

COMPOSING MODULES OF LAX FUNCTORS

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# DOUBLE CATEGORIES

1A OBJECTS, HORIZONTAL & VERTICAL ARROWS  
CELLS

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow \alpha & & \downarrow \alpha' \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{A}'
 \end{array}$$

HORIZONTAL AND VERTICAL COMPOSITION

HORIZONTALLY GET CATEGORY

VERTICALLY GET PSEUDO CATEGORY

(ASSOCIATIVITY UP TO SPECIAL ISO)

$$\begin{array}{ccc}
 A & \rightrightarrows & A \\
 \downarrow & \alpha & \downarrow \\
 \bar{A} & \rightrightarrows & \bar{A}
 \end{array}$$

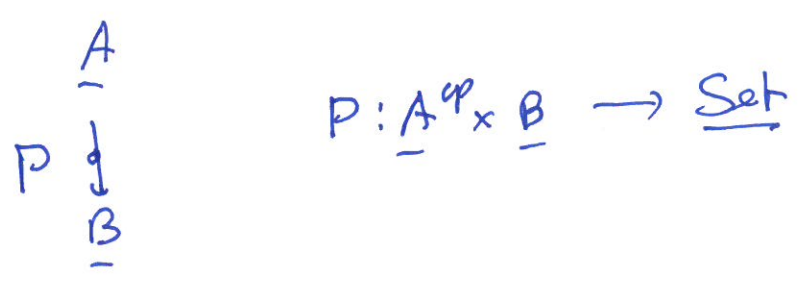
EXAMPLES • SET - SETS, FUNCTIONS, SPANS



- V-SET (V ⊗-CAT WITH COPRODUCTS)  
SETS, FUNCTIONS, V-MATRICES



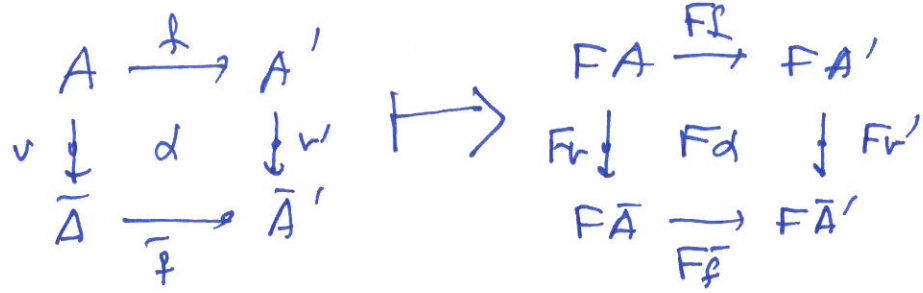
- CAT - SMALL CATEGORIES, FUNCTORS, PROFUNCTORS



- V-CAT (GOOD V)
- H|A (A Z-CAT)
- VA (A BICAT)

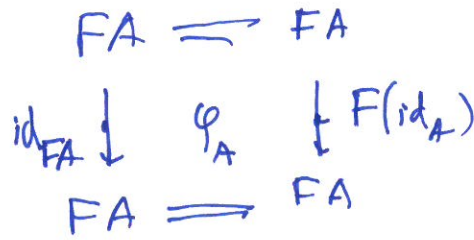
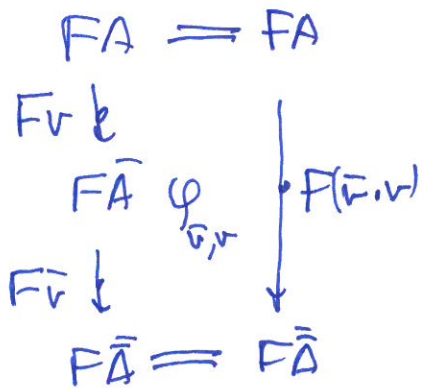
# LAX FUNCTORS

$$F : \mathcal{A} \longrightarrow \mathcal{X}$$



HORIZONTAL COMPOSITION PRESERVED

VERTICAL COMPOSITION - COMPARISON CELLS



## EXAMPLES

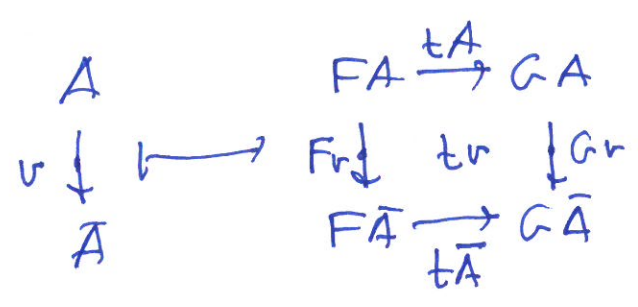
- $\underline{\mathcal{V}}_{\mathcal{A}} \rightarrow \underline{\mathcal{V}}_{\mathcal{B}}$  (LAX) MORPHISM OF BICATEGORIES
- $\underline{H}_{\mathcal{A}} \rightarrow \underline{H}_{\mathcal{B}}$  2-FUNCTORS
- $\mathbb{1} \rightarrow \text{SET}$  SMALL CATEGORY
- $\mathbb{1} \rightarrow \underline{\mathcal{V}}\text{-SET}$  SMALL  $\underline{\mathcal{V}}$ -CATEGORY
- $\mathcal{A}(-, \Delta) : \mathcal{A}^{op} \rightarrow \text{SET}$

# NATURAL TRANSFORMATIONS

$F, G: A \rightarrow X$  LAX FUNCTORS

$t: F \rightarrow G$

$$A \mapsto (FA \xrightarrow{t_A} GA)$$



- HORIZONTALLY NATURAL
- VERTICALLY FUNCTORIAL

## EXAMPLES

$\underline{V}A \xrightarrow{\Downarrow} \underline{V}B$  CO ICONS

$\underline{H}A \xrightarrow{\Downarrow} \underline{H}B$  2-NATURAL TRANS

$\mathbb{1} \xrightarrow{\Downarrow} \text{SET}$  FUNCTORS (ALSO V-FUNCTORS)

$$\frac{A(-, A) \rightarrow A(-, B)}{A \rightarrow B}$$

# MODULES

NATURAL TRANS  $t: F \rightarrow G = \text{HORIZONTAL}$

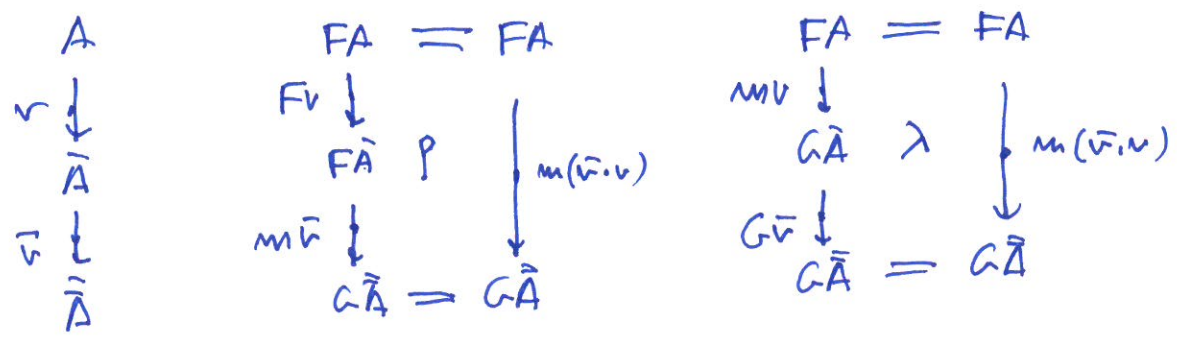
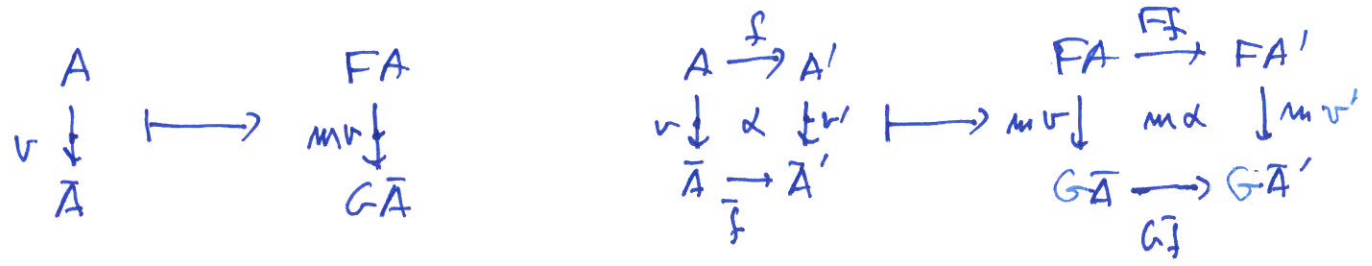
VERTICAL  $F \rightarrow G ?$

COMES UP ALL THE TIME

- YONEDA  $\mathcal{A} \rightarrow \text{Lax}(\mathcal{A}^{op}, \text{Set})$
- LIMITS  $\mathcal{A} \xrightarrow{\Delta} \text{Lax}(\mathbb{I}, \mathcal{A})$

COCKETT, KOSLOWSKI, SEELY, WOOD - MODULES (TAC)

$m: F \rightarrow G$



- HORIZONTALLY FUNCTORIAL
- ASSOCIATIVE (3x) UNITARY (2x)
- NATURAL



## EXAMPLES

- $\mathbb{1} \Downarrow \text{SET}$  PROFUNCTORS
- $\mathbb{1} \Downarrow \underline{V}\text{-SET}$   $\underline{V}$ -PROFUNCTORS
- $A \xrightarrow{v} \bar{A} \mapsto \text{MODULE } A(-, A) \xrightarrow{A(-, v)} A(-, \bar{A})$
- $\underline{V}A \Downarrow \underline{V}B$  EXACTLY [CKSW]
- $\underline{V}A \Downarrow \text{SET}$  PROFUNCTOR /  $\underline{A}$
- $\text{id}_F : F \rightarrow F$ ,  $\text{id}_F(v) = F(v) : F(A) \rightarrow F(\bar{A})$

PROP: A LAX FUNCTOR  $\forall \mathcal{D} \times \mathcal{A} \rightarrow \mathcal{X}$  IS

THE SAME AS TWO LAX FUNCTORS  $F, G : \mathcal{A} \rightarrow \mathcal{X}$

AND A MODULE  $m : F \rightarrow G$ .

# COMPANIONS AND CONJOINTS

GIVEN  $f: A \rightarrow B$  IN  $\mathcal{A}$ , A COMPANION

FOR  $f$  IS A VERTICAL  $f^*: A \rightarrow B$

WITH 2 CELLS

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 f_* \downarrow & \varepsilon & \downarrow id_B \\
 B & = & B
 \end{array}
 \quad \text{AND} \quad
 \begin{array}{ccc}
 A & = & A \\
 id_A \downarrow & \eta & \downarrow f^* \\
 A & \xrightarrow{f} & B
 \end{array}$$

SUCH THAT

$$\begin{array}{ccc}
 A & = & A \xrightarrow{f} B \\
 id_A \downarrow & \eta & \downarrow f^* \quad \varepsilon \quad \downarrow id_B \\
 A & \xrightarrow{f} & B = B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & id_f & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad \text{AND} \quad
 \begin{array}{ccc}
 A & = & A \\
 id \downarrow & \eta & \downarrow f^* \\
 A & \xrightarrow{f} & B \\
 f_* \downarrow & \varepsilon & \downarrow id \\
 B & = & B
 \end{array}
 =
 \begin{array}{ccc}
 A & = & A \\
 f_* \downarrow & \eta & \downarrow f^* \\
 B & = & B
 \end{array}$$

A CONJOINT FOR  $f$  IS  $f^*: B \rightarrow A$  DUALY DEF.

PROP: LET  $\varepsilon: F \rightarrow G$  BE A NATURAL TRANSF.

IF FOR EVERY  $A$ ,  $\varepsilon_A: FA \rightarrow GA$  HAS A COMPANION

$\varepsilon_{A^*}: FA \rightarrow GA$ , THEN DEFINING  $\varepsilon_{A^*}(v) = G(v) \circ \varepsilon_A$

GIVES A MODULE  $\varepsilon: F \rightarrow G$ .



# MODULATIONS [CKSW]

GIVEN LAX FUNCTORS  $F, G, \bar{F}, \bar{G} : A \rightarrow X$ ,

NATURAL TRANSFORMATIONS  $t, \bar{t}$ ,

MODULES  $m, m'$  AS IN

$$\begin{array}{ccc}
 F & \xrightarrow{t} & G \\
 m \downarrow & & \downarrow m \\
 \bar{F} & \xrightarrow{\bar{t}} & \bar{G}
 \end{array}$$

A MODULATION  $\mu$  ASSIGNS TO EACH  $v: A \rightarrow \bar{A}$

A CELL

$$\begin{array}{ccc}
 FA & \xrightarrow{\bar{t}_A} & GA \\
 m(v) \downarrow & & \downarrow m(v) \\
 \bar{F}\bar{A} & \xrightarrow{\bar{t}_{\bar{A}}} & \bar{G}\bar{A}
 \end{array}$$

SATISFYING

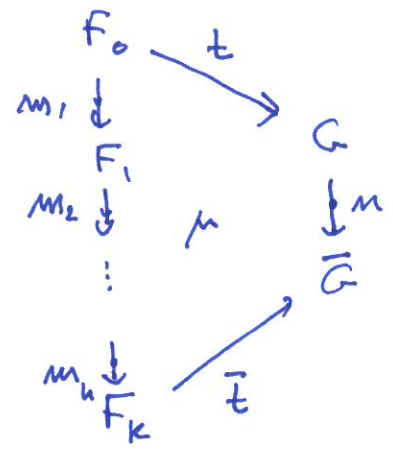
(1) HORIZONTAL NATURALITY

(2) EQUIVARIANCE

EXAMPLE  $A(-, d)$

# MULTIMODULATIONS [CKSW]

$\Sigma_N$



$F_i, G, \bar{G}$  ARE LAX FUNCTORS  $A \rightarrow X$

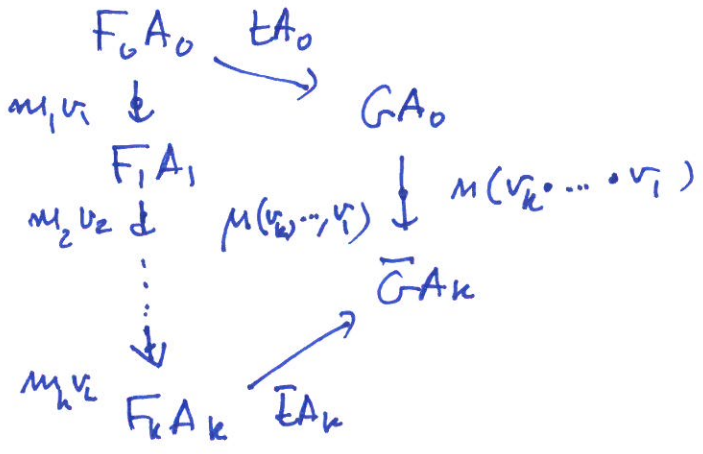
$t, \bar{t}$  NATURAL TRANSFORMATIONS

$m_i, m$  MODULES

A MULTIMODULATION  $\mu$  ASSIGNS TO EACH

PATH OF VERTICAL ARROWS  $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_k} A_k$

A CELL



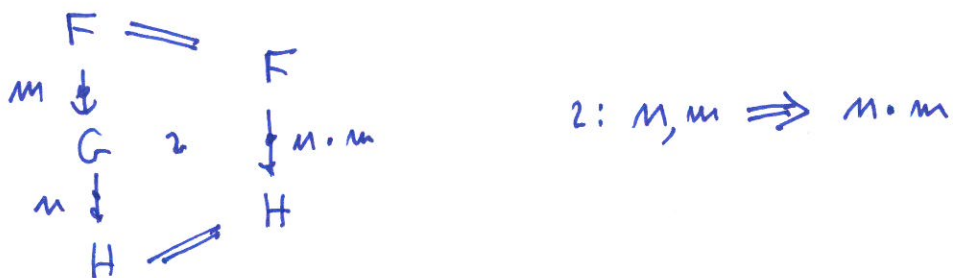
SATISFYING ① HORIZ. NAT. ② EQUIVARIANCE

# COMPOSITION OF MODULES

PROBLEM: GIVEN MODULES  $F \xrightarrow{m} G \xrightarrow{n} H$

CONSTRUCT A MODULE  $m \circ m : F \rightarrow H$

AND A MULTIMODULATION



WITH THE (STRONG) UNIVERSAL PROPERTY

$$\frac{m, m \rightarrow p}{m \circ m \rightarrow p} \quad (\text{U.P.})$$

$$\frac{\vec{x}, m, m, \vec{y} \rightarrow p}{\vec{x}, m \circ m, \vec{y} \rightarrow p} \quad (\text{STRONG U.P.})$$

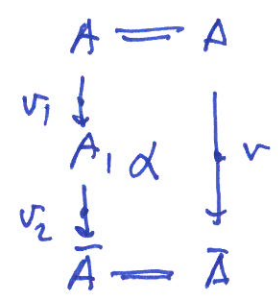
U.P. DETERMINES  $m \circ m$  UP TO SPECIAL ISO

STRONG U.P. IMPLIES ASSOCIATIVITY

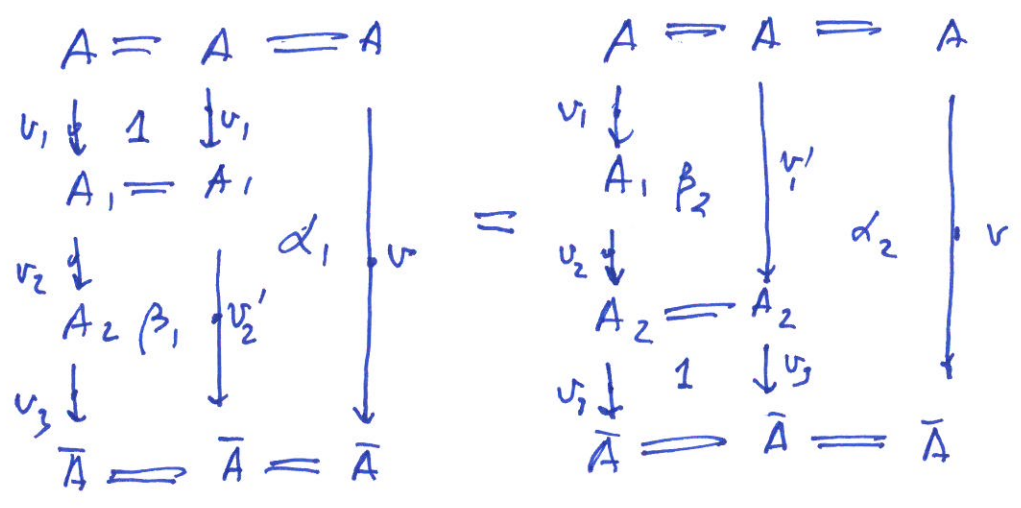
THE FORMULA FROM [CKSW]

$$\sum_J m(v_3) \cdot G(v_2) \cdot m(v_1) \Rightarrow \sum_I m(v_2) \cdot m(v_1) \rightarrow (m \cdot m)(v)$$

I IS THE SET OF DIAGRAMS



J IS THE SET OF DIAGRAMS



PROBLEMS : (1) UNWIELDY (E.G. FOR ASSOCIATIVITY)

(2) DOESN'T WORK FOR DOUBLE CATEGORIES - CAN'T DEFINE m \cdot m ON CELLS.

# THE PLAN

ENLARGE THE DIAGRAM OVER WHICH WE TAKE  $\varinjlim$

- MAKES IT MORE FUNCTORIAL
- BRINGS OUT "GEOMETRIC CONTENT"
- MAKES IT MORE GENERAL - APPLIES TO LAX DOUBLE CATEGORIES

# NOTATION

FIX DOUBLE CATEGORIES  $\mathbb{A}, \mathbb{X}$

LAX FUNCTORS  $F_i : \mathbb{A} \rightarrow \mathbb{X} \quad i = 0, \dots, k$

MODULES  $F_0 \xrightarrow{m_1} F_1 \xrightarrow{m_2} \dots \xrightarrow{m_k} F_k$

WE WILL CONSTRUCT THE COMPOSITE

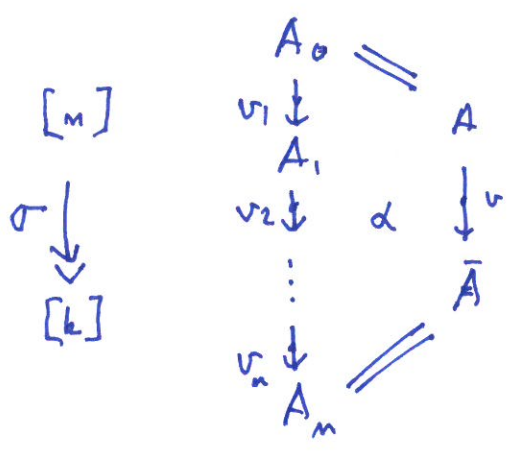
$m : F_0 \rightarrow F_k$  AS A LOCAL COLIMIT IN  $\mathbb{X}$



# THE DIAGRAM

LET  $v: A \rightarrow \bar{A}$  BE A VERTICAL ARROW IN  $\mathcal{A}$

$\underline{D}_v$  IS THE CATEGORY WITH OBJECTS  $(\sigma, \vec{v}, \alpha)$



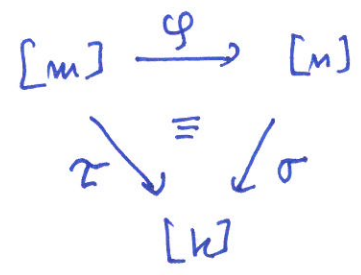
$\sigma: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, k\}$  ORDER PRESERVING SURJECTION

SERVES TO MARK  $k$  DISTINCT  $v_i$

$\sigma(i) =$  THE NUMBER OF MARKED  $v_i$ 'S UP TO  $v_i$

# MORPHISMS

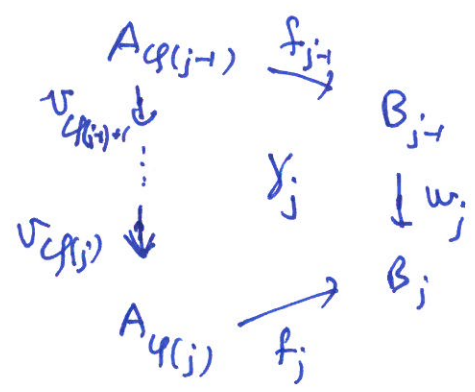
$$(\varphi, \vec{f}, \vec{\sigma}) : (\sigma, \vec{v}, \alpha) \rightarrow (\tau, \vec{w}, \beta)$$



$\varphi$  PRESERVES  
END POINTS  
AND ORDER

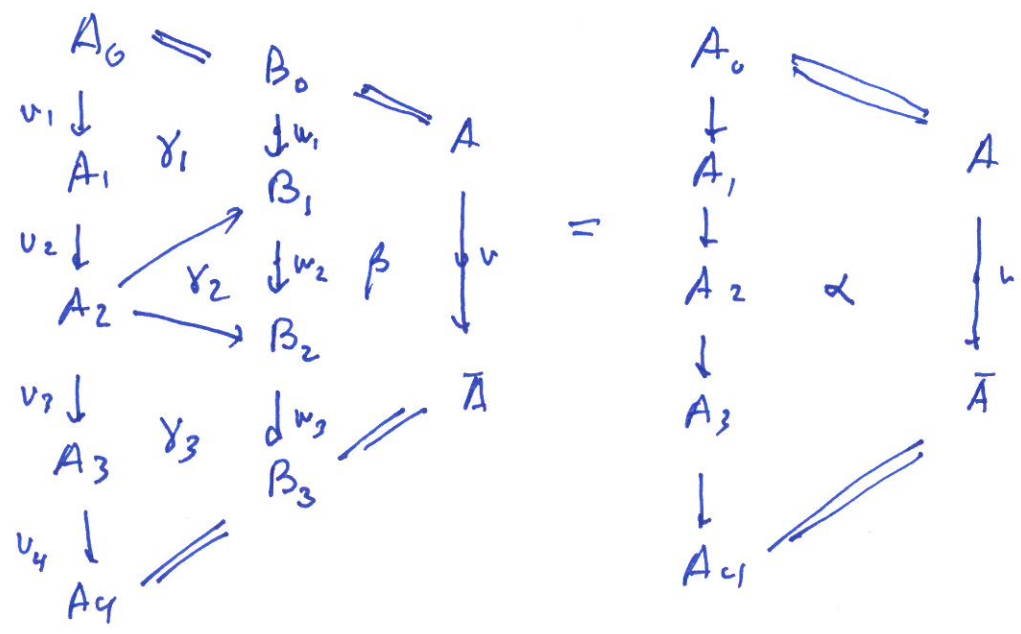
$$f_j : A_{\varphi(j)} \rightarrow B_j$$

$f_0, f_m$  IDENTITIES



$$\beta(\gamma_m \cdots \gamma_1) = \alpha$$

E.G.



# HOM CATEGORIES

THE  $\varinjlim$  WILL BE TAKEN IN THE VERTICAL

HOM CATEGORIES OF  $\mathbb{X}$  :

$\mathbb{X}_{\text{vert}}(X, Y)$  : OBJECTS  $\begin{array}{c} X \\ \mu \downarrow \\ Y \end{array}$

MORPHISMS  $\begin{array}{ccc} X & = & X \\ \mu \downarrow & \xi & \downarrow \mu' \\ Y & = & Y \end{array}$

DEFINE  $\Gamma_n : \underline{D}_v \longrightarrow \mathbb{X}_{\text{vert}}(F_0 A, F_k \bar{A})$

$$\Gamma_n(\sigma, \vec{v}, \alpha) = \begin{array}{c} F_{00} A_0 = F_0 A_0 \\ \mu_1 \downarrow \\ F_{01} A_1 \\ \mu_2 \downarrow \\ \vdots \\ \mu_n \downarrow \\ F_{0n} A_n = F_k A_n \end{array}$$

$$\mu_i = \begin{cases} F_{\sigma(i)} v_i & \text{IF } \sigma(i) = \sigma(i-1) \\ m_{\sigma(i)} v_i & \text{IF } \sigma(i) = \sigma(i-1) + 1 \end{cases}$$

ON MORPHISMS "OBVIOUS".

## LOCAL COLIMITS

$\mathcal{X}$  HAS LOCAL COLIMITS IF

1. EACH VERT HOM CATEGORY  $\mathcal{X}_{\text{vert}}(X, Y)$  HAS  $\underline{\text{LIM}}$
2. FOR EVERY  $\bar{X} \xrightarrow{u} X, Y \xrightarrow{v} \bar{Y}$  THE FUNCTOR  $\mathcal{X}_{\text{vert}}(u, v): \mathcal{X}_{\text{vert}}(X, Y) \rightarrow \mathcal{X}_{\text{vert}}(\bar{X}, \bar{Y})$  PRESERVES  $\underline{\text{LIM}}$
3. FOR EVERY  $X \xrightarrow{x} X', Y \xrightarrow{y} Y', \underline{\text{LIM}} \Gamma$  HAS THE PROPERTY:

FOR EVERY COCONE OF CELLS

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \Gamma_I \downarrow & \exists_I & \downarrow u' \\ Y & \xrightarrow{y} & Y' \end{array}$$

THERE IS A UNIQUE CELL

$$\underline{\text{LIM}} \rightarrow \begin{array}{ccc} X & \xrightarrow{x} & X' \\ \Gamma \downarrow & \exists & \downarrow u' \\ Y & \xrightarrow{y} & Y' \end{array}$$

SUCH THAT  $\exists \gamma_I = \exists_I$ .

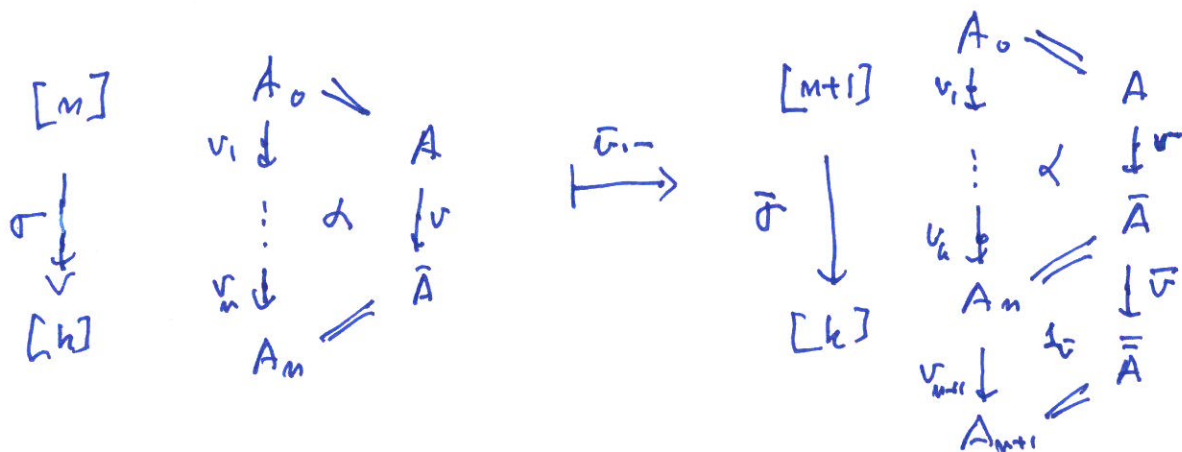
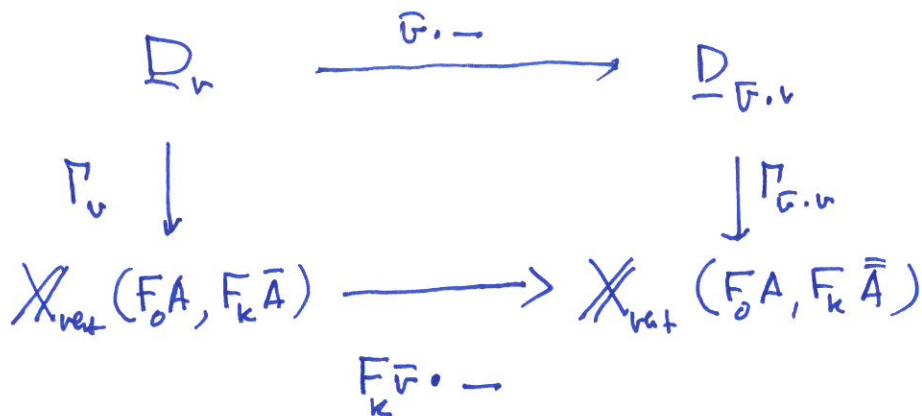
REM: 3. AUTOMATIC IF  $\mathcal{X}$  HAS COMPANIONS & CONJOINTS

# NEW [CKSW] FORMULA

ASSUME  $X$  HAS LOCAL COLIMITS

$$m(v) = \varinjlim_v \Gamma_v$$

GIVEN  $\bar{v}: \bar{A} \rightarrow \bar{A}$  WE GET



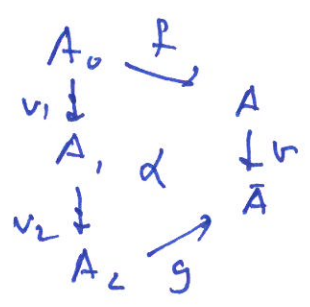
GIVES LEFT ACTION FOR  $m$



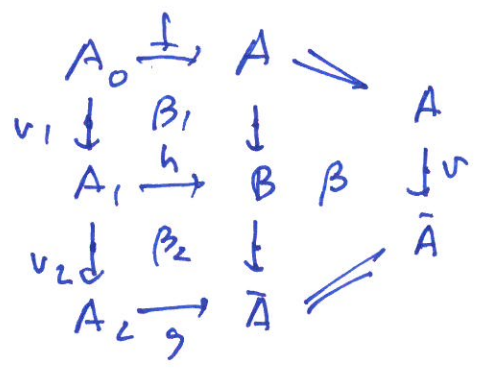
# A FACTORIZATION PROPERTY

IN ORDER TO MAKE  $m$  HORIZONTALLY FUNCTORIAL  
 WE NEED A CERTAIN FACTORIZATION  
 PROPERTY ON  $A$  !

(AFP) EVERY CELL

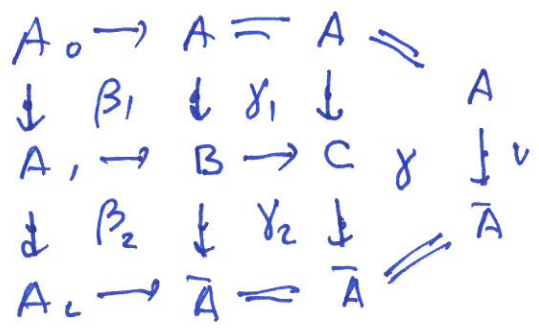


FACTORS AS



UNIQUELY UP TO THE EQUIVALENCE REL. GEN. BY

$$(\gamma, \gamma_1, \beta_1, \gamma_2, \beta_2) \sim (\gamma(\gamma_2 \cdot \gamma_1), \beta_1, \beta_2)$$



## HORIZONTAL FUNCTORIALITY

IF  $\mathcal{A}$  HAS COMPANIONS AND CONJOINTS

(E.G. A BICATEGORY) THEN IT SATISFIES (AFP)

$\mathbb{H}\underline{\mathcal{A}}$  ( $\underline{\mathcal{A}}$  CATEGORY) SATISFIES (AFP) BUT DOES

NOT HAVE COMPANIONS OR CONJOINTS

$\underline{\underline{\mathcal{A}}} = \cdot \Downarrow \cdot$  ,  $\mathbb{H}\underline{\underline{\mathcal{A}}}$  DOES NOT SATISFY (AFP)

THEOREM: IF  $\mathcal{A}$  SATISFIES (AFP) AND

$\mathcal{X}$  HAS LOCAL COLIMITS, THEN  $\mathcal{M}$

AS DEFINED ABOVE HAS THE STRONG

REPRESENTABILITY PROPERTY FOR THE

COMPOSITE  $m_k \circ \dots \circ m_1$ . THUS  $\mathbb{Lax}(\mathcal{A}, \mathcal{X})$

IS A DOUBLE CATEGORY.