end{aligned} \tag{8}$$

(mlc.3) For every multi-index $\mathbf{i}$ containing j we have a category $\mathrm{cat}_{\mathbf{i},j}(\mathsf{A})$ with objects in $A_{\mathbf{i}}$, arrows in $A_{\mathbf{i}j}$, faces ∂_j^α, identities e_j and composition $+_j$.

(mlc.4) For $i < j$ we have

$$(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u) \qquad\qquad (\textit{binary ij-interchange}), \tag{9}$$

whenever these composites make sense. (Note that the lower interchanges are already expressed above.)

More generally, for an ordered pointed set $N = (N, 0)$, an N-*indexed multiple category* A has components $A_{\mathbf{i}}$ *indexed by* (finite) *multi-indices* $\mathbf{i} \subset N$. If N is the ordinal set $\mathbf{n} = \{0, ..., n-1\}$ we obtain the n-dimensional version of a multiple category, called an n-*tuple category*. The 0-, 1- and 2-dimensional versions amount - respectively - to a set, a category or a double category.

1.3. Transversal categories. The *transversal* direction, corresponding to the index $i = 0$, is treated differently in the theory: we think of it as the 'dynamic' direction, along which 'transformation occurs', while the positive directions are viewed as the 'static' or 'geometric' ones.

A *positive* multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ (with $n \geqslant 0$ positive elements) has an 'augmented' multi-index $0\mathbf{i} = \{0, i_1, ..., i_n\}$. The *transversal category of* $\mathbf{i}$*-cubes of* A

$$\mathrm{tv}_{\mathbf{i}}(\mathsf{A}) = \mathrm{cat}_{\mathbf{i},0}(\mathsf{A}), \tag{10}$$

- has objects in $A_{\mathbf{i}}$, called $\mathbf{i}$-*cubes* and viewed as n-dimensional objects,

- has arrows $f\colon x^- \to_0 x^+$ in $A_{0\mathbf{i}}$, called $\mathbf{i}$-*maps*, with domain and codomain $\partial_0^\alpha(f) = x^\alpha$,

- has identities $1_x = \mathrm{id}(x) = e_0(x)\colon x \to_0 x$ and composition $gf = f +_0 g$.

All these items are said to be *of degree* n (though their dimension may be n or $n+1$): the degree always refers to the number of positive indices. In all of our examples, 0-composition is realised by the usual composition of mappings, while the 'positive' compositions (also called *concatenations*) are often obtained by operations (products, sums, tensor products, pullbacks, pushouts...) where reversing the order of the operands would only be confusing.

Faces and degeneracies give (ordinary) functors

$$\partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}), \qquad e_j\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \qquad (j \notin \mathbf{i},\ \alpha = 0, 1). \tag{11}$$

In particular, the unique positive multi-index of degree 0, namely $\varnothing$, gives the category $\mathrm{tv}_*(\mathsf{A})$ of *objects* of A (i.e. $\star$-cells) and their transversal maps (i.e. 0-cells).

An $\mathbf{i}$-map $f\colon x \to_0 y$ is said to be i*-special*, or *special in direction* $i \in \mathbf{i}$, if its i-faces are transversal identities (of $\mathbf{i}|j$-cubes)

$$\partial_i^\alpha f = e_0\partial_i^\alpha x = e_0\partial_i^\alpha y. \tag{12}$$

This, of course, implies that the $\mathbf{i}$-cubes x, y have the same i-faces. We say that f is ij*-special* if it is special in both directions i, j.

1.4. Multiple functors and transversal transformations. A *multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between multiple categories is a morphism of multiple sets $F = (F_{\mathbf{i}})$ that preserves all the composition laws. For an $\mathbf{i}$-map $f\colon x \to_0 y$, we use one of the following forms

$$F(f)\colon F(x) \to_0 F(y), \qquad\qquad F_{0\mathbf{i}}(f)\colon F_{\mathbf{i}}(x) \to_0 F_{\mathbf{i}}(y),$$

as it may be convenient.

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between multiple functors consists of a family of $\mathbf{i}$-maps in B (its components), for every *positive* multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$hx\colon F(x) \to_0 G(x) \qquad\qquad (h_{\mathbf{i}}x\colon F_{\mathbf{i}}(x) \to_0 G_{\mathbf{i}}(x)). \tag{13}$$

The following axioms of naturality and coherence are required:

(trt.1) $Gf.hx = hy.Ff$, for $f\colon x \to_0 y$ in A,

(trt.2) h commutes with *positive* faces, degeneracies and compositions:

$$h(\partial_j^\alpha x) = \partial_j^\alpha(hx), \qquad h(e_j z) = e_j(hz), \qquad h(x +_j y) = hx +_j hy.$$

where $\mathbf{i}$ is a positive multi-index, $j \in \mathbf{i}$, x and y are j-consecutive $\mathbf{i}$-cubes, z is an $\mathbf{i}|j$-cube.

Given two multiple categories A, B we have thus the category $\mathbf{Mlc}(\mathsf{A}, \mathsf{B})$ of their multiple functors and transversal transformations. All these form the 2-category $\mathbf{Mlc}$, in an obvious way.

More generally for any ordered pointed set $N = (N, 0)$ we have the 2-category $\mathbf{Mlc}_N$ of N-indexed multiple categories, formed of ordinary categories $\mathbf{Mlc}_N(\mathsf{A}, \mathsf{B})$.

1.5. Lift functors. For a *positive* integer j there is a *j-directed lift functor* with values in the 2-category of multiple categories indexed by the pointed set $\mathbb{N}|j = \mathbb{N} \setminus \{j\}$

$$Q_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}. \tag{14}$$

On a multiple category A, the multiple category $Q_j\mathsf{A}$ is - loosely speaking - that part of A that contains the index j, reindexed without it:

$$\begin{aligned}
&(Q_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}j},\\
&(\partial_i^\alpha\colon (Q_j\mathsf{A})_{\mathbf{i}} \to (Q_j\mathsf{A})_{\mathbf{i}|i}) = (\partial_i^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}j|i}),\\
&(e_i\colon (Q_j\mathsf{A})_{\mathbf{i}|i} \to (Q_j\mathsf{A})_{\mathbf{i}}) = (e_i\colon A_{\mathbf{i}j|i} \to A_{\mathbf{i}j}) \qquad (i \in \mathbf{i} \subset \mathbb{N}|j),
\end{aligned} \tag{15}$$

and similarly for compositions. In the same way for multiple functors $F, G\colon \mathsf{A} \to \mathsf{B}$ and a transversal transformation $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ we let

$$(Q_jF)_{\mathbf{i}} = F_{\mathbf{i}j}, \qquad (Q_jh)_{\mathbf{i}} = h_{\mathbf{i}j} \qquad (\mathbf{i} \subset \mathbb{N}|j). \tag{16}$$

There is also an obvious restriction 2-functor $R_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}$, where the multiple category $R_j\mathsf{A}$ is that part of A that does not contain the index j. The j-directed faces and degeneracies of A are not used in $Q_j\mathsf{A}$, but yield three natural transformations, also called *faces* and *degeneracy*

$$\begin{aligned}
&D_j^\alpha\colon Q_j \to R_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}, \quad (D_j^\alpha)_{\mathbf{i}} = \partial_j^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}} \quad (\mathbf{i} \subset \mathbb{N}|j),\\
&E_j\colon R_j \to Q_j\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|j}, \quad (E_j)_{\mathbf{i}} = e_j\colon A_{\mathbf{i}} \to A_{\mathbf{i}j} \quad (\mathbf{i} \subset \mathbb{N}|j),\\
&D_j^\alpha E_j = \mathrm{id}.
\end{aligned} \tag{17}$$

In particular, the objects and $\star$-maps of $Q_j(\mathsf{A})$ are the j-cubes and j-maps of A, so that

$$\begin{gathered}
\mathrm{tv}_*(Q_j(\mathsf{A})) = \mathrm{tv}_j(\mathsf{A}),\\
\mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_j(\mathsf{A}) \to \mathrm{tv}_*(\mathsf{A}), \qquad \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_*(\mathsf{A}) \to \mathrm{tv}_j(\mathsf{A}).
\end{gathered} \tag{18}$$

Plainly all the functors Q_j commute. By composing n of them in any order we get an *iterated lift functor* of degree n, in a *positive* direction $\mathbf{i} = \{i_1, ..., i_n\}$

$$Q_\mathbf{i}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}}, \qquad Q_\mathbf{i}(\mathsf{A}) = Q_{i_n}...Q_{i_1}(\mathsf{A}), \qquad \mathrm{tv}_*(Q_\mathbf{i}(\mathsf{A})) = \mathrm{tv}_\mathbf{i}(\mathsf{A}). \tag{19}$$

Again, there are faces and degeneracies (where $\mathbf{hi} = \mathbf{h} \cup \mathbf{i}$)

$$\begin{aligned} &D_j^\alpha\colon Q_{\mathbf{i}j} \to R_j Q_\mathbf{i}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}j}, \quad (D_j^\alpha)_\mathbf{h} = \partial_j^\alpha\colon A_{\mathbf{hi}j} \to A_{\mathbf{hi}} \quad (\mathbf{h} \subset \mathbb{N}|\mathbf{i}j), \\ &E_j\colon R_j Q_\mathbf{i} \to Q_{\mathbf{i}j}\colon \mathbf{Mlc} \to \mathbf{Mlc}_{\mathbb{N}|\mathbf{i}j}, \quad (E_j)_\mathbf{h} = e_j\colon A_{\mathbf{hi}} \to A_{\mathbf{hi}j} \quad (\mathbf{h} \subset \mathbb{N}|\mathbf{i}j), \end{aligned} \tag{20}$$

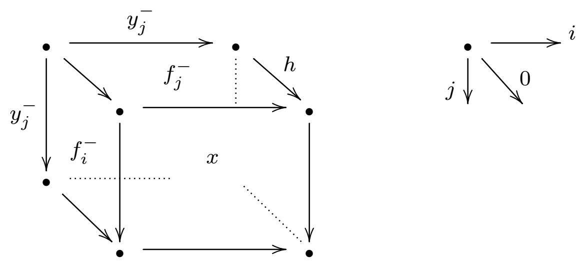
$$\mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \to \mathrm{tv}_\mathbf{i}(\mathsf{A}), \quad \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_\mathbf{i}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \quad (j \notin \mathbf{i}). \tag{21}$$

1.6. Transversal invariance. We now extend the notion of 'horizontal invariance' of double categories [GP1], obtaining a property that will be of use for multiple limits and should be expected of every 'well formed' multiple category.

We say that the multiple category A is *transversally invariant* if its cubes are 'transportable' along transversally invertible maps. Precisely:

(i) given an $\mathbf{i}$-cube x of degree n and a family of $2n$ *invertible* transversal maps $f_i^\alpha\colon y_i^\alpha \to_0 \partial_i^\alpha x$ ($i \in \mathbf{i}$, $\alpha = 0, 1$) *with consistent positive faces* (and otherwise arbitrary domains y_i^α)

$$\partial_i^\alpha(f_j^\beta) = \partial_j^\beta(f_i^\alpha) \qquad \text{(for } i \neq j \text{ in } \mathbf{i}), \tag{22}$$



$$(h = \partial_i^+(f_j^-) = \partial_j^-(f_i^+)),$$

there exists an invertible $\mathbf{i}$-map $f\colon y \to_0 x$ (a 'filler', as in the Kan extension property) with positive faces $\partial_i^\alpha f = f_i^\alpha$ (and therefore $\partial_i^\alpha y = y_i^\alpha$).

Of course this property can be equivalently stated for a family of invertible maps $g_i^\alpha\colon \partial_i^\alpha x \to_0 y_i^\alpha$.

1.7. Weak multiple categories. *Weak multiple categories* have been introduced in Part I, Section 3.

Extending weak double categories [GP1] - [GP4] and weak triple categories [GP6, GP7], the basic structure of a weak multiple category A is a multiple set with compositions in all directions. The composition laws in direction 0 are categorical and have a strict interchange with the other compositions.

On the other hand, the 'positive' compositions have transversally-invertible comparisons for unitarity, associativity and interchange (with $0 < i < j$)

$$\begin{array}{lr} \lambda_i x\colon (e_i\partial_i^- x) +_i x \to_0 x & (\textit{left } i\textit{-unitor}), \\ \rho_i x\colon x +_i (e_i\partial_i^+ x) \to_0 x & (\textit{right } i\textit{-unitor}), \\ \kappa_i(x,y,z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z & (i\textit{-associator}), \\ \chi_{ij}(x,y,z,u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) & (ij\textit{-interchanger}), \end{array} \tag{23}$$

under coherence conditions listed in I.3.3 and I.3.4.

Our main infinite-dimensional examples are of a *cubical type* (see I.3.5). Essentially, this means that components, faces and degeneracies are *invariant under renaming positive indices*, in the same order. An **i**-cube can thus be indexed by $[n] = \{1, ..., n\}$ and called an *n-cube*; an **i**-map can be indexed by $0[n] = \{0, 1, ..., n\}$ and called an *n-map*; again, such items are of *order n* and *dimension n* or $n + 1$, respectively. (The examples below are also *symmetric*, by a natural action of each symmetric group S_n on the sets of n-cubes and n-maps, permuting the *positive* directions; see Part I.)

(a) The strict symmetric cubical category $\omega\mathsf{Cub}(\mathbf{C})$ *of commutative cubes* over a category $\mathbf{C}$. An n-cube is a functor $x\colon \mathbf{2}^n \to \mathbf{C}$ ($n \geqslant 0$), where $\mathbf{2}$ is the ordinal category $\bullet \to \bullet$; an n-map is a natural transformation of such functors. Applications of this multiple category (and its truncations) to algebraic K-theory can be found in [Sh].

(b) The weak symmetric cubical category $\omega\mathsf{Cosp}(\mathbf{C})$ *of cubical cospans* over a category $\mathbf{C}$ with (a fixed choice) of pushouts has been constructed in [G1], to deal with higher-dimensional cobordism. An n-cube is a functor $x\colon \wedge^n \to \mathbf{C}$, where $\wedge$ is the formal-cospan category $\bullet \to \bullet \leftarrow \bullet$; again, an n-map is a natural transformation of such functors.

(c) The weak symmetric cubical category $\omega\mathsf{Span}(\mathbf{C})$ *of cubical span*, over a category $\mathbf{C}$ with pullbacks, is similarly constructed over $\vee = \wedge^{\mathrm{op}}$, the formal-span category $\bullet \leftarrow \bullet \to \bullet$ (see [G1]). It is *transversally dual* to $\omega\mathsf{Cosp}(\mathbf{C}^{\mathrm{op}})$.

(d) The weak symmetric cubical category of *cubical bispans*, or *cubical diamonds* $\omega\mathsf{Bisp}(\mathbf{C})$, over a category $\mathbf{C}$ with pullbacks and pushouts, is similarly constructed over a formal diamond category (see [G1]).

1.8. Chiral multiple categories and intercategories. Our main structure here is more general and partially lax.

A *chiral*, or *χ-lax*, *multiple category* A (see I.3.7) has the same data and axioms of a weak multiple category, except for the fact that the interchange comparisons χ_{ij} ($0 < i < j$) recalled above (in 1.7) are not supposed to be invertible.

Examples are constructed in [GP7] and Part I, Section 4. For instance, if the category $\mathbf{C}$ has pullbacks and pushouts, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$, of arrows and spans of $\mathbf{C}$, can be 'amalgamated' with the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$, of arrows and cospans of $\mathbf{C}$, to form a 3-dimensional structure: the chiral triple category $\mathsf{SC}(\mathbf{C})$ whose 0-, 1- *and* 2-directed arrows are the arrows, spans and cospans of $\mathbf{C}$, *in this order* (as required by the 12-interchanger). For higher dimensional examples, like $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$ and $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (and the corresponding *left-chiral* cases) see I.4.4; the latter structure is indexed by all integers, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

Chiral multiple categories, with their strict multiple functors and transversal transformations, form the 2-category **Cmc**.

As defined in I.3.9, a *lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between chiral multiple categories, or *lax functor* for short, has components $F_\mathbf{i}\colon A_\mathbf{i} \to B_\mathbf{i}$ for all multi-indices $\mathbf{i}$ (often written as F) that agree with all faces, 0-degeneracies and 0-composition. Moreover F is equipped with comparison $\mathbf{i}$-maps, for every *positive* multi-index $\mathbf{i}$ and $i \in \mathbf{i}$, that will be denoted as $\underline{F}_i$

$$\begin{aligned} &\underline{F}_i(x)\colon e_iF(x) \to_0 F(e_ix) && \text{(for } x \in A_{\mathbf{i}|i}\text{)},\\ &\underline{F}_i(x,y)\colon F(x) +_i F(y) \to_0 F(x +_i y) && \text{(for } i\text{-consecutive cubes } x, y \text{ in } A_\mathbf{i}\text{)}. \end{aligned} \tag{24}$$

These comparisons have to satisfy some axioms. We write down the naturality conditions (lmf.1-2), frequently used below, while the coherence conditions (lmf.3-5) can be found in *loc. cit.*

(lmf.1) (*Naturality of unit comparisons*) For an $\mathbf{i}|i$-map $f\colon x \to_0 y$ in A we have:

$$Fe_i(f).\underline{F}_i(x) = \underline{F}_i(y).e_i(Ff)\colon\ e_iF(x) \to_0 F(e_iy). \tag{25}$$

(lmf.2) (*Naturality of composition comparisons*) For two i-consecutive $\mathbf{i}$-maps $f\colon x \to_0 x'$ and $g\colon y \to_0 y'$ in A we have:

$$F(f +_i g).\underline{F}_i(x,y) = \underline{F}_i(x',y').(F(f) +_i F(g))\colon\ F(x) +_i F(y) \to_0 F(x' +_i y'). \tag{26}$$

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between lax functors consists of a family of $\mathbf{i}$-maps in B (its components), one for every positive multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$hx\colon F(x) \to_0 G(x) \qquad\qquad (h_\mathbf{i}x\colon F_\mathbf{i}(x) \to_0 G_\mathbf{i}(x)), \tag{27}$$

under the axioms (trt.1) and (trt.2L) of I.3.9

(trt.1) $Gf.hx = hy.Ff$ (for $f\colon x \to_0 y$ in A),

(trt.2L) for every positive multi-index $\mathbf{i}$ and $j \in \mathbf{i}$:

$$\begin{aligned} &h(\partial_j^\alpha x) = \partial_j^\alpha(hx), && (x \in A_\mathbf{i}),\\ &h(e_jx)\underline{F}_j(x) = \underline{G}_j(x).e_j(hx)\colon e_jF(x) \to_0 G(e_jx) && (x \in A_{\mathbf{i}|j}),\\ &h(z).\underline{F}_j(x,y) = \underline{G}_j(x,y).(hx +_j hy)\colon F(x) +_j F(y) \to_0 G(z) && (z = x +_j y \text{ in } A_\mathbf{i}). \end{aligned}$$

We have thus the 2-category Lx**Cmc** of chiral multiple categories, lax functors and their transversal transformations.

The *lift functor* and *restriction functor* in direction j (cf. 1.5) are extended in the same form:

$$\begin{aligned} &Q_j \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \qquad && (Q_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}j}, \\ &R_j \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \qquad && (R_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}} \qquad (j > 0,\ j \notin \mathbf{i}). \end{aligned} \tag{28}$$

Similarly one defines the 2-category Cx**Cmc** for the colax case, where the comparisons of *colax (multiple) functors* have the opposite direction. A *pseudo (multiple) functor* is a lax functor whose comparisons are invertible (and is made colax by inverting its comparisons); such functors are the arrows of the 2-category Ps**Cmc**.

1.9. Intercategories. The more general case of *intercategories*, studied in [GP6, GP7] and Part I (Sections 5 and 6), will only be considered here in a marginal way.

Let us recall that an intercategory A has other directed ij-interchangers besides χ_{ij} (for $0 < i < j$):

(a) $\tau_{ij}(x) \colon e_j e_i(x) \to_0 e_i e_j(x)$ *(for i- and j-identities),*

(b) $\mu_{ij}(x, y) \colon e_i(x) +_j e_i(y) \to_0 e_i(x +_j y)$ *(for i-identities and j-composition),*

(c) $\delta_{ij}(x, y) \colon e_j(x +_i y) \to_0 e_j(x) +_i e_j(y)$ *(for i-composition and j-identities).*

As proved in [GP7], three-dimensional intercategories comprise under a common form various structures previously studied, like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories. Literature on the these structures can be found in *loc. cit.*; the inspiring case of duoidal (or 2-monoidal) categories can be found in [AM, BS, St2].

As already noted in Part I, various 'anomalies' appear with respect to the chiral case, that make problems for a theory of multiple limits in this setting. These will be briefly considered below (see 2.3 and 3.1), without further investigating a situation for which we do not yet have examples sufficiently rich to have good limits.

Some anomalies can already be remarked here. First, an intercategory A *is no longer a multiple set* (unless each τ_{ij} is the identity). Second, a degeneracy e_i ($i > 0$) is now *lax* with respect to every higher j-composition (for $j > i$, via τ_{ij} and μ_{ij}) but *colax* with respect to every lower j-composition (for $0 < j < i$, via τ_{ji} and δ_{ji}). Therefore, in the truncated n-dimensional case e_1 is lax with respect to all other compositions and e_n is colax, but the other degeneracies (if any, i.e. for $n > 3$) are neither lax nor colax.

2. Multiple level limits

We begin our study of limits with the simple case of **i**-*level limits*, for a positive multi-index **i**.

In a chiral multiple category A, **i**-level limits are ordinary limits in the transversal category $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ (as in the cubical case, see [G2]). When all these exist, and are preserved by faces and degeneracies, we say that A has *level multiple limits*; of course they are 'generated' by multiple products and multiple equalisers.

Examples are given in the chiral triple category $\mathsf{SC}(\mathbf{C})$ recalled in 1.8; they can be easily adapted to the weak multiple categories $\omega\mathsf{Cub}(\mathbf{C})$, $\omega\mathsf{Cosp}(\mathbf{C})$, $\omega\mathsf{Span}(\mathbf{C})$ and $\omega\mathsf{Bisp}(\mathbf{C})$ of 1.7, and to the chiral multiple categories $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$ and $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ recalled in 1.8. Note that all of these are transversally invariant, a property of interest for limits as we show below, in 2.3 and 2.4.

Level limits can be extended to intercategories with the same definitions (see 1.9). But Proposition 2.3 and its consequences in 2.4 would partially fail.

Non-level limits, where the diagram and the limit cube are not confined to a transversal category, will be studied in the next two sections.

2.1. Products. Let us begin by examining various kinds of products in the chiral triple category $\mathsf{A} = \mathsf{SC}(\mathbf{C})$.

Supposing that $\mathbf{C}$ has products, the same is true of its categories of diagrams, and (using the formal-span category $\vee$ and the formal cospan $\wedge$ recalled in 1.7) we have four types of products in A (indexed by a small set Λ):

- products of objects (in $\mathbf{C}$), with projections in A_0: $\quad C = \Pi_\lambda C_\lambda, \quad p_\lambda \colon C \to_0 C_\lambda$,
- products of 1-arrows (in $\mathbf{C}^\vee$), with projections in A_{01}: $\quad f = \Pi_\lambda f_\lambda, \quad p_\lambda \colon f \to_0 f_\lambda$,
- products of 2-arrows (in $\mathbf{C}^\wedge$), with projections in A_{02}: $\quad u = \Pi_\lambda u_\lambda, \quad p_\lambda \colon u \to_0 u_\lambda$,
- products of 12-cells (in $\mathbf{C}^{\vee\times\wedge}$), with projections in A_{012}: $\quad \pi = \Pi_\lambda \pi_\lambda, \quad p_\lambda \colon \pi \to_0 \pi_\lambda$.

Faces and degeneracies preserve these products. Saying that the triple category $\mathsf{SC}(\mathbf{C})$ *has triple products* we mean all this. It is important to note that *this is a global condition*: we shall *not* define when, in a chiral triple category, a *single* product of objects $\Pi_\lambda C_\lambda$ should be called 'a triple product'.

It is now simpler and clearer to work in a chiral *multiple* category A, rather than in a truncated case, as above.

Let $n \geqslant 0$ and let $\mathbf{i}$ be a positive multi-index (possibly empty). An $\mathbf{i}$-*product* $a = \Pi_{\lambda\in\Lambda} a_\lambda$ will be an ordinary product in the transversal category $\mathrm{tv}_\mathbf{i}(\mathsf{A})$ of $\mathbf{i}$-cubes of A (recalled in Section 1). It comes with a family $p_\lambda \colon a \to_0 a_\lambda$ of $\mathbf{i}$-maps (i.e. cells of $A_{0\mathbf{i}}$) that satisfies the obvious universal property.

We say that A:

(i) *has* $\mathbf{i}$-*products*, or products of type $\mathbf{i}$, if all these products (indexed by an arbitrary small set Λ) exist,

(ii) *has products* if it has $\mathbf{i}$-products for all positive multi-indices $\mathbf{i}$,

(iii) *has multiple products* if it has all products, and these are preserved by faces and degeneracies, viewed as (ordinary) functors (cf. (11))

$$\partial_j^\alpha \colon \mathrm{tv}_\mathbf{i}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}), \qquad e_j \colon \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}) \to \mathrm{tv}_\mathbf{i}(\mathsf{A}) \qquad (j \in \mathbf{i},\ \alpha = 0,1). \tag{29}$$

Of course this preservation is meant in the usual sense, up to isomorphism (i.e. invertible transversal maps); however, if this holds and A is transversally invariant (cf. 1.6), one can construct a choice of products that is *strictly* preserved by faces and degeneracies, starting from $\star$-products and going up. This will be proved, more generally, in Proposition 2.3.

A $\star$-product is also called a *product of degree* 0.

2.2. Level limits. We now let Λ be a small category and consider the functors $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ with values in the transversal category of $\mathbf{i}$-cubes of A, for a positive multi-index $\mathbf{i}$.

There is an obvious chiral multiple category A^Λ whose $\mathbf{i}$-cubes are the functors $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ and whose $\mathbf{i}$-maps are their natural transformations, composed as such. The positive faces, degeneracies and compositions are pointwise (as well as their comparisons):

$$(\partial_i^\alpha F)(\lambda) = \partial_i^\alpha(F(\lambda)), \qquad (e_iF)(\lambda) = e_i(F(\lambda)), \qquad (F +_i G)(\lambda) = F(\lambda) +_i G(\lambda).$$

The diagonal functor $D\colon \mathsf{A} \to \mathsf{A}^\Lambda$ takes each $\mathbf{i}$-cube a to the constant a-valued functor $Da\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, and each $\mathbf{i}$-map $h\colon a \to_0 b$ to the constant h-valued natural transformation $Dh\colon Da \to Db\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$.

The limit of the functor F, called an $\mathbf{i}$-*level limit* in A, is an $\mathbf{i}$-cube $L \in A_{\mathbf{i}}$ equipped with a universal natural transformation $t\colon DL \to F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, where $DL\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ is the constant functor at L. It is an $\mathbf{i}$-product if Λ is discrete and an $\mathbf{i}$-*equaliser* if Λ is the category $0 \rightrightarrows 1$.

We say that A:

(i) *has* $\mathbf{i}$-*level limits on* Λ if all the functors $\Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ have a limit,

(ii) *has level limits on* Λ if it has such limits for all positive multi-indices $\mathbf{i}$,

(iii) *has level multiple limits on* Λ if it has such level limits, and these are preserved by faces and degeneracies (as specified in (29)),

(iv) *has level multiple limits* if the previous property holds for every small category Λ.

Obviously, the multiple category A has level multiple limits if and only if it has multiple products and multiple equalisers. *Finite level limits* work 'in the same way', with finite multiple products.

In particular, a $\star$-*level limit* is a limit in the transversal category $\mathrm{tv}_*(\mathsf{A})$, associated to the multi-index $\varnothing$, of degree 0; it will also be called a *level limit of degree 0.*

Extending the case of multiple products considered in 2.1, if the category $\mathbf{C}$ is complete (or finitely complete) so are its categories of diagrams, and the chiral triple category $\mathsf{SC}(\mathbf{C})$ has level triple limits (or the finite ones).

2.3. Proposition. [Level limits and invariance] *Let* Λ *be a category and* A *a* transversally invariant *chiral multiple category (cf. 1.6). If* A *has level multiple limits on* Λ*, one can find a* consistent choice of such limits. *More precisely, one can fix for every positive multi-index* $\mathbf{i}$ *and every functor* $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ *a limit of* F

$$L(F) \in A_{\mathbf{i}}, \qquad\qquad t(F)\colon DL(F) \to F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}), \qquad (30)$$

so that these choices are strictly preserved by faces and degeneracies:

$$\begin{array}{lll} \partial_i^\alpha(L(F)) = L(\partial_i^\alpha F), & \partial_i^\alpha(t(F)) = t(\partial_i^\alpha F) & (i \in \mathbf{i}), \\ e_i(L(F)) = L(e_iF), & e_i(t(F)) = t(e_iF) & (i \notin \mathbf{i}). \end{array} \tag{31}$$

Proof. We proceed by induction on the degree n of positive multi-indices. For $n = 0$ we just fix a choice $(L(F), t(F))$ of $\star$-level limits on Λ, for all $F\colon \Lambda \to \mathrm{tv}_*(\mathsf{A})$. Then, for $n \geqslant 1$, we suppose to have a consistent choice for all positive multi-indices of degree up to $n-1$ and extend this choice to degree n, as follows.

For a functor $F\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ of degree n, we already have a choice $(L(\partial_i^\alpha F), t(\partial_i^\alpha F))$ of the limit of each of its faces. Let (L, t) be an arbitrary limit of F; since faces preserve limits (in the usual, non-strict sense), there is a unique family of transversal isomorphisms h_i^α coherent with the limit cones (of degree $n-1$)

$$h_i^\alpha\colon L(\partial_i^\alpha F) \to_0 \partial_i^\alpha L, \qquad t(\partial_i^\alpha F) = (\partial_i^\alpha t).h_i^\alpha \qquad (i \in \mathbf{i},\ \alpha = 0, 1), \tag{32}$$

and this family has consistent faces (cf. (22)), as it follows easily from their coherence with the limit cones of a lower degree (when $n \geqslant 2$, otherwise the consistency condition is void).

Now, because of the hypothesis of transversal invariance, this family can be filled with a transversal isomorphism h, *yielding a choice* for $L(F)$ and $t(F)$

$$h\colon L(F) \to_0 L, \qquad t(F) = t.Dh\colon DL(F) \to F. \tag{33}$$

By construction this extension of L is strictly preserved by all faces. To make it also consistent with degeneracies, we assume that - in the previous construction - the following constraint has been followed: for an i-degenerate functor $F = e_iG\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ we always choose the pair $(e_iL(G), e_it(G))$ as its limit (L, t). This allows us to take $h_i^\alpha = \mathrm{id}(L(G))$ (for all $i \in \mathbf{i}$ and $\alpha = 0, 1$), and finally $h = \mathrm{id}(L)$, that is

$$L(F) = e_iL(G), \qquad t(F) = e_it(G)\colon DL(F) \to F. \tag{34}$$

If F is also j-degenerate, then $F = e_ie_jH = e_je_iH$; therefore, by the inductive assumption, both procedures give the same result: $e_iL(G) = e_ie_jL(H) = e_je_iL(H) = e_jL(e_iH)$.

Note that this point would fail in an intercategory with $e_ie_j \neq e_je_i$. ■

2.4. Level limits as unitary lax functors. The previous proposition shows that, if the chiral multiple category A is transversally invariant and has level multiple limits on the small category Λ, we can form a *unitary* lax functor L and a transversal transformation t

$$L\colon \mathsf{A}^\Lambda \to \mathsf{A}, \qquad t\colon DL \to 1\colon \mathsf{A}^\Lambda \to \mathsf{A}^\Lambda, \tag{35}$$

such that, on every $\mathbf{i}$-cube F, the pair $(L(F), t(F))$ is the level limit of the functor F, as in (30).

Indeed, after defining L and t on $\mathbf{i}$-cubes, *by a consistent choice* (which is possible by the proposition itself), we define $L(h)$ for every natural transformation $h\colon F \to G\colon \Lambda \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$. By the universal property of limits, there is precisely one $\mathbf{i}$-map $L(h)$ such that

$$L(h)\colon L(F) \to_0 L(G), \qquad\qquad h.t(F) = t(G).DL(h), \tag{36}$$

and this extension on $\mathbf{i}$-maps is obviously the only one that makes the family $t(F)\colon DL(F) \to F$ into a transversal transformation $DL \to 1$. The lax comparison for i-composition (with $i \in \mathbf{i}$)

$$\underline{L}(F,G)\colon L(F) +_i L(G) \to_0 L(F +_i G), \qquad t(F +_i G).D\underline{L}(F,G) = t(F) +_i t(G), \tag{37}$$

comes from the universal property of $L(F +_i G)$ as a limit.

2.5. Level limits and liftings. Let us recall (from (19) and 1.8) that, for a positive multi-index $\mathbf{i}$, the chiral multiple category A has a lifting $Q_{\mathbf{i}}(\mathsf{A})$ such that

$$\mathrm{tv}_*(Q_{\mathbf{i}}(\mathsf{A})) = \mathrm{tv}_{\mathbf{i}}(\mathsf{A}). \tag{38}$$

Therefore, an $\mathbf{i}$-level limit in A is the same as a $\star$-level limit in $Q_{\mathbf{i}}(\mathsf{A})$. The chiral multiple category A

(i) *has* $\mathbf{i}$*-level limits on* Λ if and only if its lifting $Q_{\mathbf{i}}(\mathsf{A})$ has $\star$-level limits on Λ,

(ii) *has level limits on* Λ if and only if all its liftings $Q_{\mathbf{i}}(\mathsf{A})$ have $\star$-level limits,

(iii) *has level multiple limits on* Λ if and only if all its liftings $Q_{\mathbf{i}}(\mathsf{A})$ have $\star$-level limits, and these are preserved by faces and degeneracies, namely the multiple functors $D_j^\alpha = D_j^\alpha(\mathsf{A})$ and $E_j = E_j(\mathsf{A})$ for $j \notin \mathbf{i}$ and $\alpha = 0, 1$ (cf. 1.5)

$$\begin{array}{ll} D_j^\alpha\colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_j Q_{\mathbf{i}}(\mathsf{A}), & E_j\colon R_j Q_{\mathbf{i}}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A}), \\ \mathrm{tv}_*(D_j^\alpha) = \partial_j^\alpha\colon \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}), & \mathrm{tv}_*(E_j) = e_j\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}j}(\mathsf{A}) \end{array} \tag{39}$$

(iv) *has level multiple limits* if the previous property holds for every small category Λ.

2.6. Level limits in weak double categories. Let $\mathbb{A}$ be a weak double category, viewed as the weak multiple category $\mathrm{sk}_2(\mathbb{A})$, by adding degenerate items of all the missing types (cf. I.2.7).

The present $\star$-level limits in $\mathbb{A}$, i.e. limits of ordinary functors $\Lambda \to \mathrm{tv}_*(\mathbb{A})$, correspond to the 'limits of horizontal functors' in [GP1]. There are slight differences in terminology, essentially because the '2-dimensional universal property' of double limits (see [GP1], 4.2) here is not required from the start but comes out of a condition of preservation by degeneracies. We think that the present terminology is preferable.

As a particular case of the definitions in 2.2, we have the following cases.

(i) $\mathbb{A}$ *has* $\star$*-level limits on* a (small) category Λ if all the functors $\Lambda \to \mathrm{tv}_*(\mathbb{A})$ have a limit. By the usual theorem on ordinary limits, all of them can be constructed from:

- small products ΠA_λ of objects,
- equalisers of pairs $f, g\colon A \to B$ of parallel horizontal arrows.

(i′) $\mathbb{A}$ *has* 1-*level limits on* Λ if all the functors $\Lambda \to \mathrm{tv}_1(\mathsf{A})$ have a limit. All of them can be constructed from:
- small products Πu_λ of vertical arrows,
- equalisers of pairs $a, b\colon u \to v$ of double cells (between the same vertical arrows).

(ii) $\mathbb{A}$ *has level limits on* Λ if it has $\star$- and 1-level limits on Λ.

(iii) $\mathbb{A}$ *has level double limits on* Λ if it has such level limits, preserved by faces and degeneracies.

(iv) $\mathbb{A}$ *has level double limits* if the previous property holds for every small category Λ; this is equivalent to the existence of small *double products* and *double equalisers*.

Let us note again, as in 2.1 that the existence of (say) double products is now a *global condition*: it means the existence of products of objects *and* vertical arrows, *consistently* with faces and degeneracies. Here we are *not* defining when a *single* product ΠA_λ should be called a 'double product' (while in [GP1] this meant a product of objects preserved by vertical identities).

In [GP1] case (i) would be expressed saying that $\mathbb{A}$ has 1-*dimensional* limits of horizontal functors on Λ. Case (iii) (resp. (iv)) would be expressed saying that $\mathbb{A}$ can be given a lax choice of double limits for all horizontal functors defined on Λ (resp. defined on some small category).

3. Multiple limits of degree zero

We now define 'multiple limits' of degree zero - *namely those limits that produce objects*. They extend the previous *level limits of degree zero* (or $\star$-level limits), and are generated by the latter together with *tabulators of degree zero* (Theorem 3.6). The general case - *limits that produce cubes of arbitrary dimension* - will be treated in the next section.

3.1. The diagonal functor. Let X and A be chiral multiple categories, and let X be small. Consider the *diagonal* functor (of ordinary categories)

$$D\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Ps}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}). \tag{40}$$

D takes each object A of A to a unitary pseudo functor, 'constant' at A, via the family of the total $\mathbf{i}$-degeneracies (cf. (7))

$$\begin{aligned}
&DA\colon \mathsf{X} \to \mathsf{A},\\
&DA(x) = e_{\mathbf{i}}(A) \qquad DA(f) = \mathrm{id}(e_{\mathbf{i}}A) && \text{(for } x \text{ and } f \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}),\\
&\underline{DA}_i(x) = \mathrm{id}(e_{\mathbf{i}}A)\colon e_i(DA(x)) \to DA(e_i x) && \text{(for } x \text{ in } X_{\mathbf{i}|i}),\\
&\underline{DA}_i(x,y) = \lambda_i\colon e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(A) && \text{(for } i\text{-consecutive cubes } x, y \text{ in } X_{\mathbf{i}}).
\end{aligned} \tag{41}$$

In fact, as required by axiom (lmf.3) of lax multiple functors (in I.3.9), the comparison $\underline{DA}_i(x, y)$ above is (necessarily) the unitor $\lambda_i(e_{\mathbf{i}}A) = \rho_i(e_{\mathbf{i}}A)$ of A, equivalently left or right (see I.3.3), that will generally be written as λ_i for short.

Similarly, a $\star$-map $h\colon A \to B$ in A is sent to the constant transversal transformation

$$Dh\colon DA \to DB\colon \mathsf{X} \to \mathsf{A}, \qquad (Dh)(x) = e_{\mathbf{i}}(h)\colon e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(B) \qquad (x \text{ in } X_{\mathbf{i}}). \qquad (42)$$

DA is a *strict* multiple functor whenever A is pre-unitary (cf. I.3.2).

Note also that the definition of the diagonal functor D depends on the commutativity of degeneracies in A, *which holds in the present chiral case.* For a general 3-dimensional *intercategory* A one could define two functors

$$D_{12}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}), \qquad D_{21}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Cx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}), \qquad (43)$$

where $D_{ij}(A)$ sends a 12-cube x to $D_{ij}(A)(x) = e_ie_j(A)$ (and any lower $\mathbf{i}$-cube to $e_{\mathbf{i}}(A)$). In higher dimension the situation is even more complex.

Still, in an intercategory we have *level* limits, defined as in Section 2, and some simple non-level limits that can be defined ad hoc, like the e_1e_2-*tabulator* and the e_2e_1-*tabulator* of a 12-cube considered in Part I, Section 6.

3.2. Cones. Let $F\colon \mathsf{X} \to \mathsf{A}$ be a lax functor. A *(transversal) cone* of F will be a pair $(A, h\colon DA \to F)$ comprising an object A of A (the *vertex* of the cone) and a transversal transformation of lax functors $h\colon DA \to F\colon \mathsf{X} \to \mathsf{A}$; in other words, it is an object of the ordinary comma category $(D \downarrow F)$, where F is viewed as an object of the category $\mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A})$.

By definition (cf. 1.8), the transversal transformation h amounts to assigning the following data:

- a transversal $\mathbf{i}$-map $hx\colon e_{\mathbf{i}}(A) \to Fx$, for every $\mathbf{i}$-cube x in X, subject to the following axioms of naturality and coherence:

(tc.1) $Ff.hx = hy$ (for every $\mathbf{i}$-map $f\colon x \to_0 y$ in X),

(tc.2) h commutes with positive faces, and agrees with positive degeneracies and compositions:

$$h(\partial_i^\alpha x) = \partial_i^\alpha(hx), \qquad \text{(for } x \text{ in } X_{\mathbf{i}}),$$
$$h(e_ix) = \underline{F}_i(x).e_i(hx)\colon e_{\mathbf{i}}(A) \to_0 F(e_ix) \qquad \text{(for } x \text{ in } X_{\mathbf{i}|i}),$$
$$h(z) = \underline{F}_i(x, y).(hx +_i hy).\lambda_i^{-1}\colon e_{\mathbf{i}}(A) \to_0 F(z) \qquad \text{(for } z = x +_i y \text{ in } X_{\mathbf{i}}),$$

where $\lambda_i\colon e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(A)$ is a unitor of A (cf. (41)).

3.3. Definition (Limits of degree zero). Given a lax functor $F\colon \mathsf{X} \to \mathsf{A}$ between chiral multiple categories, the *(transversal) limit of degree zero* $\lim(F) = (L, t)$ is a universal cone $(L, t\colon DL \to F)$.

In other words:

(tl.0) L is an object of A and $t\colon DL \to F$ is a transversal transformation of lax functors,

(tl.1) for every cone $(A, h\colon DA \to F)$ there is precisely one $\star$-map $f\colon A \to L$ in A such that $t.Df = h$.

We say that A *has limits of degree zero* on X if all these exist. In particular, if X is the multiple category freely generated by a category Λ, *at level* 0, then A has 0-degree limits on X if and only if it has 0-degree *level* limits on Λ (cf. 2.2).

3.4. Tabulators of degree zero. A is always a chiral multiple category. Let us recall that every positive multi-index $\mathbf{i}$ gives a 'total' degeneracy

$$e_{\mathbf{i}} = e_{i_1}...e_{i_n}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}. \tag{44}$$

An $\mathbf{i}$-cube x of A can be viewed as a unitary pseudo functor $x\colon \mathsf{u_i} \to \mathsf{A}$ where $\mathsf{u_i}$ is the strict multiple category freely generated by one $\mathbf{i}$-cube $u_{\mathbf{i}}$. The pseudo functor x sends $u_{\mathbf{i}}$ to x, and has comparisons $\underline{x}_i$ for i-composites that derive from the unitors of A, as in the following

$$\underline{x}_i(e_i\partial_i^-u_{\mathbf{i}}, u_{\mathbf{i}}) = \lambda_i(x)\colon e_i\partial_i^-x +_i x \to x, \qquad \underline{x}_i(u_{\mathbf{i}}, e_i\partial_i^+u_{\mathbf{i}}) = \rho_i(x)\colon x +_i e_i\partial_i^+x \to x.$$

(All such pseudo functors x are strict precisely when A is unitary.)

The *tabulator of degree zero* of x in A will be the limit of this pseudo functor $x\colon \mathsf{u_i} \to \mathsf{A}$; we also speak of the *total tabulator*, or $\mathbf{i}$-*tabulator*, of x.

The tabulator is thus an object $T = \top x$ $(= \top_{\mathbf{i}}x)$ equipped with an $\mathbf{i}$-map $t_x\colon e_{\mathbf{i}}(T) \to_0 x$ such that the pair $(T, t_x\colon e_{\mathbf{i}}(T) \to_0 x)$ is a universal arrow from the functor $e_{\mathbf{i}}\colon \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}$ to the object x of $\mathrm{tv}_{\mathbf{i}}\mathsf{A}$. Explicitly, this means that, for every object A and every $\mathbf{i}$-map $h\colon e_{\mathbf{i}}(A) \to_0 x$ there is a unique $\star$-map f such that

$$\begin{array}{ccc} e_{\mathbf{i}}(A) & \xrightarrow{e_{\mathbf{i}}(f)} & e_{\mathbf{i}}(T) \\ & \searrow^{h} & \downarrow t_x \\ & & x \end{array} \qquad \begin{array}{l} f\colon A \to_0 T, \\ \\ t_x.e_{\mathbf{i}}(f) = h. \end{array} \tag{45}$$

We say that A *has tabulators of degree zero* if all these exist, for every positive multi-index $\mathbf{i}$. Obviously, the tabulator of an object always exists and is the object itself.

When such tabulators exist, we can form for every positive multi-index $\mathbf{i}$ a right adjoint functor

$$\top_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}\mathsf{A} \to \mathrm{tv}_*\mathsf{A}, \qquad e_{\mathbf{i}} \dashv \top_{\mathbf{i}}, \tag{46}$$

which is just the identity for $\mathbf{i} = \varnothing$.

Assuming that the tabulators of $x \in A_{\mathbf{i}}$ and $z = \partial_j^\alpha x$ exist (for $j \in \mathbf{i}$), the *projection* $p_j^\alpha x$ of $\top x$ $(= \top_{\mathbf{i}}x)$ will be the following $\star$-map of A

$$\begin{array}{ccc} e_{\mathbf{i}|j}\top x & \xrightarrow{e_{\mathbf{i}|j}(p_j^\alpha x)} & e_{\mathbf{i}|j}\top(\partial_j^\alpha x) \\ & \searrow^{\partial_j^\alpha(t_x)} & \downarrow t_z \\ & & z = \partial_j^\alpha x \end{array} \qquad \begin{array}{l} p_j^\alpha x\colon \top x \to_0 \top(\partial_j^\alpha x), \\ \\ t_z.e_{\mathbf{i}|j}(p_j^\alpha x) = \partial_j^\alpha(t_x). \end{array} \tag{47}$$

3.5. Tabulators and concatenation. We now examine the relationship between tabulators of $\mathbf{i}$-cubes and (zero-ary or binary) j-concatenation, for $j \in \mathbf{i}$.

(a) If the degenerate $\mathbf{i}$-cube $x = e_j z$ and the $\mathbf{i}|j$-cube z have total tabulators in A, they are linked by a *diagonal* transversal $\star$-map $d_j z$, defined as follows

$$\begin{array}{ccc} e_{\mathbf{i}}(\top z) & \xrightarrow{e_{\mathbf{i}}(d_j z)} & e_{\mathbf{i}}(\top(e_j z)) \\ & \searrow^{e_j t_z} & \downarrow t_x \\ & & x = e_j z \end{array} \qquad \begin{array}{l} d_j z\colon \top z \to_0 \top(e_j z), \\ \\ t_x.e_{\mathbf{i}}(d_j z) = e_j(t_z). \end{array} \tag{48}$$

This $\star$-map $d_j z$ is a section of both projections $p_j^\alpha x$ (defined above) because

$$t_z.e_{\mathbf{i}|j}(p_j^\alpha x.d_j z) = \partial_j^\alpha(t_x).e_{\mathbf{i}|j}(d_j z) = \partial_j^\alpha(t_x.e_{\mathbf{i}}(d_j z)) = \partial_j^\alpha(e_j(t_z)) = t_z.$$

(b) For a concatenation $z = x +_j y$ of $\mathbf{i}$-cubes, the three total tabulators of x, y, z are also related. The link goes through the ordinary pullback $\top_j(x, y)$ of the objects $\top x$ and $\top y$, over the tabulator $\top w$ of the $\mathbf{i}|j$-cube $w = \partial_j^+ x = \partial_j^- y$ (*provided* all these tabulators and such a pullback exist)

$$\begin{array}{ccccc} & \nearrow^{p_j(x,y)} & \top x & \searrow^{p_j^+ x} & \\ \top_j(x, y) & & & & \top w \\ & \searrow_{q_j(x,y)} & \top y & \nearrow_{p_j^- y} & \end{array} \qquad \begin{array}{l} t_w.e_{\mathbf{i}|j}(p_j^+ x) = \partial_j^+(t_x), \\ \\ t_w.e_{\mathbf{i}|j}(p_j^- y) = \partial_j^-(t_y). \end{array} \tag{49}$$

We now have a *diagonal* transversal $\star$-map $d_j(x, y)$ given by the universal property of $\top z$

$$d_j(x, y)\colon \top_j(x, y) \to_0 \top z, \qquad t_z.e_{\mathbf{i}}(d_j(x, y)) = t_x.e_{\mathbf{i}} p_j(x, y) +_j t_y.e_{\mathbf{i}} q_j(x, y). \tag{50}$$

The j-composition above is legitimate, by construction

$$\begin{aligned} \partial_j^+(t_x.e_{\mathbf{i}} p_j(x, y)) &= \partial_j^+(t_x).e_{\mathbf{i}|j}(p_j(x, y)) = t_w.e_{\mathbf{i}|j}(p_j^+ x).e_{\mathbf{i}|j}(p_j(x, y)) \\ &= t_w.e_{\mathbf{i}|j}(p_j^- y).e_{\mathbf{i}|j}(q_j(x, y)) = \partial_j^-(t_y).e_{\mathbf{i}|j}(q_j(x, y)) = \partial_j^-(t_y.e_{\mathbf{i}} q_j(x, y)). \end{aligned}$$

It is easy to show (and it also follows from the proof of the theorem below) that $\top_j(x, y)$ is the transversal limit of the diagram 'formed' by $z = x +_j y$ (based on the multiple category freely generated by two j-consecutive $\mathbf{i}$-cubes).

3.6. Theorem. [Construction and preservation of 0-degree limits] *Let* A *and* B *be chiral multiple categories.*

(a) All limits of degree zero in A *can be constructed from level limits of degree zero and tabulators of degree zero, or also from products, equalisers and tabulators - all of degree zero.*

(b) If A *has all limits of degree zero, a lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ *preserves them if and only if it preserves products, equalisers and tabulators of degree zero.*

Proof. See Section 5. ■

3.7. Examples. In the chiral triple category $\mathsf{SC}(\mathbf{C})$ (over a category $\mathbf{C}$ with pullbacks and pushouts) we have the following three kinds of tabulators of degree zero (apart from the trivial tabulator of an object), already described in I.4.3.

(a) The tabulator of a 1-arrow f (i.e. a span) is an object $\top_1 f$ with a universal 1-map $e_1(\top_1 f) \to_0 f$; the solution is the (trivial) limit of the span f, i.e. its middle object.

(b) The tabulator of a 2-arrow u (a cospan) is an object $\top_2 u$ with a universal 2-map $e_2(\top_2 u) \to_0 u$; the solution is the pullback of u.

(c) The *total tabulator* of a 12-cell π (a span of cospans) is an object $\top_{12}\pi$ with a universal 12-map $e_{12}(\top_{12}\pi) \to_0 \pi$; the solution is the limit of the diagram, i.e. the pullback of its middle cospan.

The two (non total) tabulators *of degree* 1 of the 12-cell π will be reviewed below, in 4.6.

4. Multiple limits of arbitrary degree

We now introduce general limits in a chiral multiple category A, taking advantage of the iterated lift functors $Q_{\mathbf{i}}$ (cf. 1.5), where $\mathbf{i}$ is a positive multi-index of degree $n \geqslant 0$. X is always a small chiral multiple category.

Let us recall that $\mathsf{u}_{\mathbf{i}}$ denotes the multiple category freely generated by one $\mathbf{i}$-cube $u_{\mathbf{i}}$ (cf. 3.4).

4.1. A motivation. For a positive multi-index $\mathbf{i}$ of degree $n \geqslant 0$, the limits (of degree 0) of multiple functors with values in the lifted chiral multiple category $Q_{\mathbf{i}}\mathsf{A}$ will be called *multiple limits of type* $\mathbf{i}$ (*and degree* n) in A; their results are thus $\mathbf{i}$-cubes of A. They extend the limits of degree zero considered above, for $\mathbf{i} = \varnothing$ and $Q_*\mathsf{A} = \mathsf{A}$.

Let us begin with some simple examples, based on a 2-dimensional cube $x \in A_{12}$, introducing definitions that will be made precise below.

(a) The cube $x \in A_{12}$ is the same as a unitary pseudo functor $x\colon \mathsf{u}_{12} \to \mathsf{A}$. We have already considered its *tabulator of degree zero*, namely an object $\top x = \top_{12} x$ with a universal 12-map $t\colon e_{12}(\top_{12} x) \to_0 x$ (where $e_{12} = e_1 e_2 = e_2 e_1\colon A_* \to A_{12}$ is the composed degeneracy).

(b) But x can also be viewed as a (1-dimensional) 1-cube of $Q_2\mathsf{A}$, i.e. a unitary pseudo functor $x\colon \mathsf{u}_1 \to Q_2\mathsf{A}$. Its e_1-*tabulator* (of degree 1) will be the total tabulator of x as a 1-cube of $Q_2\mathsf{A}$; this amounts to a 2-cube $\top_1 x$ of A with a universal 12-map $t\colon e_1(\top_1 x) \to_0 x$ (where $e_1\colon A_2 \to A_{12}$ is the degeneracy $e_1\colon (Q_2\mathsf{A})_* \to (Q_2\mathsf{A})_1$).

(c) Symmetrically, x can be viewed as a (1-dimensional) 2-cube of $Q_1\mathsf{A}$, i.e. a unitary pseudo functor $x\colon \mathsf{u}_2 \to Q_1\mathsf{A}$. Its e_2-*tabulator* (of degree 1, again) will be the total tabulator of x as a 2-cube of $Q_1\mathsf{A}$; this amounts to a 1-cube $\top_2 x$ of A with a universal 12-map $t\colon e_2(\top_2 x) \to_0 x$ (where $e_2\colon A_1 \to A_{12}$ is the degeneracy $e_2\colon (Q_1\mathsf{A})_* \to (Q_1\mathsf{A})_2$).

(d) The 2-dimensional cube x is also an object of $Q_{12}\mathsf{A}$. Its *tabulator of degree two* is x itself. This is a (trivial) level limit, while the previous limits are not level, i.e. are not limits in some transversal category of A.

4.2. General tabulators. An $\mathbf{i}$-cube $x \in A_\mathbf{i}$ is a unitary pseudo functor $x\colon \mathsf{u}_\mathbf{i} \to \mathsf{A}$. For every $\mathbf{k} \subset \mathbf{i}$ we can also view x as a pseudo functor $\mathsf{u}_\mathbf{j} \to Q_\mathbf{k}\mathsf{A}$, where $\mathbf{j} = \mathbf{i}\setminus\mathbf{k}$, so that x can have an $e_\mathbf{j}$-*tabulator*, namely a $\mathbf{k}$-cube $T = \top_\mathbf{j}x \in A_\mathbf{k}$ with a universal $\mathbf{i}$-map $t_x\colon e_\mathbf{j}(\top_\mathbf{j}x) \to_0 x$. (Total tabulators correspond to $\mathbf{j} = \mathbf{i}$, while $\mathbf{j} = \varnothing$ gives the trivial case $\top_\varnothing x = x$.)

The universal property says now that, for every $\mathbf{k}$-cube A and every $\mathbf{i}$-map $h\colon e_\mathbf{j}(A) \to_0 x$ there is a unique $\mathbf{k}$-map u such that

$$\begin{array}{ccc} e_\mathbf{j}(A) & \xrightarrow{e_\mathbf{j}(u)} & e_\mathbf{j}(T) \\ & \searrow^{h} & \downarrow t_x \\ & & x \end{array} \qquad \begin{array}{l} u\colon A \to_0 T, \\ \\ t_x.e_\mathbf{j}(u) = h. \end{array} \tag{51}$$

We say that the chiral multiple category A *has tabulators of all degrees* if every $\mathbf{i}$-cube $x \in A_\mathbf{i}$ has all $\mathbf{j}$-tabulators $\top_\mathbf{j}x \in A_\mathbf{k}$ (for $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, disjoint union). We say that A *has multiple tabulators* if it has *tabulators of all degrees*, preserved by faces and degeneracies.

In this case, if A is transversally invariant, one can always make a choice of multiple tabulators such that this preservation is strict (as we have already seen in various examples of Part I):

$$\partial_i^\alpha(\top_\mathbf{j}x) = \top_\mathbf{j}(\partial_i^\alpha x), \qquad \top_\mathbf{j}(e_i(y)) = e_i(\top_\mathbf{j}(y)) \qquad (\mathbf{j} \subset \mathbf{i},\ i \in \mathbf{i}\setminus\mathbf{j}), \tag{52}$$

for $x \in A_\mathbf{i}$ and $y \in A_{\mathbf{i}|i}$.

Note that these conditions are trivial if $\mathbf{j} = \varnothing$ or $\mathbf{j} = \mathbf{i}$, *whence for all weak double categories* (where there is only one positive index). This remark will be important when reconsidering double limits, in 4.7.

4.3. Lemma. [Basic properties of tabulators] *Let* A *be a chiral multiple category.*

(a) For an $\mathbf{i}$*-cube* x *and a disjoint union* $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ *we have*

$$\top_\mathbf{i}x = \top_\mathbf{k}\top_\mathbf{j}x, \tag{53}$$

provided that the two tabulators on the right exist.

(b) A *has tabulators of all degrees if and only it has all* elementary tabulators $\top_j x$ *(for every positive multi-index* $\mathbf{i}$*, every* $j \in \mathbf{i}$ *and every* $\mathbf{i}$*-cube* x*).*

(c) If all e_j*-tabulators of* $\mathbf{i}$*-cubes exist in* A *there is an ordinary adjunction*

$$e_j\colon \mathrm{tv}_{\mathbf{i}|j}(\mathsf{A}) \rightleftarrows \mathrm{tv}_\mathbf{i}(\mathsf{A}) : \top_j, \qquad e_j \dashv \top_j \qquad (j \in \mathbf{i}), \tag{54}$$

and $e_j\colon \mathrm{tv}_{\mathbf{i}|j}\mathsf{A} \to \mathrm{tv}_\mathbf{i}\mathsf{A}$ *preserves colimits.*

*(d) If all e_j-*cotabulators *of* $\mathbf{i}$*-cubes exist in* A*, then* $e_j\colon \mathrm{tv}_{\mathbf{i}|j}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}$ *is a right adjoint and preserves the existing limits (so that a condition on multiple* level *limits in 2.2(iii) is automatically satisfied).*

(e) In a weak double category $\mathbb{A}$ *the existence of cotabulators of vertical arrows implies that all ordinary limits in* $\mathrm{tv}_*(\mathbb{A})$ *are preserved by vertical identities. (This has already been used in I.5.5.)*

Proof. (a) Follows from a composition of universal arrows for

$$e_{\mathbf{i}} = e_{\mathbf{j}}e_{\mathbf{k}}\colon\ \mathrm{tv}_*\mathsf{A} \to \mathrm{tv}_{\mathbf{k}}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{A}.$$

The rest is obvious. ■

4.4. Definition (Multiple limits). We are now ready for a general definition of multiple limits in a chiral multiple category A.

(a) For a positive multi-index $\mathbf{i} \subset \mathbb{N}$ and a chiral multiple category X we say that A has *limits of type* $\mathbf{i}$ on X if $Q_{\mathbf{i}}\mathsf{A}$ has limits of degree zero on X.

(b) We say that A *has limits of type* $\mathbf{i}$ if this happens for all small chiral multiple categories X.

(c) We say that A *has limits of all degrees* (or *all types*) if this happens for all positive multi-indices $\mathbf{i}$.

(d) We say that A *has multiple limits of all degrees* if all the previous limits exist and are preserved by the multiple functors (cf. 1.5)

$$D_j^\alpha\colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_jQ_{\mathbf{i}}(\mathsf{A}), \qquad E_j\colon R_jQ_{\mathbf{i}}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A}) \qquad (j \notin \mathbf{i}). \tag{55}$$

In this case, if A is transversally invariant, one can always operate a choice of multiple limits such that this preservation is strict (working as in Proposition 2.3).

We do not speak here of *completeness*: this notion should also involve the existence of 'companions' and 'adjoints' for all transversal maps, as shown by our study of Kan extensions in the domain of weak double categories [GP3, GP4].

4.5. Main Theorem. [Construction and preservation of multiple limits, II] *Let* A *and* B *be chiral multiple categories.*

(a) All multiple limits in A *can be constructed from level multiple limits and multiple tabulators, or also from multiple products, multiple equalisers and multiple tabulators.*

(b) If A *has all multiple limits, a lax multiple functor* $S\colon \mathsf{A} \to \mathsf{B}$ *preserves them if and only if it preserves multiple products, multiple equalisers and multiple tabulators.*

Similarly for finite limits and finite products.

Proof. Follows from Theorem 3.6, applied to the family of chiral multiple categories $Q_{\mathbf{i}}\mathsf{A}$, together with the multiple functors of faces and degeneracies (cf. (55)) and the multiple functors $Q_{\mathbf{i}}S\colon Q_{\mathbf{i}}\mathsf{A} \to Q_{\mathbf{i}}\mathsf{B}$. ■

4.6. Examples. For a category $\mathbf{C}$ with pushouts and pullbacks we complete the discussion of tabulators in the chiral triple category $\mathsf{SC}(\mathbf{C})$, after the three types of tabulators of degree zero examined in 3.7. We start again from a 12-cube $\pi\colon \vee\times\wedge \to \mathbf{C}$ (a span of cospans in $\mathbf{C}$).

(a) The e_1-tabulator of π is a 2-arrow $\top_1\pi$ (a cospan) with a universal 12-map $e_1(\top_1\pi) \to_0 \pi$; the solution is the middle cospan of π.

(b) The e_2-tabulator of π is a 1-arrow $\top_2\pi$ with a universal 12-map $e_2(\top_2\pi) \to_0 \pi$; the solution is the obvious span whose objects are the pullbacks of the three cospans of π.

These limits are preserved by faces and degeneracies. For instance:

- $\partial_1^-(\top_2\pi) = \top_2(\partial_1^-\pi)$, which means that the domain of the span $\top_2\pi$ (described above) is the pullback of the cospan $\partial_1^-\pi$,

- $\top_2(e_1u) = e_1(\top_2u)$, i.e. the e_2-tabulator of the 1-degenerate cell e_1u (on the cospan u) is the degenerate span on the pullback of u.

Finally, putting together the previous results (in 2.2 and 3.7): if $\mathbf{C}$ is a complete (or finitely complete) category with pushouts, then the chiral triple category $\mathsf{SC}(\mathbf{C})$ has multiple limits (or the finite ones).

4.7. Limits in weak double categories. We now complete the discussion of limits in a weak double category $\mathbb{A}$, after the case of level limits examined in 2.6.

Here a strong difference appears between the present analysis and that of [GP1]. In that paper tabulators were assumed to satisfy also a 'two-dimensional universal property'. On the other hand we already remarked here, at the end of 4.2, that multiple tabulators are subject to preservation properties that only become non-trivial in dimension three or higher; the examples above (in 4.6) clearly show that at least two positive indices are required to formulate non-trivial conditions of this type.

In other words, with the present terminology, tabulators in a weak double category $\mathbb{A}$ are automatically double tabulators, and the only limits that must be preserved by faces and degeneracies are the level ones, generated by products and equalisers of objects or vertical arrows of $\mathbb{A}$.

We think that the present terminology for a weak double category $\mathbb{A}$, a particular case of the definitions in 4.2 and 4.4, is preferable; it can be summarised as follows.

(a) $\mathbb{A}$ *has tabulators* if every vertical arrow u (a 1-cube) has an object $\top u = \top_1 u$ with a universal double cell $e_1(\top_1 u) \to u$. (This is what we now consider to be the correct definition of a double tabulator.)

(b) $\mathbb{A}$ *has limits of degree zero* (namely the limits that produce *objects*) if all the functors $\mathbb{X} \to \mathbb{A}$ (defined on a small weak double category) have a limit. Theorem 3.6 says that this condition amounts to the existence of:
- all products ΠA_λ of objects,
- all equalisers of pairs $f, g\colon A \to B$ of parallel horizontal arrows,
- all tabulators $\top u$ of vertical arrows.

(c) $\mathbb{A}$ *has limits of degree 1* (namely the limits that produce *vertical arrows*) if all the functors $\Lambda \to \mathrm{tv}_1(\mathbb{A})$ defined on a small category) have a limit. By the usual theorem on ordinary limits, this condition amounts to the existence of:
- products $\Pi\, u_\lambda$ of vertical arrows,
- equalisers of pairs $a, b\colon u \to v$ of double cells (between the same vertical arrows).

Let us recall that such limits can be viewed as limits of degree zero in the lifted category $\mathbf{C} = Q_1(\mathbb{A})$, with objects in $C_* = A_1$ (the vertical arrows of $\mathbb{A}$) and arrows in $C_0 = A_{01}$ (the double cells of $\mathbb{A}$). On the other hand $R_1\mathbb{A} = \mathrm{tv}_*\mathbb{A}$ is the category of objects and horizontal arrows of $\mathbb{A}$.

(d) $\mathbb{A}$ *has limits of all degrees* if both conditions (b) and (c) are satisfied.

(e) $\mathbb{A}$ *has double limits* if all the previous limits exist and are preserved by the ordinary functors

$$D_1^\alpha\colon \mathbf{C} \to \mathrm{tv}_*\mathbb{A}, \qquad\qquad E_1\colon \mathrm{tv}_*\mathbb{A} \to \mathbf{C}, \tag{56}$$

inasmuch as this makes sense (i.e. for ordinary limits in $\mathrm{tv}_*\mathbb{A}$ and $\mathbf{C}$, which amount to 0- and 1-level limits of $\mathbb{A}$).

Theorem 4.5 says that $\mathbb{A}$ has double limits if and only if it has: double products, double equalisers and tabulators. Concretely, this amounts to the existence of the limits listed in (b) and (c), together with the conditions:
- products preserve domain, codomain and vertical identities,
- equalisers preserve domain, codomain and vertical identities.

If this holds *and* $\mathbb{A}$ is transversally invariant ('horizontally invariant' in [GP1]), Proposition 2.3 says one can always choose double limits such that this preservation is strict. For products this means that:
- for a family of vertical arrows $u_\lambda\colon A_\lambda \to B_\lambda$ we have $\Pi\, u_\lambda\colon \Pi\, A_\lambda \to \Pi\, B_\lambda$,
- for a family of objects A_λ the product of their vertical identities is the vertical identity of $\Pi\, A_\lambda$.

4.8. The symmetric cubical case. As analysed in [G1], weak symmetric cubical categories (with lax cubical functors) have a path endofunctor

$$\begin{gathered} P\colon \mathrm{Lx}\mathbf{Wsc} \to \mathrm{Lx}\mathbf{Wsc}, \\ P((\mathrm{tv}_n\mathsf{A}), (\partial_i^\alpha), (e_i), (+_i), (s_i), ...) = ((\mathrm{tv}_{n+1}\mathsf{A}), (\partial_{i+1}^\alpha), (e_{i+1}), (+_{i+1}), (s_{i+1}), ...), \end{gathered} \tag{57}$$

which lifts all components of one degree and discards 1-indexed faces, degeneracies, transpositions and comparisons (the latter are omitted above). The discarded faces and degeneracy yield three natural transformations

$$\partial_1^\alpha\colon P \rightleftarrows 1 : e_1, \qquad\qquad \partial_1^\alpha.e_1 = \mathrm{id}, \tag{58}$$

which make P into a *path endofunctor*, from a structural point of view. The role of symmetries is crucial (without them we would have two *non-isomorphic* path-functors, and a plethora of higher path functors, their composites, see [G1]).

This situation cannot be extended to chiral multiple categories: the path endofunctor was replaced by the lift functors $Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ and the restriction functors $R_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ of 1.8, with faces and degeneracy

$$D_j^\alpha\colon Q_j \;\overrightarrow{\overleftarrow{\rightarrow}}\; R_j : E_j, \qquad\qquad D_j^\alpha.E_j = \mathrm{id}. \tag{59}$$

The whole system is consistent, by means of commutative squares

$$\begin{array}{ccc} \mathrm{Lx}\mathbf{Wsc} & \xrightarrow{P} & \mathrm{Lx}\mathbf{Wsc} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U_j} \\ \mathrm{Lx}\mathbf{Cmc} & \xrightarrow[Q_j]{} & \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Lx}\mathbf{Wsc} & \xrightarrow{1} & \mathrm{Lx}\mathbf{Wsc} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U_j} \\ \mathrm{Lx}\mathbf{Cmc} & \xrightarrow[R_j]{} & \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j} \end{array} \tag{60}$$

where $U\colon \mathrm{Lx}\mathbf{Wsc} \to \mathrm{Lx}\mathbf{Cmc}$ is the embedding described in I.2.8 (that gives rise to weak multiple categories of a symmetric cubical type) and $U_j = R_jU$.

In this way, cubical limits in weak symmetric cubical categories, dealt with in [G2], agree with multiple limits as presented here.

5. Proof of the theorem on the construction of multiple limits

We now prove Theorem 3.6. The argument is similar to the proof of the corresponding theorem for double limits [GP1], or its extension to cubical limits [G2].

5.1. Comments. Of course we only have to prove the 'sufficiency' part of the statement. We write down the argument for the construction of limits; the preservation property is proved in the same way.

The chiral multiple category A is supposed to have all level limits of degree zero and all tabulators of degree zero (or total tabulators). The proof works by transforming a lax functor $F\colon \mathsf{X} \to \mathsf{A}$ of chiral multiple categories into a graph-morphism $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ and taking the limit of the latter. The (directed) graph $\mathbf{X}$ is a sort of 'transversal subdivision' of X, where every $\mathbf{i}$-cube of X is replaced with an object *simulating its total tabulator*.

The procedure is similar to computing the end of a functor $S\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ as the limit of the associated functor $S^\S\colon \mathbf{C}^\S \to \mathbf{D}$ based on Kan's *subdivision category* of $\mathbf{C}$ ([Ka], 1.10; [Ma], IX.5).

5.2. Transversal subdivision. The *transversal subdivision* $\mathbf{X}$ of X is a graph, formed by the following objects and arrows, for an arbitrary positive multi-index $\mathbf{i}$ of degree $n \geqslant 0$, with arbitrary $j \in \mathbf{i}$ and $\alpha \in \{0, 1\}$. (Note that this graph is finite whenever X is.)

(a) For every $\mathbf{i}$-cell x of X there is an object x in $\mathbf{X}$. For every $\mathbf{i}$-map $f\colon x \to y$ of X there is an arrow $f\colon x \to y$ in $\mathbf{X}$.

(b) For every $\mathbf{i}$-cell x of X, we also add $2n$ arrows $p_j^\alpha x\colon x \to \partial_j^\alpha x$ (that simulate the projections (47) of the total tabulator of x, for $j \in \mathbf{i}$ and $\alpha = 0, 1$).

(c) If $x = e_j z$ is degenerate (in direction j) we also add an arrow $d_j z\colon z \to e_j z$ (that simulates the diagonal map (48)).

(d) For every j-concatenation of $\mathbf{i}$-cells $z = x +_j y$ in X, we also add an object $(x, y)_j$ in $\mathbf{X}$ and three arrows

$$p_j(x,y)\colon (x,y)_j \to x, \qquad q_j(x,y)\colon (x,y)_j \to y, \qquad d_j(x,y)\colon (x,y)_j \to z, \tag{61}$$

that simulate the pullback-object $\top_j(x, y)$ of (49), with its projections and the diagonal map (50).

5.3. The associated morphism of graphs. We now construct a graph-morphism $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ that naturally comes from F and the existence of level limits and tabulators (of degree zero) in A.

(a) For every $\mathbf{i}$-cell x of X, we define Gx as the following total tabulator (a $\star$-cube) of A

$$G(x) = \top(Fx) \qquad\qquad (t_{Fx}\colon e_{\mathbf{i}}G(x) \to_0 F(x)). \tag{62}$$

For every $\mathbf{i}$-map $f\colon x \to_0 y$ of X, we define Gf as the transversal map of A determined by the universal property of t_{Fy}, as follows

$$\begin{array}{ccc} e_{\mathbf{i}}\top(Fx) & \xrightarrow{e_{\mathbf{i}}(Gf)} & e_{\mathbf{i}}\top(Fy) \\ {\scriptstyle t_{Fx}}\downarrow & & \downarrow{\scriptstyle t_{Fy}} \\ Fx & \xrightarrow[Ff]{} & Fy \end{array} \qquad \begin{array}{l} Gf\colon \top(Fx) \to_0 \top(Fy), \\ \\ t_{Fy}.e_{\mathbf{i}}(Gf) = Ff.t_{Fx}. \end{array} \tag{63}$$

(b) For $z = \partial_j^\alpha x$ we define $G(p_j^\alpha x)\colon Gx \to_0 Gz$ as the following transversal map of A

$$\begin{array}{ccc} e_{\mathbf{i}|j}\top(Fx) & \xrightarrow{e_{\mathbf{i}|j}(Gp_j^\alpha x)} & e_{\mathbf{i}|j}\top(Fz) \\ & {\scriptstyle \partial_j^\alpha(t_{Fx})}\searrow & \downarrow{\scriptstyle t_{Fz}} \\ & & Fz \end{array} \qquad \begin{array}{l} G(p_j^\alpha x)\colon \top(Fx) \to_0 \top(Fz), \\ \\ t_{Fz}.e_{\mathbf{i}|j}(G(p_j^\alpha x)) = \partial_j^\alpha(t_{Fx}). \end{array} \tag{64}$$

(c) For a degenerate $\mathbf{i}$-cube $x = e_j z$ (where z is an $\mathbf{i}|j$-cube) the map $G(d_j z)\colon Gz \to_0 G(e_j z)$ is defined as follows

$$\begin{array}{ccc} e_{\mathbf{i}}(\top Fz) & \xrightarrow{e_{\mathbf{i}}(Gd_j z)} & e_{\mathbf{i}}(\top(Fe_j z)) \\ {\scriptstyle e_j t_{Fz}}\downarrow & & \downarrow{\scriptstyle t_{Fx}} \\ e_j Fz & \xrightarrow[\underline{F}_j(z)]{} & F(e_j z) = Fx \end{array} \qquad \begin{array}{l} G(d_j z)\colon \top Fz \to_0 \top(Fe_j z), \\ \\ t_{Fx}.e_{\mathbf{i}}(G(d_j z)) = \underline{F}_j(z).e_j(t_{Fz}). \end{array} \tag{65}$$

(d) For a concatenation $z = x +_j y$ of $\mathbf{i}$-cubes, the object $G(x, y)_j = \top_j(Fx, Fy)$ is the pullback of the objects $\top Fx$ and $\top Fy$, over the tabulator $\top Fw$ associated to the $\mathbf{i}|j$-cube $w = \partial_j^+ x = \partial_j^- y$ (cf. 3.5).

The arrows $p_j(x,y)\colon (x,y)_j \to x$ and $q_j(x,y)\colon (x,y)_j \to y$ of $\mathbf{X}$ are taken by G to the projections (49) of $\top_j(Fx, Fy)$

$$G(p_j(x,y))\colon G(x,y)_j \to_0 Gx, \qquad G(q_j(x,y))\colon G(x,y)_j \to_0 Gy, \tag{66}$$

so that $(G(x,y)_j; Gp_j(x,y), Gq_j(x,y))$ is the pullback of $(p_j^+(Fx), p_j^-(Fy))$ in $\mathrm{tv}_*\mathsf{A}$.

Finally, the arrow $d_j(x,y)\colon (x,y)_j \to z$ of $\mathbf{X}$ is sent by G to the diagonal (50) of $G(x,y)_i = \top_j(Fx, Fy)$, determined as follows

$$\begin{gathered} G(d_j(x,y))\colon \top_j(Fx,Fy) \to_0 \top F(z), \\ t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y)) = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}G(p_j(x,y)) +_j t_{Fy}.e_{\mathbf{i}}G(q_j(x,y))).\lambda_j^{-1}, \end{gathered} \tag{67}$$

$$\begin{array}{ccccc} e_{\mathbf{i}}(\top_j(Fx,Fy)) & \xrightarrow{e_{\mathbf{i}}(G(d_j(x,y))} & e_{\mathbf{i}}(\top(Fz)) & \xrightarrow{t_{Fz}} & Fz \\ \downarrow{\scriptstyle \lambda_j^{-1}} & & & & \uparrow{\scriptstyle \underline{F}_j(x,y)} \\ e_{\mathbf{i}}(\top_j(Fx,Fy)) +_j e_{\mathbf{i}}(\top_j(Fx,Fy)) & & \xrightarrow[t_{Fx}.e_{\mathbf{i}}Gp_j(x,y) +_j t_{Fy}.e_{\mathbf{i}}Gq_j(x,y)]{} & & Fx +_j Fy \end{array}$$

The limit of this diagram $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$ exists, by hypothesis.

5.4. From multiple cones to cones. In order to prove that the limit of G gives the limit of degree 0 of F we construct an isomorphism

$$(D{\downarrow}F) \to (D'{\downarrow}G),$$

from the comma category of transversal cones of the lax functor F to the comma category of ordinary cones of the graph-morphism G. We proceed first in this direction, and then backwards.

Let $(A, h\colon DA \to F)$ be a cone of F. For every $\mathbf{i}$-cube x of X, we define $k(x)\colon A \to_0 Gx = \top(Fx)$ as the $\star$-map of A determined by the $\mathbf{i}$-map hx, via the tabulator property

$$t_{Fx}.e_{\mathbf{i}}(kx) = hx. \tag{68}$$

Further, we define $k(x,y)_j\colon A \to_0 G(x,y)_j$ by means of the pullback-property of $G(x,y)_j$

$$p_j(x,y).k(x,y)_j = kx\colon A \to_0 Gx, \qquad q_j(x,y).k(x,y)_j = ky\colon A \to_0 Gy. \tag{69}$$

Let us verify that this family k is indeed a cone of $G\colon \mathbf{X} \to \mathrm{tv}_*\mathsf{A}$.

(a) Coherence with an $\mathbf{i}$-map $f\colon x \to_0 y$ (viewed as an arrow of $\mathbf{X}$) means that $Gf.kx = ky$, which follows from the cancellation property of t_{Fy}

$$t_{Fy}.e_{\mathbf{i}}(Gf.kx) = Ff.t_{Fx}.e_{\mathbf{i}}(kx) = Ff.hx = hy = t_{Fy}.e_{\mathbf{i}}(ky). \tag{70}$$

(b) - (c). Coherence with the **X**-arrows $p_j^\alpha(x)\colon x \to \partial_j^\alpha x$ and $d_j z\colon z \to e_j z = x$ follows from (64) and (65)

$$\begin{aligned} G(p_j^\alpha(x)).kx &= k(\partial_j^\alpha x), \\ t_{Fx}.e_{\mathbf{i}}(G(d_j z).kz) = \underline{F}_j(z).e_j(t_{Fz}).e_{\mathbf{i}}(kz) &= \underline{F}_j(z).e_j(t_{Fz}.e_{\mathbf{i}|j}(kz)) \\ = \underline{F}_j(z).e_j(hz) &= h(e_j z) = h(x) = t_{Fx}.e_{\mathbf{i}}(kx). \end{aligned} \tag{71}$$

(d) Coherence with the **X**-arrows $p_j(x,y)$ and $q_j(x,y)$ holds by construction (see (66)). For $d_j(x,y)$ and $z = x +_j y$ we have

$$\begin{aligned} & t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y).k(x,y)_j) \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j(x,y) +_j t_{Fy}.e_{\mathbf{i}}q_j(x,y)).\lambda_j^{-1}.e_{\mathbf{i}}k(x,y)_j \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j(x,y) +_j t_{Fy}.e_{\mathbf{i}}q_j(x,y)).(e_{\mathbf{i}}k(x,y)_j +_j e_{\mathbf{i}}k(x,y)_j).\lambda_j^{-1} \\ &\quad = \underline{F}_j(x,y).(hx +_j hy).\lambda_j^{-1} = hz = t_{Fz}.e_{\mathbf{i}}(kx). \end{aligned} \tag{72}$$

Finally, a map of multiple cones $f\colon (A, h\colon DA \to F) \to (A', h'\colon DA' \to F)$ determines a map of G-cones $f\colon (A,k) \to (A',k')$, since

$$t_{Fx}.e_{\mathbf{i}}(k'x.f) = h'x.e_{\mathbf{i}}(f) = hx = t_{Fx}.e_{\mathbf{i}}(kx). \tag{73}$$

5.5. From cones to multiple cones. In the reverse direction $(D'{\downarrow}G) \to (D{\downarrow}F)$ we just specify the procedure on cones. Given an ordinary cone $(A, k\colon D'A \to G)$ of G, one forms a multiple cone $(A, h\colon DA \to F)$ by letting

$$hx = t_{Fx}.e_{\mathbf{i}}(kx)\colon e_{\mathbf{i}}(A) \to x \qquad\qquad (x \in A_{\mathbf{i}}). \tag{74}$$

This satisfies (tc.1) (cf. 3.2) since, for $f\colon x \to_0 y$ in X

$$Ff.hx = Ff.t_{Fx}.e_{\mathbf{i}}(kx) = t_{Fy}.e_{\mathbf{i}}(Gf.kx) = t_{Fy}.e_{\mathbf{i}}(ky) = hy. \tag{75}$$

Finally, to verify the condition (tc.2) for j-units and j-composition in X we operate much as above (with $x = e_j z$ in the first case and $z = x +_j y$ in the second)

$$\begin{aligned} \underline{F}_j(z).e_j(hz) = \underline{F}_j(z).e_j(t_{Fz}.e_{\mathbf{i}|j}(kz)) &= \underline{F}_j(z).e_j(t_{Fz}).e_{\mathbf{i}}(kz) \\ = t_{Fx}.e_{\mathbf{i}}(G(d_j z).kz) &= t_{Fx}.e_{\mathbf{i}}(kx) = hx. \end{aligned} \tag{76}$$

$$\begin{aligned} & hz = t_{Fz}.e_{\mathbf{i}}(kz) = t_{Fz}.e_{\mathbf{i}}(G(d_j(x,y)).k(x,y)_j) = \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j(x,y) +_j t_{Fy}.e_{\mathbf{i}}q_j(x,y)).\lambda_j^{-1}.e_{\mathbf{i}}k(x,y)_j \\ &\quad = \underline{F}_j(x,y).(t_{Fx}.e_{\mathbf{i}}p_j(x,y) +_j t_{Fy}.e_{\mathbf{i}}q_j(x,y)).(e_{\mathbf{i}}k(x,y)_j +_j e_{\mathbf{i}}k(x,y)_j).\lambda_j^{-1} \\ &\quad = \underline{F}_j(x,y).(hx +_j hy).\lambda_j^{-1}. \end{aligned} \tag{77}$$

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