The horizontal/vertical synergy of double categories

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Category Theory Seminar

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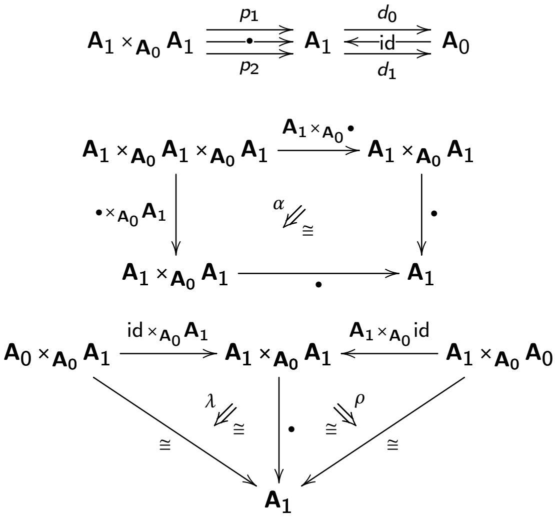
2-Categories vs bicategories

- 2-categories are categories enriched in **Cat**
- bicategories are “monoidal categories with several objects”
- different focus
- double categories combine the two
- exploit the rich interrelationship

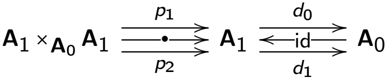
(Weak) double categories

A *(weak) double category* is a weak category object in $\mathscr{C}at$



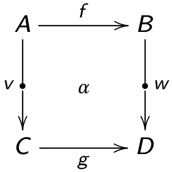
satisfying the usual coherence conditions (pentagon, etc.)

Think inside the box



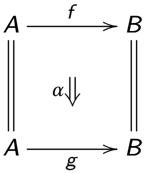
$\mathbf{A}_0$ – Objects and horizontal arrows

$\mathbf{A}_1$ – Vertical arrows and (double) cells

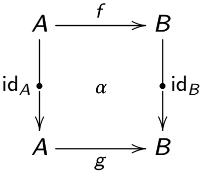


2-Categories

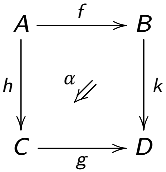
- $\mathscr{A} \rightsquigarrow \mathbb{H}\mathrm{or}\,\mathscr{A}$



- $\mathbb{A} \rightsquigarrow \mathscr{H}\mathit{or}\,\mathbb{A}$

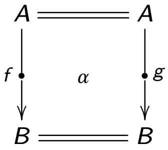


- $\mathscr{A} \rightsquigarrow \mathbb{Q}\mathscr{A}$

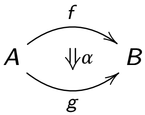


Bicategories

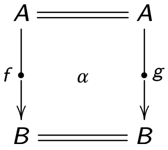
- $\mathscr{B} \rightsquigarrow \mathbb{V}\mathrm{ert}\mathscr{B}$



- $\mathbb{B} \rightsquigarrow \mathscr{V}ert\,\mathbb{B}$



is



Relations

$\mathbb{R}\mathrm{el}(\mathbf{A})$ for $\mathbf{A}$ a regular category

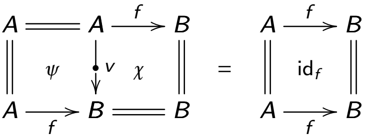
$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle R}\,\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow\,{\scriptstyle S} \\ B & \xrightarrow[g]{} & D \end{array} \qquad \text{iff} \qquad \begin{array}{ccc} R & \overset{\alpha}{\dashrightarrow} & S \\ \big\downarrow & & \big\downarrow \\ A \times B & \xrightarrow[f \times g]{} & C \times D \end{array}$$

Every morphism $f : A \longrightarrow B$ gives rise to two relations

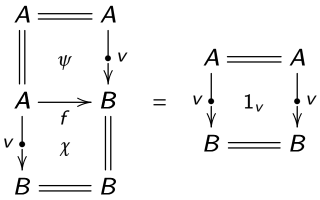
- its graph, $\langle 1_A, f \rangle : A \rightarrowtail A \times B$, called its companion $f_* : A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$
- the reverse of its graph, $\langle f, 1_A \rangle : A \rightarrowtail B \times A$, called its conjoint $f^* : B \longrightarrow\!\!\!\!\!\!\bullet\;\; A$

Companions

v is *companion* to f



$$\chi\psi = \mathrm{id}_f$$



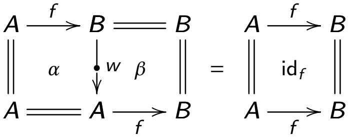
$$\chi \bullet \psi = 1_v$$

Proposition

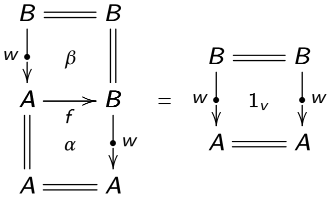
(1) If f has a companion it's unique up to isomorphism: write $v = f_*$
(2) $(1_A)_* \cong \mathrm{id}_A$
(3) $(gf)_* \cong g_* f_*$

Conjoints

w is *conjoint* to f



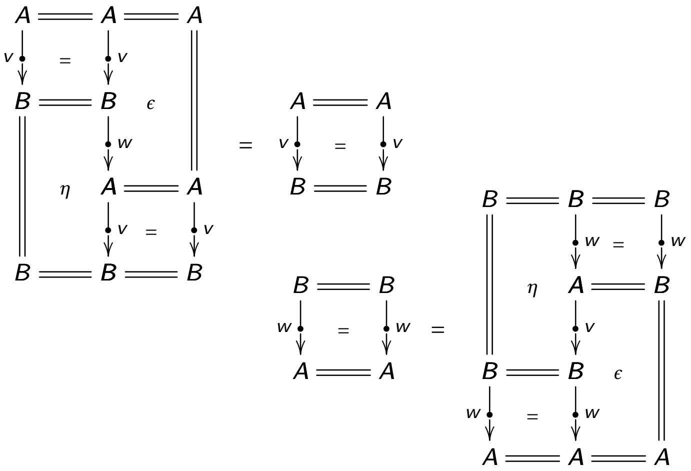
$$\beta\alpha = \mathrm{id}_f$$



$$\alpha \bullet \beta = 1_w$$

- Unique up to iso: write $w = f^*$
- $1_A^* \cong \mathrm{id}_A$
- $(gf)^* \cong f^* g^*$

Adjoints



$w \dashv v$

Companions, conjoints, adjoints

Theorem

Any two of the following conditions imply the third:

(1) $w = f_*$

(2) $v = f^*$

(3) $w \dashv v$

Theorem

In $\mathbb{R}\mathrm{el}(\mathbf{A})$

(1) Every f *has a companion:* $f_* = (A \overset{\langle 1_A, f\rangle}{\rightarrowtail} A \times B)$

(2) Every f *has a conjoint:* $f^* = (A \overset{\langle f, 1_A\rangle}{\rightarrowtail} B \times A)$

(3) Every adjoint pair $R \dashv S$ *is of the form* $f_* \dashv f^*$

Quintets

Proposition

In $\mathbb{Q}\mathscr{A}$

(1) Every horizontal arrow has a companion

(2) A horizontal arrow has a conjoint iff it has a left adjoint

$\mathbb{Q}\mathscr{A}$ is the free double category with companions generated by $\mathscr{A}$

It is also the cofree "cogenerated" by $\mathscr{A}^{tr}$

Theorem

(1) the identity $\mathscr{A} \longrightarrow \mathscr{H}or\mathbb{Q}\mathscr{A}$ is the unit for a biadjunction

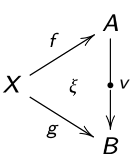
$$\mathbb{Q} \dashv \mathscr{H}or : \mathscr{D}oub_* \longrightarrow 2\text{-}\mathscr{C}at$$

(2) The identity $\mathscr{V}ert^{co}\mathbb{Q}\mathscr{A} \longrightarrow \mathscr{A}$ is the counit for a biadjunction

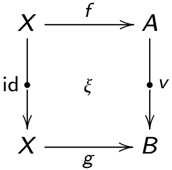
$$\mathscr{V}ert^{co} \dashv \mathbb{Q} : 2\text{-}\mathscr{C}at \longrightarrow st\mathscr{D}oub_*$$

Tabulators

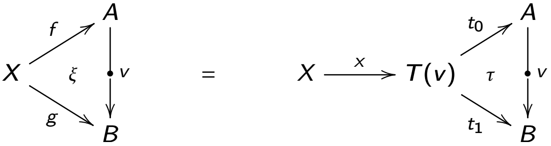
The *tabulator* of v is a universal cell of the form



i.e.



$$\forall \xi \, \exists ! x (\xi = \tau x)$$



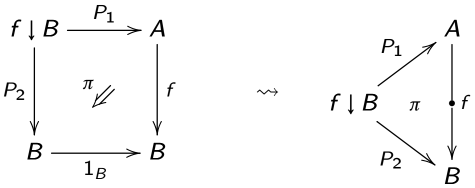
$T(v)$ is *effective* if t_1 has a companion, t_0 has a conjoint and $v \cong t_{1*} \bullet t_0^*$

- $\mathbb{R}\mathrm{el}(\mathbf{A})$ has effective tabulators

Tabulators for quintets

Theorem

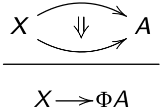
The tabulator of $f\colon A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$ in $\mathbb{Q}\mathscr{A}$ is the comma object $f \downarrow B$ (if it exists). It is effective



Corollary

If $\mathscr{A}$ has comma objects, $\mathbb{H}\mathrm{or}\mathscr{A}$ has tabulators

This is what Gray calls a *representable 2-category*



Monoidal categories

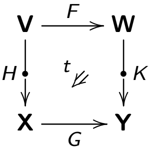
$\mathbb{M}\mathrm{on}\,\mathbb{C}\mathrm{at}$

Objects – monoidal categories

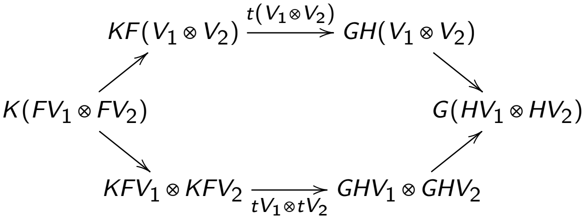
Horizontal arrows – monoidal functors $(FV_1 \otimes FV_2 \longrightarrow F(V_1 \otimes V_2), \ldots)$

Vertical arrows – comonoidal functors $(H(V_1 \otimes V_2) \longrightarrow HV_1 \otimes HV_2, \ldots)$

Cells



$t\colon KF \longrightarrow GH$ natural transformation



(and a pentagon for I as well)

Properties of $\mathbb{M}\mathrm{on}\,\mathbb{C}\mathrm{at}$

Theorem

For a monoidal functor $(F, \phi, \phi_0) \colon \mathbf{V} \longrightarrow \mathbf{W}$

(1) F has a companion iff ϕ *and* ϕ_0 *are isos, i.e. F is a strong monoidal functor*

(2) F has a conjoint iff F has a left adjoint

(3) $\mathbb{M}\mathrm{on}\,\mathbb{C}\mathrm{at}$ *has effective tabulators*

The tabulator of a vertical arrow $H \colon \mathbf{V} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{X}$ is $H \downarrow \mathbf{X}$

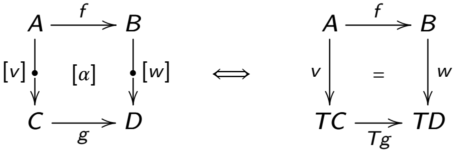
$$(V_1, HV_1 \longrightarrow X_1) \otimes (V_2, HV_2 \longrightarrow X_2)$$

$$= (V_1 \otimes V_2, H(V_1 \otimes V_2) \longrightarrow HV_1 \otimes HV_2 \longrightarrow X_1 \otimes X_2)$$

$P_1 \colon H \downarrow \mathbf{X} \longrightarrow \mathbf{V}$ has a left adjoint $L(V) = (V, HV \longrightarrow HV)$

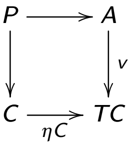
Kleisli double categories

$\mathbb{K}\mathrm{l}(T)$ for T a monad on $\mathbf{A}$



Theorem

(1) Every horizontal arrow $f: A \to B$ has a companion $[\eta B \cdot f]$
(2) $f: A \to B$ has a conjoint iff Tf iso
(3) $[v]: A \to C$ has a tabulator iff the pullback of v along ηC exists



2-monads

$\mathbb{K}\mathrm{l}(\mathbb{T})$ for $\mathscr{A}$ a 2-category, $\mathbb{T}$ a 2-monad on $\mathscr{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle [v]}\downarrow\!\!\bullet & [\alpha] & \bullet\!\!\downarrow{\scriptstyle [w]} \\ C & \xrightarrow[g]{} & D \end{array} \quad \text{is} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle w} \\ TC & \xrightarrow[Tg]{} & TD \end{array}$$

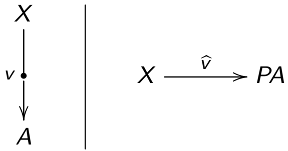
Theorem

(1) Every horizontal arrow f has a companion $f_* = [\eta B \cdot f]$
(2) f has a conjoint iff Tf has a left adjoint
(3) $[v]$ has a tabulator iff the comma object of v with ηC exists

$$\begin{array}{ccc} v \downarrow \eta C & \longrightarrow & A \\ \downarrow & \swarrow & \downarrow{\scriptstyle v} \\ C & \xrightarrow[\eta C]{} & TC \end{array}$$

Representability of vertical arrows

$\mathbb{A}$ has *representable vertical arrows* if for every A there is an object PA and a natural bijection



Examples:

(1) $\mathbb{K}\mathrm{l}(\mathbb{T})$

(2) $\mathbb{R}\mathrm{el}(\mathbf{A})$ iff $\mathbf{A}$ topos

(3) $\mathbb{P}\mathrm{ar}(\mathbf{A})$ iff $\mathbf{A}$ topos

Naturality?

Naturality uses companions

$$
\begin{array}{c}
Y \\
{\scriptstyle x_*}\big\downarrow\!\bullet \\
X \\
{\scriptstyle v}\big\downarrow\!\bullet \\
A
\end{array}
\quad\Bigg|\quad
Y \xrightarrow{\;x\;} X \xrightarrow{\;\widehat{v}\;} PA
\qquad\qquad
(\widehat{v}x = \widehat{v \cdot x_*})
$$

We want to define representable vertical arrows by the condition that

$$(\ )_* : \mathbf{Hor}\mathbb{A} \longrightarrow \mathbf{Vert}\mathbb{A}$$

have a right adjoint P

Functorial choice of companions

For this we need, first of all, that $\mathbb{A}$ be strict (α, λ, ρ) identities

$\mathbb{A}$ has a *functorial choice of companions* if there is a functor

$$(\;)_* : \mathbf{Hor}\,\mathbb{A} \longrightarrow \mathbf{Vert}\,\mathbb{A}$$

with binding cells

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle f_*}\big\downarrow & \chi_f & \big\| \\ A & = & A \end{array} \qquad \begin{array}{ccc} B & = & B \\ \big\| & \psi_f & \big\downarrow{\scriptstyle f_*} \\ B & \xrightarrow[f]{} & A \end{array}$$

such that

$$\chi_{fg} = \chi_f \bullet (\mathrm{id}_f \, \chi_g)$$
$$\psi_{fg} = (\psi_f \, \mathrm{id}_g) \bullet \psi_g$$

Strict representability

Assume that $\mathbb{A}$

- Is strict
- Has a canonical choice of companions
- Companions are functorial

We say that vertical arrows are *strictly representable* if

$$(\ )_* : \mathbf{Hor}\,\mathbb{A} \longrightarrow \mathbf{Vert}\,\mathbb{A}$$

has a right adjoint P

This means that for every A in $\mathbb{A}$ there is an object PA and a vertical arrow $eA\colon PA \longrightarrow\!\!\bullet\!\!\longrightarrow A$ such that for every vertical $v\colon X \longrightarrow\!\!\bullet\!\!\longrightarrow A$ there exists a unique horizontal $\widehat{v}\colon X \longrightarrow PA$ such that

$$v = e\mathbb{A} \bullet \widehat{v}_*$$

Examples:

- $\mathbb{Q}\mathscr{A}$
- $\mathbb{K}\mathrm{l}(\mathbb{T})$ a 2-monad
- $\mathbb{R}\mathrm{el}\,\mathbf{E}$ – ($\Leftrightarrow$ $\mathbf{E}$ a topos)
- $\mathbb{P}\mathrm{ar}\,\mathbf{E}$ – ($\Leftrightarrow$ $\mathbf{E}$ a topos)

Chaos

Example: Any category **A** gives a "chaotic" double category of squares $\mathbb{X}\mathbf{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\big\downarrow & ! & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

$$\mathbf{Hor}\mathbb{X}\mathbf{A} = \mathbf{Vert}\mathbb{X}\mathbf{A} = \mathbf{A}$$

A functorial choice of companions is any identity on objects functor

$$(\ )_* = F \colon \mathbf{A} \longrightarrow \mathbf{A}$$

E.g. **A** = the category of sets with pairs of functions as morphisms

$$1_{\mathbf{A}} \colon \mathbf{A} \longrightarrow \mathbf{A}$$

$$F(f,g) = (g,f)$$

$$G(f,g) = (f,f)$$

Crank it up a notch

If $\mathbb{A}$ has companions we get a pseudo-functor

$$(\ )_* \colon \mathbf{Hor}\,\mathbb{A} \longrightarrow \mathscr{V}ert\,\mathbb{A}$$

In fact all double categories with companions arise this way

If $\mathbf{A}$ is a category, $\mathscr{B}$ a bicategory and $\Phi\colon \mathbf{A} \longrightarrow \mathscr{B}$ a pseudo-functor, then we get a double category $\mathbb{Q}(\Phi)$

- Objects and horizontal arrows: $\mathbf{A}$
- Vertical arrows $A \longrightarrow\!\!\!\!\!\!\bullet\;\; A'$ are $\Phi A \longrightarrow \Phi A'$ in $\mathscr{B}$
- Cells

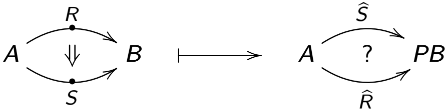
$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle [v]}\big\downarrow\bullet & \beta & \bullet\big\downarrow{\scriptstyle [w]} \\ A' & \xrightarrow[f']{} & C' \end{array} \quad \text{are} \quad \begin{array}{ccc} \Phi A & \xrightarrow{\Phi f} & \Phi C \\ {\scriptstyle v}\big\downarrow & \overset{\beta}{\swarrow} & \big\downarrow{\scriptstyle w} \\ \Phi A' & \xrightarrow[\Phi f']{} & \Phi C' \end{array}$$

Bi-adjoints are not the answer

$$(\ )_* : (\mathscr{H}or\,\mathbb{A})^{co} \longrightarrow \mathscr{V}ert\,\mathbb{A}$$

$$P : \mathscr{V}ert\,\mathbb{A} \longrightarrow (\mathscr{H}or\,\mathbb{A})^{co} \ ?$$

Let $\mathbb{A} = \mathbb{R}\mathrm{el} = \mathbb{R}\mathrm{el}(\mathbf{Set})$



$\mathbb{K}\mathrm{l}(\mathbb{T})$ doesn't work either

Indexed categories show the way

An indexed category, i.e. a pseudo-functor

$$\Phi \colon \mathbf{S}^{op} \longrightarrow \mathscr{C}at$$

is *essentially small* if there is a category object $\mathbb{C}$ in $\mathbf{S}$ such that

$$\Phi \simeq \mathbf{S}(-, \mathbb{C})$$

Explicitly

- There are

(1) $C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0$ in $\mathbf{S}$

(2) A generic object $G \in \Phi(C_0)$

(3) A generic morphism $g\colon \Phi(d_0)(G) \longrightarrow \Phi(d_1)(G)$

- such that

(4) For every I in $\mathbf{S}$ and A in $\Phi(I)$ there exist $\alpha\colon I \longrightarrow C_0$ and an isomorphism $A \cong \Phi(\alpha)(G)$

(5) For all $\alpha, \beta\colon I \longrightarrow C_0$ in $\mathbf{S}$ and $f\colon \Phi(\alpha)(G) \longrightarrow \Phi(\beta)(G)$ there exists a *unique* $\gamma\colon I \longrightarrow C_1$ such that

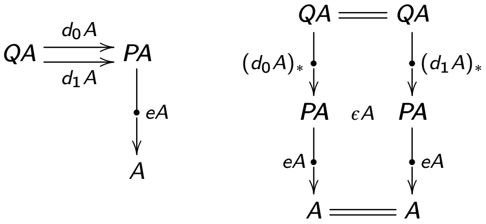
$$\begin{array}{ccc} & & C_0 \\ & \overset{\alpha}{\nearrow} & \uparrow d_0 \\ I & \overset{\gamma}{\longrightarrow} & C_1 \\ & \underset{\beta}{\searrow} & \downarrow d_1 \\ & & C_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Phi(\gamma)\Phi(d_0)(G) & \xrightarrow{\Phi(\gamma)} & \Phi(\gamma)\Phi(d_1)(G) \\ \cong \downarrow & & \downarrow \cong \\ \Phi(\alpha)(G) & \xrightarrow[f]{} & \Phi(\beta)(G) \end{array}$$

Back to representing vertical arrows

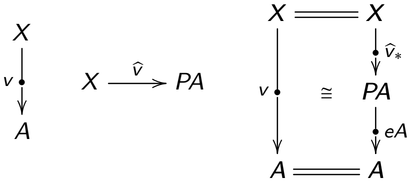
Apply this to

$$\Phi\colon \mathbf{Hor}(\mathbb{A})^{op} \xrightarrow{(\ )_*^{op}} \mathcal{V}ert(\mathbb{A})^{op} \xrightarrow{\mathcal{V}ert(\mathbb{A})(-,A)} \mathscr{C}at$$

Vertical morphisms into A are representable if there are

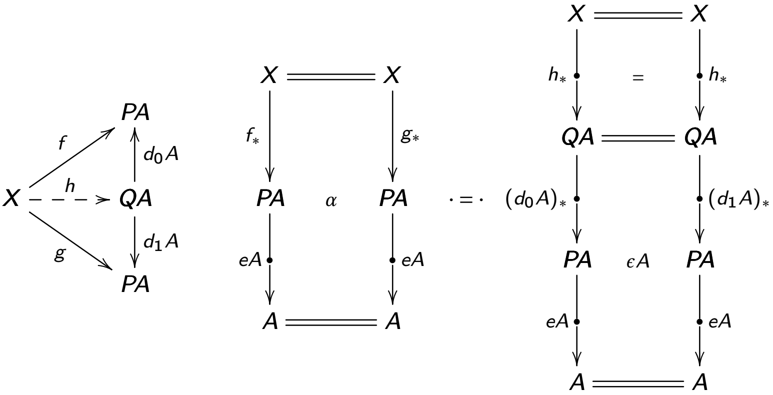


such that for every v there exist a $\widehat{v}$ and an iso $v \cong eA \bullet \widehat{v}_*$



Condition on cells

For every f, g and α as below there exists a unique h such that $\alpha = \epsilon A \bullet 1_{h_*}$



Our motivating examples

- $\mathbb{R}\mathrm{el}(\mathbf{E})$, $\mathbf{E}$ a topos

  PA is the power object $(= \Omega^A)$
  QA is the order on PA

- $\mathbb{P}\mathrm{ar}(\mathbf{E})$, $\mathbf{E}$ a topos

  PA is the partial morphism classifier $(= \widetilde{A})$
  QA is the order on it

- $\mathbb{K}\mathrm{l}(\mathbb{T})$, $\mathbb{T}$ a 2-monad on a 2-category $\mathscr{A}$

  $PA = TA$
  $QA = \Phi TA$, provided $\mathscr{A}$ is Gray representable

A different example

$\mathbb{M}\mathrm{on}\mathbb{C}\mathrm{at}_{str}$ – same as $\mathbb{M}\mathrm{on}\mathbb{C}\mathrm{at}$ but horizontal arrows are strong
Vertical arrows are corepresentable

$$\begin{array}{c|c} \mathbf{V} \\ H \downarrow\!\!\bullet \\ \mathbf{X} \end{array} \qquad P\mathbf{V} \xrightarrow{\widehat{H}} \mathbf{X}$$

$P\mathbf{V}$:

- Objects of $P\mathbf{V}$ are finite sequences of objects of $\mathbf{V}$

$$\langle V_1, V_2, \ldots, V_n \rangle$$

- Morphisms

$$(\alpha, \langle f_i \rangle_{i \in [n]}) : \langle V_i \rangle_{i \in [n]} \longrightarrow \langle V'_j \rangle_{j \in [m]}$$

$$\alpha : [m] \longrightarrow [n] \text{ order preserving}$$

$$f_i : V_i \longrightarrow \otimes_{\alpha(j)=i} V'_j$$

- $\otimes$ concatenation
- $Q\mathbf{V} = P(\mathbf{2} \times \mathbf{V})$?

Concluding remarks

- All of our examples were strict double categories
- The non-strict case is perhaps more important but harder
- Still unresolved (tantalizing) questions

E.g., $\mathbb{R}\mathrm{ing}$ – the double category of rings, homomorphisms, bimodules

$$\frac{\text{Vertical arrow } M\colon R \longrightarrow\!\!\!\!\!\bullet\;\; S}{\text{Additive functor} \quad R \longrightarrow S\text{-}\mathbf{Mod}}$$

Similarly
$\mathbb{C}\mathrm{at}$ – the double category of categories, functors, profunctors

$$\frac{\text{Vertical arrow } P\colon \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\;\; \mathbf{B}}{\text{Functor} \quad \mathbf{A} \to (\mathbf{Set}^{\mathbf{B}})^{op}}$$

$\mathbb{S}\mathrm{pan}\,\mathbf{A}$ – ??

Much more work to be done...

... and miles to go before I sleep

Thank you!