COMPOSING MODULES OF LAX FUNCTORS

ROBERT PARÉ

OTTAWA
OCTOBER 23, 2011

DOUBLE CATEGORIES

$\mathbb{A}$ OBJECTS, HORIZONTAL & VERTICAL ARROWS

CELLS

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}'
\end{tikzcd}
$$

HORIZONTAL AND VERTICAL COMPOSITION

HORIZONTALLY GET CATEGORY

VERTICALLY GET PSEUDO CATEGORY

(ASSOCIATIVITY UP TO SPECIAL ISO)

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\bullet"] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "\bullet"] \\
\bar{A} \arrow[r, equal] & \bar{A}
\end{tikzcd}
$$

<u>EXAMPLES</u>

- $\mathbb{S}\mathrm{ET}$ – SETS, FUNCTIONS, SPANS

$$\begin{tikzcd} X \arrow[d, "S"', "\bullet"] \\ Y \end{tikzcd} \quad \text{IS} \quad \begin{tikzcd} & X \\ S \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd}$$

- $\underline{V}$-$\mathbb{S}\mathrm{ET}$ ($\underline{V}$ $\otimes$-CAT WITH COPRODUCTS)

SETS, FUNCTIONS, $\underline{V}$-MATRICES

$$\begin{tikzcd} X \arrow[d, "{[V_{x,y}]}"', "\bullet"] \\ Y \end{tikzcd} \qquad V_{x,y} \in \underline{V}$$

- $\mathbb{C}\mathrm{AT}$ – SMALL CATEGORIES, FUNCTORS, PROFUNCTORS

$$\begin{tikzcd} \underline{A} \arrow[d, "P"', "\bullet"] \\ \underline{B} \end{tikzcd} \qquad P : \underline{A}^{op} \times \underline{B} \longrightarrow \underline{\mathrm{Set}}$$

- $\underline{V}$-$\mathbb{C}\mathrm{AT}$ (GOOD $\underline{V}$)

- $\mathbb{H}\underline{\underline{A}}$ ($\underline{\underline{A}}$ 2-CAT)

- $\mathbb{V}\underline{\underline{A}}$ ($\underline{\underline{A}}$ BICAT)

LAX FUNCTORS $\quad F : \mathbb{A} \longrightarrow \mathbb{X}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\ \bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fv"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'"] \\ F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}' \end{tikzcd}$$

HORIZONTAL COMPOSITION PRESERVED

VERTICAL COMPOSITION - COMPARISON CELLS

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "F(\bar{v}\cdot v)"] \\ F\bar{A} \arrow[d, "F\bar{v}"'] \arrow[r, phantom, "\varphi_{\bar{v},v}"] & \\ F\bar{\bar{A}} \arrow[r, equal] & F\bar{\bar{A}} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "id_{FA}"'] \arrow[dr, phantom, "\varphi_A"] & FA \arrow[d, "F(id_A)"] \\ FA \arrow[r, equal] & FA \end{tikzcd}$$

EXAMPLES

- $\mathbb{V}\underline{\underline{A}} \to \mathbb{V}\underline{\underline{B}}$ (LAX) MORPHISM OF BICATEGORIES
- $\mathbb{H}\underline{\underline{A}} \to \mathbb{H}\underline{\underline{B}}$ 2-FUNCTORS
- $\mathbb{1} \longrightarrow \mathbb{S}\mathrm{ET}$ SMALL CATEGORY
- $\mathbb{1} \longrightarrow \underline{V}\text{-}\mathbb{S}\mathrm{ET}$ SMALL $\underline{V}$-CATEGORY
- $\mathbb{A}(-, A) : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{ET}$

NATURAL TRANSFORMATIONS

$F, G : \mathbb{A} \longrightarrow \mathbb{X}$ LAX FUNCTORS

$t : F \longrightarrow G$

$A \longmapsto \left(FA \xrightarrow{tA} GA \right)$

$$\begin{tikzcd} A \arrow[d, "v"'] & \longmapsto & FA \arrow[r, "tA"] \arrow[d, "Fv"'] \arrow[dr, phantom, "tv"] & GA \arrow[d, "Gv"] \\ \bar{A} & & F\bar{A} \arrow[r, "t\bar{A}"'] & G\bar{A} \end{tikzcd}$$

- HORIZONTALLY NATURAL
- VERTICALLY FUNCTORIAL

EXAMPLES

$$\begin{tikzcd} \mathbb{V}\underline{\underline{A}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{V}\underline{\underline{B}} \end{tikzcd}$$ co ICONS

$$\begin{tikzcd} \mathbb{H}\underline{\underline{A}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{H}\underline{\underline{B}} \end{tikzcd}$$ 2- NATURAL TRANS

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{S}\mathrm{ET} \end{tikzcd}$$ FUNCTORS (ALSO $\underline{V}$-FUNCTORS)

$$\frac{\mathbb{A}(-, A) \longrightarrow \mathbb{A}(-, B)}{A \longrightarrow B}$$

MODULES

NATURAL TRANS $t : F \to G$ = HORIZONTAL

VERTICAL $F \to\!\!\!\!\!\!\circ\;\; G$?

COMES UP ALL THE TIME

- YONEDA $\mathbb{A} \longrightarrow \mathbb{L}\mathrm{ax}(\mathbb{A}^{op}, \mathbb{S}\mathrm{et})$
- LIMITS $\mathbb{A} \xrightarrow{\Delta} \mathbb{L}\mathrm{ax}(\mathbb{I}, \mathbb{A})$

COCKETT, KOSLOWSKI, SEELY, WOOD - MODULES (TAC)

$m : F \to\!\!\!\!\!\!\circ\;\; G$

$$\begin{tikzcd} A \arrow[d, "v"'] \\ \bar{A} \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[d, "mv"'] \\ G\bar{A} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'"] \\ \bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "mv"'] \arrow[dr, phantom, "m\alpha"] & FA' \arrow[d, "mv'"] \\ G\bar{A} \arrow[r, "G\bar{f}"'] & G\bar{A}' \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "v"'] \\ \bar{A} \arrow[d, "\bar{v}"'] \\ \bar{\bar{A}} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "m(\bar{v}\cdot v)"] \\ F\bar{A} \arrow[d, "m\bar{v}"'] \arrow[r, phantom, "\rho"] & \\ G\bar{\bar{A}} \arrow[r, equal] & G\bar{\bar{A}} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "mv"'] & FA \arrow[dd, "m(\bar{v}\cdot v)"] \\ G\bar{A} \arrow[d, "G\bar{v}"'] \arrow[r, phantom, "\lambda"] & \\ G\bar{\bar{A}} \arrow[r, equal] & G\bar{\bar{A}} \end{tikzcd}$$

- HORIZONTALLY FUNCTORIAL
- ASSOCIATIVE (3x) UNITARY (2x)
- NATURAL

EXAMPLES

- $\mathbb{1} \underset{\Downarrow}{\rightrightarrows} \mathbb{S}\mathrm{ET}$ PROFUNCTORS
- $\mathbb{1} \underset{\Downarrow}{\rightrightarrows} \underline{V}\text{-}\mathbb{S}\mathrm{ET}$ $\underline{V}$-PROFUNCTORS
- $A \overset{v}{\looparrowright} \bar{A} \longmapsto$ MODULE $\mathbb{A}(-,A) \xrightarrow{\mathbb{A}(-,v)} \mathbb{A}(-,\bar{A})$
- $\mathbb{V}\underline{\underline{A}} \underset{\Downarrow}{\rightrightarrows} \mathbb{V}\underline{\underline{B}}$ EXACTLY [CKSW]
- $\mathbb{V}\underline{A} \underset{\Downarrow}{\rightrightarrows} \mathbb{S}\mathrm{ET}$ PROFUNCTOR / $\underline{A}$
- $id_F : F \looparrowright F$, $id_F(v) = F(v) : F(A) \looparrowright F(\bar{A})$

PROP: A LAX FUNCTOR $\mathbb{V}2 \times \mathbb{A} \to \mathbb{X}$ IS THE SAME AS TWO LAX FUNCTORS $F, G : \mathbb{A} \to \mathbb{X}$ AND A MODULE $m : F \looparrowright G$.

COMPANIONS AND CONJOINTS

GIVEN $f : A \to B$ IN $\mathbb{A}$, A <u>COMPANION</u> FOR f IS A VERTICAL $f_* : A \to B$ WITH 2 CELLS

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "id_B"] \\ B \arrow[r, equal] & B \end{tikzcd} \quad \text{AND} \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "id_A"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f_*"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

SUCH THAT

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "id_A"'] \arrow[dr, phantom, "\eta"] & A \arrow[r, "f"] \arrow[d, "f_*"] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "id_B"] \\ A \arrow[r, "f"'] & B \arrow[r, equal] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "id_f"] & B \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \text{AND} \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "id"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f_*"] \\ A \arrow[r, "f"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "id"] \\ B \arrow[r, equal] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f_*"'] \arrow[dr, phantom, "1_{f_*}"] & A \arrow[d, "f_*"] \\ B \arrow[r, equal] & B \end{tikzcd}.$$

A <u>CONJOINT</u> FOR f IS $f^* : B \to A$ DUALLY DEF.

<u>PROP</u>: LET $t : F \to G$ BE A NATURAL TRANSF. IF FOR EVERY A, $tA : FA \to GA$ HAS A COMPANION $tA_* : FA \to GA$, THEN DEFINING $t_*(v) = G(v) \cdot tA_*$ GIVES A MODULE $t_* : F \to G$.

MODULATIONS [CKSW]

GIVEN LAX FUNCTORS $F, G, \bar{F}, \bar{G} : \mathbb{A} \to \mathbb{X}$,

NATURAL TRANSFORMATIONS $t, \bar{t}$,

MODULES m, m' AS IN

$$\begin{tikzcd} F \arrow[r, "t"] \arrow[d, "m"'] \arrow[dr, phantom, "\mu"] & G \arrow[d, "m"] \\ \bar{F} \arrow[r, "\bar{t}"'] & \bar{G} \end{tikzcd}$$

A <u>MODULATION</u> μ ASSIGNS TO EACH $v : A \to \bar{A}$

A CELL

$$\begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "m(v)"'] \arrow[dr, phantom, "\mu(v)"] & GA \arrow[d, "m(v)"] \\ \bar{F}\bar{A} \arrow[r, "\bar{t}\bar{A}"'] & \bar{G}\bar{A} \end{tikzcd}$$

SATISFYING

(1) HORIZONTAL NATURALITY

(2) EQUIVARIANCE

EXAMPLE $\mathbb{A}(-, \alpha)$

MULTIMODULATIONS [CKSW]

IN

$$\begin{tikzcd}
F_0 \arrow[d, "m_1"'] \arrow[drr, "t"] & & \\
F_1 \arrow[d, "m_2"'] & & G \arrow[d, "m"] \\
\vdots \arrow[d, "m_k"'] & \mu & \bar{G} \\
F_k \arrow[urr, "\bar{t}"'] & &
\end{tikzcd}$$

$F_i, G, \bar{G}$ ARE LAX FUNCTORS $\mathbb{A} \to \mathbb{X}$

$t, \bar{t}$ NATURAL TRASFORMATIONS

m_i, m MODULES

A <u>MULTIMODULATION</u> μ ASSIGNS TO EACH PATH OF VERTICAL ARROWS $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \cdots \xrightarrow{v_k} A_k$

A CELL

$$\begin{tikzcd}
F_0A_0 \arrow[d, "m_1 v_1"'] \arrow[drr, "tA_0"] & & \\
F_1A_1 \arrow[d, "m_2 v_2"'] & & GA_0 \arrow[d, "m(v_k \cdot \dots \cdot v_1)"] \\
\vdots \arrow[d, "m_k v_k"'] & \mu(v_k, \dots, v_1) & \bar{G}A_k \\
F_kA_k \arrow[urr, "\bar{t}A_k"'] & &
\end{tikzcd}$$

SATISFYING ① HORIZ. NAT. ② EQUIVARIANCE

COMPOSITION OF MODULES

PROBLEM: GIVEN MODULES $F \xrightarrow{m} G \xrightarrow{n} H$

CONSTRUCT A MODULE $n \cdot m : F \longrightarrow H$

AND A MULTIMODULATION

$$\begin{tikzcd} F \arrow[d, "m"'] \arrow[r, equal] & F \arrow[dd, "n \cdot m"] \\ G \arrow[d, "n"'] \arrow[r, phantom, "2"] & \\ H \arrow[r, equal] & H \end{tikzcd} \qquad 2 : n, m \Longrightarrow n \cdot m$$

WITH THE (STRONG) UNIVERSAL PROPERTY

$$\frac{n, m \longrightarrow p}{n \cdot m \longrightarrow p} \qquad (\text{U.P.})$$

$$\frac{\vec{x}, n, m, \vec{y} \longrightarrow p}{\vec{x}, n \cdot m, \vec{y} \longrightarrow p} \qquad (\text{STRONG U.P.})$$

U.P. DETERMINES $n \cdot m$ UP TO SPECIAL ISO

STRONG U.P. IMPLIES ASSOCIATIVITY

THE FORMULA FROM [CKSW]

$$\sum_{J} M(v_3)\cdot G(v_2)\cdot m(v_1) \rightrightarrows \sum_{I} M(v_2)\cdot m(v_1) \twoheadrightarrow (M\cdot m)(v)$$

I IS THE SET OF DIAGRAMS

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v_1"'] & A \arrow[dd, "v"] \\
A_1 \arrow[d, "v_2"'] \arrow[r, phantom, "\alpha"] & \\
\bar{A} \arrow[r, equal] & \bar{A}
\end{tikzcd}$$

J IS THE SET OF DIAGRAMS

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v_1"'] \arrow[dr, phantom, "1"] & A \arrow[r, equal] \arrow[d, "v_1"] & A \arrow[ddd, "v"] \\
A_1 \arrow[r, equal] \arrow[d, "v_2"'] & A_1 \arrow[dd, "v_2'"] \arrow[r, phantom, "\alpha_1"] & \\
A_2 \arrow[d, "v_3"'] \arrow[r, phantom, "\beta_1"] & & \\
\bar{A} \arrow[r, equal] & \bar{A} \arrow[r, equal] & \bar{A}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v_1"'] & A \arrow[r, equal] \arrow[dd, "v_1'"] & A \arrow[ddd, "v"] \\
A_1 \arrow[d, "v_2"'] \arrow[r, phantom, "\beta_2"] & & \\
A_2 \arrow[r, equal] \arrow[d, "v_3"'] \arrow[dr, phantom, "1"] & A_2 \arrow[d, "v_3"] \arrow[uu, phantom, "\alpha_2"'] \arrow[r, phantom, "\alpha_2"] & \\
\bar{A} \arrow[r, equal] & \bar{A} \arrow[r, equal] & \bar{A}
\end{tikzcd}$$

PROBLEMS: (1) UNWIELDY (E.G. FOR ASSOCIATIVITY)

(2) DOESN'T WORK FOR DOUBLE CATEGORIES — CAN'T DEFINE $M\cdot m$ ON CELLS.

THE PLAN

ENLARGE THE DIAGRAM OVER WHICH WE TAKE $\underrightarrow{\mathrm{LIM}}$

- MAKES IT MORE FUNCTORIAL
- BRINGS OUT "GEOMETRIC CONTENT"
- MAKES IT MORE GENERAL — APPLIES TO LAX DOUBLE CATEGORIES

NOTATION

FIX DOUBLE CATEGORIES $\mathbb{A}, \mathbb{X}$

LAX FUNCTORS $F_i : \mathbb{A} \longrightarrow \mathbb{X}$ $\quad i = 0, \dots, k$

MODULES $F_0 \overset{m_1}{\longrightarrow} F_1 \overset{m_2}{\longrightarrow} \cdots \overset{m_k}{\longrightarrow} F_k$

WE WILL CONSTRUCT THE COMPOSITE

$m : F_0 \longrightarrow F_k$ AS A LOCAL COLIMIT IN $\mathbb{X}$

THE DIAGRAM

LET $v : A \to \bar{A}$ BE A VERTICAL ARROW IN $\mathbb{A}$

$\underline{D}_v$ IS THE CATEGORY WITH OBJECTS $(\sigma, \vec{v}, \alpha)$

$$
\begin{tikzcd}
{[m]} \arrow[d, two heads, "\sigma"'] \\
{[k]}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[d, "v_1"'] \arrow[dr, equal] & \\
A_1 \arrow[d, "v_2"'] & A \arrow[d, "v"] \\
\vdots \arrow[d, "v_m"'] \arrow[r, phantom, "\alpha"] & \bar{A} \\
A_m \arrow[ur, equal] &
\end{tikzcd}
$$

$\sigma : \{0, 1, \dots, m\} \to \{0, 1, \dots, k\}$ ORDER PRESERVING SURJECTION

SERVES TO MARK k DISTINCT v_i

$\sigma(i)$ = THE NUMBER OF MARKED v's UP TO v_i

MORPHISMS

$$(\varphi, \vec{f}, \vec{\gamma}) : (\sigma, \vec{v}, \alpha) \longrightarrow (\tau, \vec{w}, \beta)$$

$$\begin{tikzcd}
{[m]} \arrow[rr, "\varphi"] \arrow[dr, "\tau"'] \arrow[rr, phantom, shift right=6, "\equiv"] & & {[n]} \arrow[dl, "\sigma"] \\
& {[k]} &
\end{tikzcd}$$

φ PRESERVES END POINTS AND ORDER

$f_j : A_{\varphi(j)} \longrightarrow B_j$ $\qquad$ f_0, f_m IDENTITIES

$$\begin{tikzcd}
A_{\varphi(j-1)} \arrow[r, "f_{j-1}"] \arrow[d, "v_{\varphi(j-1)+1}"'] & B_{j-1} \arrow[dd, "w_j"] \\
\vdots \arrow[d, "v_{\varphi(j)}"'] \arrow[r, phantom, "\gamma_j"] & \\
A_{\varphi(j)} \arrow[r, "f_j"'] & B_j
\end{tikzcd}$$

$$\beta(\gamma_m \circ \cdots \circ \gamma_1) = \alpha$$

E.G.

$$\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[dd, "v_1"'] & B_0 \arrow[dr, equal] \arrow[dd, "w_1"] & \\
& & A \arrow[ddd, "v"] \\
A_1 \arrow[d, "v_2"'] \arrow[r, phantom, "\gamma_1"] & B_1 \arrow[dd, "w_2"] & \\
A_2 \arrow[ur] \arrow[dr] \arrow[dd, "v_3"'] \arrow[r, phantom, "\gamma_2"] & {} \arrow[r, phantom, "\beta"] & {} \\
& B_2 \arrow[d, "w_3"] & \bar{A} \\
A_3 \arrow[d, "v_4"'] \arrow[r, phantom, "\gamma_3"] & B_3 \arrow[ur, equal] & \\
A_4 \arrow[ur, equal] & &
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[dr, equal] \arrow[dd] & \\
& A \arrow[ddd, "v"] \\
A_1 \arrow[d] & \\
A_2 \arrow[dd] \arrow[r, phantom, "\alpha"] & {} \\
& \bar{A} \\
A_3 \arrow[d] & \\
A_4 \arrow[uur, equal] &
\end{tikzcd}$$

HOM CATEGORIES

THE $\varinjlim$ WILL BE TAKEN IN THE VERTICAL HOM CATEGORIES OF $\mathbb{X}$:

$\mathbb{X}_{vert}(X,Y)$: OBJECTS
$$\begin{tikzcd} X \arrow[d, "u"'] \\ Y \end{tikzcd}$$

MORPHISMS
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\xi"] & X \arrow[d, "u'"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

DEFINE $\Gamma_v : \underline{D}_v \longrightarrow \mathbb{X}_{vert}(F_0 A, F_k \bar{A})$

$$\Gamma_v(\sigma, \vec{v}, \alpha) = \begin{tikzcd} F_{\sigma 0} A_0 \arrow[r, equal] \arrow[d, "u_1"'] & F_0 A_0 \\ F_{\sigma 1} A_1 \arrow[d, "u_2"'] & \\ \vdots \arrow[d, "u_n"'] & \\ F_{\sigma n} A_n \arrow[r, equal] & F_k A_n \end{tikzcd}$$

$$u_i = \begin{cases} F_{\sigma i} v_i & \text{IF } \sigma(i) = \sigma(i-1) \\ m_{\sigma i} v_i & \text{IF } \sigma(i) = \sigma(i-1)+1 \end{cases}$$

ON MORPHISMS "OBVIOUS".

LOCAL COLIMITS

$\mathbb{X}$ HAS <u>LOCAL COLIMITS</u> IF

1. EACH VERT HOM CATEGORY $\mathbb{X}_{vert}(X,Y)$ HAS $\varinjlim$

2. FOR EVERY $\bar{X} \xrightarrow{u} X$, $Y \xrightarrow{v} \bar{Y}$ THE FUNCTOR

   $\mathbb{X}_{vert}(u,v) : \mathbb{X}_{vert}(X,Y) \longrightarrow \mathbb{X}_{vert}(\bar{X},\bar{Y})$ PRESERVES $\varinjlim$

3. FOR EVERY $X \xrightarrow{x} X'$, $Y \xrightarrow{y} Y'$, $\varinjlim \Gamma$ HAS THE PROPERTY:

   FOR EVERY COCONE OF CELLS

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "\Gamma_I"'] \arrow[dr, phantom, "\xi_I"] & X' \arrow[d, "u'"] \\ Y \arrow[r, "y"'] & Y' \end{tikzcd}$$

   THERE IS A UNIQUE CELL

$$\varinjlim \quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\xi"] & X' \arrow[d, "u'"] \\ Y \arrow[r, "y"'] & Y' \end{tikzcd}$$

   SUCH THAT $\xi \gamma_I = \xi_I$.

<u>REM</u>: 3. AUTOMATIC IF $\mathbb{X}$ HAS COMPANIONS & CONJOINTS

NEW [CKSW] FORMULA

ASSUME $\mathbb{X}$ HAS LOCAL COLIMITS

$$m(v) = \varinjlim \Gamma_v$$

GIVEN $\bar{v} : \bar{A} \longrightarrow \bar{\bar{A}}$ WE GET

$$\begin{tikzcd}
\underline{D}_v \arrow[r, "\bar{v}\circ -"] \arrow[d, "\Gamma_v"'] & \underline{D}_{\bar{v}\circ v} \arrow[d, "\Gamma_{\bar{v}\circ v}"] \\
\mathbb{X}_{\mathrm{rel}}(F_0 A, F_k \bar{A}) \arrow[r, "F_k\bar{v}\circ -"'] & \mathbb{X}_{\mathrm{rel}}(F_0 A, F_k \bar{\bar{A}})
\end{tikzcd}$$

$$\begin{tikzcd}
{[n]} \arrow[d, "\sigma"'] & A_0 \arrow[d, "v_1"'] \arrow[dr] & \\
{[k]} & \vdots \arrow[d, "v_n"'] \arrow[r, phantom, "\alpha"] & A \arrow[d, "v"] \\
& A_n \arrow[r, equal] & \bar{A}
\end{tikzcd}
\overset{\bar{v}\circ -}{\longmapsto}
\begin{tikzcd}
{[n+1]} \arrow[d, "\bar{\sigma}"'] & A_0 \arrow[d, "v_1"'] \arrow[r, equal] & A \arrow[d, "v"] \\
{[k]} & \vdots \arrow[d, "v_k"'] \arrow[r, phantom, "\alpha"] & \bar{A} \\
& A_n \arrow[d, "v_{n+1}"'] \arrow[ur, equal] \arrow[r, phantom, "1_{\bar{v}}"] & \bar{A} \arrow[d, "\bar{v}"] \\
& A_{n+1} \arrow[r, equal] & \bar{\bar{A}}
\end{tikzcd}$$

GIVES LEFT ACTION FOR m

A FACTORIZATION PROPERTY

IN ORDER TO MAKE m HORIZONTALLY FUNCTORIAL WE NEED A CERTAIN FACTORIZATION PROPERTY ON $\mathbb{A}$!

(AFP) EVERY CELL

$$\begin{tikzcd}
A_0 \arrow[d, "v_1"'] \arrow[rr, bend left, "f"] & & A \arrow[dd, "v"] \\
A_1 \arrow[d, "v_2"'] \arrow[rr, phantom, "\alpha"] & & \\
A_2 \arrow[rr, bend right, "g"'] & & \bar{A}
\end{tikzcd}$$

FACTORS AS

$$\begin{tikzcd}
A_0 \arrow[r, "f"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\beta_1"] & A \arrow[d] \arrow[r, equal, bend left] & A \arrow[dd, "v"] \\
A_1 \arrow[r, "h"] \arrow[d, "v_2"'] \arrow[dr, phantom, "\beta_2"] & B \arrow[d] \arrow[r, phantom, "\beta"] & \\
A_2 \arrow[r, "g"'] & \bar{A} \arrow[r, equal, bend right] & \bar{A}
\end{tikzcd}$$

UNIQUELY UP TO THE EQUIVALENCE REL. GEN. BY

$$(\gamma, \gamma_1\beta_1, \gamma_2\beta_2) \sim (\gamma(\gamma_2 \cdot \gamma_1), \beta_1, \beta_2)$$

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1"] & A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\gamma_1"] & A \arrow[d] \arrow[r, equal, bend left] & A \arrow[dd, "v"] \\
A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma_2"] & C \arrow[d] \arrow[r, phantom, "\gamma"] & \\
A_2 \arrow[r] & \bar{A} \arrow[r, equal] & \bar{A} \arrow[r, equal, bend right] & \bar{A}
\end{tikzcd}$$

HORIZONTAL FUNCTORIALITY

IF $\mathbb{A}$ HAS COMPANIONS AND CONJOINTS (E.G. A BICATEGORY) THEN IT SATISFIES (AFP)

$\mathbb{H}\underline{A}$ ($\underline{A}$ CATEGORY) SATISFIES (AFP) BUT DOES NOT HAVE COMPANIONS OR CONJOINTS

$\underline{\underline{A}} =$
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$
, $\mathbb{H}\underline{\underline{A}}$ DOES NOT SATISFY (AFP)

THEOREM: IF $\mathbb{A}$ SATISFIES (AFP) AND $\mathbb{X}$ HAS LOCAL COLIMITS, THEN m AS DEFINED ABOVE HAS THE STRONG REPRESENTABILITY PROPERTY FOR THE COMPOSITE $m_k \bullet \cdots \bullet m_1$. THUS $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{X})$ IS A DOUBLE CATEGORY.